Large gaps between zeros of GL(2) L-functions

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Abstract

The distribution of critical zeros of the Riemann zeta function $\zeta(s)$ and other L-functions lies at the heart some of the most central problems in number theory. The Euler product of $\zeta(s)$ translates information about its zeros into knowledge about the distribution of the prime numbers; similar arithmetically important results (such as the ranks of elliptic curves and the size of the class number) hold for other L-functions. A natural question to ask is how often large gaps occur between critical zeros of L-functions relative to the normalized average gap size. A striking connection to random matrix theory suggests that the spacing distributions between zeros of many classes of L-functions behave similarly to the spacing distributions of eigenvalues of large Hermitian matrices. In particular, it is believed that arbitrarily large gaps between zeros occur infinitely often. However, few nontrivial results in this direction have been established.

Through the work of many researchers, the best result to date for $\zeta(s)$ is that gaps at least 2.69 times the mean spacing occur infinitely often, assuming the Riemann hypothesis. In the present work, we prove the first nontrivial result on the occurrence of large gaps between critical zeros of L-functions associated to primitive holomorphic cusp forms f of level one. Combining mean value estimates from Montgomery and Vaughan and extending a method of Ramachandra, we develop a procedure to compute shifted second moments, which are of interest to other questions besides our own. Using the mixed second moments of derivatives of $L(1/2+it, f)$, we prove that there are infinitely many gaps between consecutive zeros of $L(s, f)$ on the critical line which are at least $\sqrt{3}$ times the average spacing. Our techniques are general and promise similar results for other primitive GL(2) L-functions such as L-functions associated to Maass forms.

1. Zeros of L-functions: background and motivation

The motivating case of $\zeta(s)$

The Riemann zeta function $\zeta(s)$ is given for $\Re(s) > 1$ by the following absolutely convergent Dirichlet series and Euler product:

$$\zeta(s) = \sum_n n^{-s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \operatorname{Re}(s) > 1$$

Riemann Hypothesis: All non-trivial zeros have $\Re(s) = \frac{1}{2}$.

Critical zeros of L-functions are of central importance to many problems in number theory.

- The Euler product of $\zeta(s)$ translates knowledge about zeros of $\zeta(s)$ to knowledge about the distribution of prime numbers.
- Other classes of L-functions encode information about many mathematical objects, e.g., ranks of elliptic curves and class numbers of imaginary quadratic fields.

Spacings between zeros

- Classical question: how are the spacings between consecutive critical zeros distributed?
- Numerical observation: spacings between zeros behave statistically similarly to spacings between eigenvalues of large complex Hermitian matrices.

Conjecture 1.1. *Gaps that are arbitrarily large, relative to the average gap size, appear infinitely often.*

$$\text{Letting} \quad \lambda = \limsup \frac{(\gamma' - \gamma)\log\gamma}{2\pi},$$

Conjecture 1.1 is equivalent to $\lambda = \infty$. Few nontrivial results have been established. Even for the Riemann zeta function, unconditionally it is only known that

$$\lambda > 2.69.$$

2. GL(2) L-functions

What, really, is an L-function?

- Not any old Dirichlet series.
- Axiomatic definition of the Selberg class $\mathcal{S}$: $\mathcal{L}$ is in $\mathcal{S}$ if it is given by a Dirichlet series $\mathcal{L}(s) = \sum_{n=1}^{\infty} a(n)n^{-s}$ absolutely convergent for $\Re(s) > 1$ that satisfies the following:
  - $\mathcal{L}(s)$ admits an Euler product over primes in terms of local factors for $\Re(s) > 1$:
    $$\mathcal{L}(s) = \prod_p \mathcal{L}_p(s).$$
  - $\mathcal{L}(s)$ admits an analytic continuation to a meromorphic function on **C** and is of finite order.
  - There exists a 'local (gamma) factor at infinity' $\mathcal{L}_\infty(s)$ s.t. the completed L-function $\Lambda(s) = \mathcal{L}_\infty(s)\mathcal{L}(s)$ obeys the functional equation $\Lambda(s) = \Lambda(1-s)$.
  - Ramanujan conjecture: $a(1) = 1$ and $a(n) \ll n^{o(1)}$.

Another way that we characterize L-functions depends on a notion of *degree*. An analytic way to characterize degree is to look at the order in p^{-s} of the local $\mathcal{L}_p$ factors in the Euler product (equal to the number of Satake parameters).

$$\zeta_p(s) = \left(1 - \frac{1}{p^s}\right)^{-1} \leftarrow \text{degree 1 L-function}$$

$$L_p(s, f) = \left(1 - \frac{\alpha_f(p)}{p^s}\right)^{-1}\left(1 - \frac{\beta_f(p)}{p^s}\right)^{-1} \leftarrow \text{degree 2 L-function}$$

A *primitive* L-function is one that cannot be written as the product of two L-functions. (For example, the Dedekind zeta function for a quadratic number field K is not primitive because it factors as $\zeta_K(s) = \zeta(s)L(s, \chi_d)$.

We expect the following to hold for primitive GL(2) L-functions:

- Ramanujan conjecture ($\forall L \in \mathcal{S}$): $a(n) \ll n^{o(1)}$.
- Convexity bound for growth in the critical strip – order of growth is a function of degree of $\mathcal{L}$.
- Conjecture: Rankin-Selberg convolution $\sum_n \frac{a(n)^2}{n^s}$ has a simple pole at $s = 1 \Leftrightarrow \mathcal{L}(s)$ primitive. (The residue of this pole contributes to the main term in our shifted moment result for newforms.)
- For GL(2) L-functions, we have the conjectural asymptotics $\sum_{n\leqslant x}|a(n)| = o(x)$ and $\sum_{n\leqslant x}|a(n)|^2 = O(x)$.
- Selberg's orthogonality conjectures: For GL(2) L-functions, we have the asymptotic $\sum_{n\leqslant x}\frac{|a(p)|^2}{p} = \log\log x + O(1)$.

3. Results

Our approach is sufficiently general to apply to any primitive GL(2) L-function satisfying the properties described in the previous section. As a first example, we considered the specific example the family of L-functions associated to holomorphic cusp forms of weight k that are newforms for the full modular group in the sense of Atkin-Lehner theory. We denote the L-function attached to any such newform $f \in H_k^*(1)$ by $L(s, f)$. We have proved the following unconditional theorem:

Theorem 3.1. *Let $\{\gamma_1, \gamma_2, ..., \gamma_N\}$ be the set of distinct zeros of $L\left(\frac{1}{2}+it, f\right)$ in the interval $[T, 2T]$. Let*

$$\kappa_T = \max\{\gamma_{n+1} - \gamma_n : T + 1 \leqslant \gamma_n \leqslant 2T - 1\}.$$

Then

$$\kappa_T \geqslant \frac{\sqrt{3}\pi}{\log T}\left(1 + O\left(\frac{1}{c_f}(\log T)^{-\delta}\right)\right).$$

In order to prove our theorem, we use a method due to R.R. Hall, along with the following shifted moment result:

Theorem 3.2.

$$\int_T^{2T} L\left(\frac{1}{2}+it+\alpha, f\right) L\left(\frac{1}{2}-it+\beta, f\right) dt = c_f T \sum_{n\geqslant 0} \frac{(-1)^n 2^{n+1}(\alpha+\beta)^n(\log T)^{n+1}}{(n+1)!} + O(T(\log T)^{1-\delta})$$

Obtaining this shifted moments result for L-functions of newforms of the full modular group was our first major task.

To arrive at Theorem 3.2, we considered the product

$$L(s+\alpha, f)L(1-s+\beta, f),$$

where both $\alpha, \beta \in \mathbf{C}$ and $\alpha, \beta \ll 1/\log T$. This product of shifted L-functions gives rise to 16 products, each of which required estimation. We found that the contribution to our main term was given by two such products, whereas the others were absorbed into the error term.

The shifted moment result allows us to deduce lower order terms and moments of derivatives of L-functions by means of differentiation and the Cauchy Integral Formula. That is, we desired an expression for

$$\int_T^{2T} L^{(\mu)}\left(\frac{1}{2}+it, f\right) L^{(\nu)}\left(\frac{1}{2}-it, f\right) dt,$$

where $T \geqslant 2$ and $\mu, \nu \in \mathbf{Z}_+$. We use this result in Hall's method to obtain the lower bound stated in Theorem 3.1.

Using Hall's method, we bound the gaps between zeroes. This requires the following result, due to Wirtinger and modified by Bredberg:

Lemma 3.3. *Let $y : [a, b] \to \mathbf{C}$ be a continuously differentiable function and suppose that $y(a) = y(b) = 0$. Then*

$$\int_a^b |y(x)|^2 dx \leqslant \left(\frac{b-a}{\pi}\right)^2 \int_a^b |y'(x)|^2 dx.$$

We define the function

$$g(t) := e^{i\rho\log T} L\left(\frac{1}{2}+it, f\right)$$

where ρ is a real constant established later. We fix f and let $\tilde{\gamma}$ denote an ordinate zero of $L(s, f)$ on the critical line $\Re e(s) = 1/2$. The crucial property of this function is that it has the same zeroes as $L(s, f)$, namely $g(t) = 0$ when $t = \tilde{\gamma}$. We use this function in the modified Wirtinger's inequality.

We apply sub-convexity bounds along the critical line to establish:

$$\int_T^{2T} |g(t)|^2 dt \leqslant \frac{\kappa_T^2}{\pi^2}\int_T^{2T} |g'(t)|^2 dt + O\left(T^{\frac{2}{3}}(\log T)^{\frac{5}{6}}\right)$$

Noting that our $g(t)$ and $g'(t)$ may be expressed in terms of $L\left(\frac{1}{2}+it, f\right)$, we can write our inequality explicitly in terms of formulægiven by our theorem(s) for moments of L-functions.

After substituting our formulæ, we have the inequality:

$$\frac{\kappa_T^2}{\pi^2} \geqslant \frac{3}{3\rho^2 - 6\rho + 4}(\log T)^{-2}\left(1 + O(\log T)^{-\delta}\right).$$

We are able to minimize this by setting $\rho = 1$, so we have our desired result

$$\kappa_T \geqslant \frac{\sqrt{3}\pi}{\log T}\left(1 + O\left(\frac{1}{c_f}(\log T)^{-\delta}\right)\right).$$

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