Biases in Elliptic Curve Families

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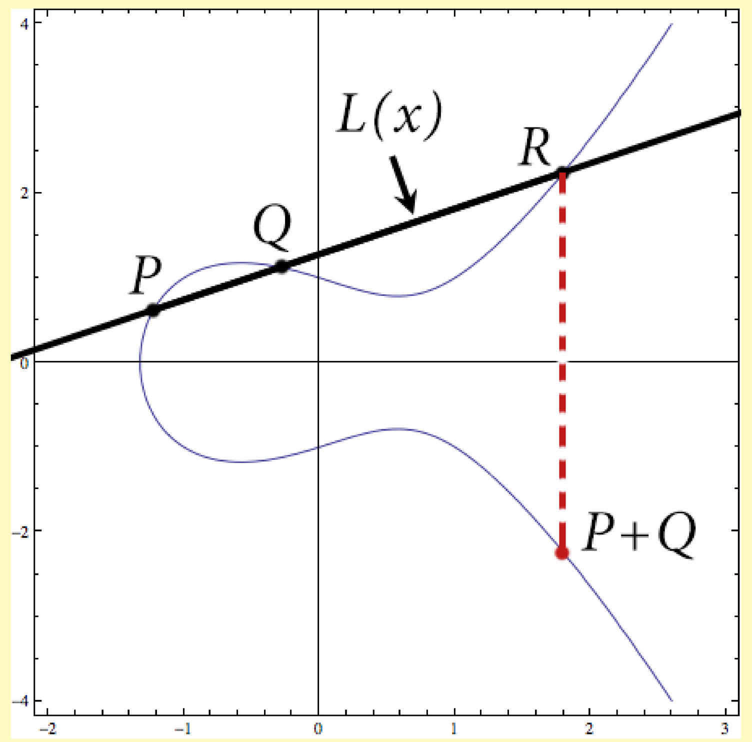
Introduction

An elliptic curve $E_{a,b}$ over $\mathbb{Q}$ are all the solutions (x, y) of $y^2 = x^3 + ax + b$ for a fixed pair $a, b \in \mathbb{Z}$. We write the number of solutions modulo p as $p - a_E(p)$. We can use these coefficients to build an L-function by setting $L(s, E_{a,b}) = \sum_n a_E(n)/n^s$, and many properties of the elliptic curve are encoded in this function. We study a one-parameter family of curves over $\mathbb{Q}(T)$: $\mathcal{E} : y^2 = x^3 + A(T)x + B(T)$, where now $A(T), B(T)$ are polynomials in $\mathbb{Z}[T]$ and each specialization of T to an integer t gives an elliptic curve over $\mathbb{Q}$. Let $A_{r,\mathcal{E}}(p) := \sum_{t \bmod p} a_{E_t}(p)^r$ be the r^{th} moment of the Fourier coefficients of the associated L-functions. As the first moments are related to the rank of the family over $\mathbb{Q}(T)$, it is natural to explore the distribution and consequences of the second moment.

The Group Law



The Biases Conjecture

Michel proved that $A_{2,\mathcal{E}}(p) = p^2 + O(p^{3/2})$ for families without complex multiplication, and cohomological arguments prove that the lower order terms are of sizes $p^{3/2}, p, p^{1/2}$ and 1. We have extensively studied thousands of families numerically and theoretically, and in each case the first term in the second moment expansion that does not average to zero has always had a negative bias. We conjecture that this bias always exists, and using methods from algebraic geometry and the theory of Legendre sums, we are able to prove this claim for many families. In particular, we consider families with rank and families with unusual distributions of signs. These nontrivial cases strongly support our bias conjecture.

The observed and proven negative bias of the lower order terms has implications towards the excess rank conjecture and the behavior of the zeros near the central point of elliptic curve L-functions. In 1998 Rosen and Silverman proved a conjecture of Nagao that the first moment $A_{1,\mathcal{E}}(p)$ is related to the rank; we end by formulating an analogous conjecture for the second moment, which we prove in some cases.

Proven Cases

We have proven the conjecture for a variety of specific families and some restricted cases. We list a few of these cases below. The average bias refers to the average value of the coefficient of the largest lower order term not averaging to 0 (which in all of our cases is the p term).

Consider elliptic curve families of the form $y^2 = ax^3 + bx^2 + cx + d + et$. These families have rank 0 over $\mathbb{Q}(t)$, and for primes $p > 3$ with $p \nmid a, e$ and $p \nmid b^2 - 3ac$,

$$A_2(p) = p^2 - p\left(1 + \left(\frac{b^2 - 3ac}{p}\right) + \left(\frac{-3}{p}\right)\right) \tag{1}$$

These families obey the Bias Conjecture with an average bias of -1.

Consider families of the form $y^2 = ax^3 + bx^2 + (ct + d)x$. These families have rank 0, and for primes $p > 3$ with $p \nmid a, b, c$,

$$A_2(p) = p^2 - p\left(1 + \left(\frac{-1}{p}\right)\right) \tag{2}$$

These families obey the Bias Conjecture with an average bias of -1.

Consider families of the form $y^2 = x^3 + t^n x$. These families have rank 0, and for primes $p > 3$,

$$A_2(p) = \begin{cases} (p-1)\left(\sum_{x(p)}\left(\frac{x^3+x}{p}\right)\right)^2 \\ (p^2 - p)\left(1 + \left(\frac{-1}{p}\right)\right) \end{cases} \tag{3}$$

Where the first line is for even n and the second line for odd n. These families obey the Bias Conjecture with an average bias of $-4/3$ for $n \equiv 0(2)$ and -1 for $n \equiv 1(2)$.

Numerical Data

Family	Average$(c_1(p))$	Average$(c_0(p))$
$y^2 = 4x^3 - 7x^2 + 4tx + 4$	0.0068	0.974
$y^2 = 4x^3 + 5x^2 + (4t - 2)x + 1$	-0.0176	1.005
$y^2 = 4x^3 + 5x^2 + (4t + 2)x + 1$	-0.0174	1.005
$y^2 = 4x^3 + x^2 + (4t + 2)x + 1$	0.0399	0.993
$y^2 = 4x^3 + x^2 + 4tx + 4$	0.0068	0.985
$y^2 = 4x^3 + x^2 + (4t + 6)x + 9$	-0.0113	1.988
$y^2 = 4x^3 + 4x^2 + 4tx + 1$	0.0072	0.974
$y^2 = 4x^3 + 5x^2 + (4t + 4)x + 4$	0.0035	1.012
$y^2 = 4x^3 + 4x^2 + 4tx + 9$	0.0256	1.005
$y^2 = 4x^3 + 5x^2 + 4tx + 4$	0.0043	1.005
$y^2 = 4x^3 + 5x^2 + (4t + 6)x + 9$	-0.0143	1.037

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