Summand Minimality of Generalized Zeckendorf Representations of Non-Negative Linear Recurrence Relations

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Background and Motivation

Zeckendorf Representation and Summand Minimality: A celebrated theorem of Zeckendorf states that every number can be uniquely represented as a sum of non-consecutive Fibonacci numbers. Further, of all the decompositions of an integer as a sum of Fibonacci numbers, the Zeckendorf decomposition is minimal in that no other decomposition has fewer summands.

Generalized Zeckendorf Representations for Non-Negative Linear Recurrences: The Zeckendorf theorem has an alternate statement paralleling the notion of a base d representation. Just as each number has a unique representation base d composed of digits from the set $\{[0], [1], \ldots, [d-1]\}$, each number has a unique representation using the Fibonacci numbers (with initial condition $F_1 = 0, F_2 = 1$), composed of "digits" from the set $\{[0], [1, 0]\}$. Miller et al. [3], and independently Hamlin [2], proved that given a non-negative linear recurrence with positive leading coefficient, each number has a unique representation with respect to the recurrence sequence composed of "digits" from a finite list of allowable "digits."

Generalizing Summand Minimality: There are several articles proving the Zeckendorf decomposition is minimal among binary decompositions, or a related far-difference decomposition, for both the original Fibonacci sequence as well as some generalizations (see [1], for example). It is hence natural for us to ask:

Motivating Questions

1. Are generalized Zeckendorf representations for non-negative linear recurrence relations always summand minimal?
2. If not, what are the necessary and sufficient conditions for a non-negative linear recurrence relation to guarantee summand minimal generalized Zeckendorf representations of all integers?

References

1. H. Alpert, *Differences of multiple Fibonacci numbers*, Integers: Electronic Journal of Combinatorial Number Theory **9** (2009).
2. N. Hamlin, *Representing Positive Integers as a Sum of Linear Recurrence Sequences*, Fibonacci Quarterly **50** (2012), no. 2.
3. S. J. Miller and Y. Wang, *From Fibonacci numbers to Central Limit Type Theorems*, Journal of Combinatorial Theory, Series A **119** (2012).

Methods

Our treatment is built on the terminology developed by Hamlin.

Definition 1. Suppose a recurrence is given by $H_n = c_1 H_{n-1} + \ldots + c_t H_{n-t}$. Then $(c_1, c_2, \ldots, c_t)$ is called the **signature** of the recurrence. Furthermore, we call $\underbrace{0, 0, \ldots, 0}_{t-1}, 1$ ideal initial condition.

Definition 2. Consider the signature $(c_1, \ldots, c_t)$. A valid **block** is a string of the form $[c_1, \ldots, c_k, x]$ where $k < t$ and $x < c_{k+1}$.

Definition 3. Consider a recurrence sequence with signature and initial condition as in Definition 1. Let $[a_1, \ldots, a_n, \underbrace{\infty, \cdots, \infty}_{t-1}]$ (ask us what those ∞'s are for) represent the integer

$$\sum_{i=1}^{n} a_i H_{n-i}.$$

Then $[a_1, \ldots, a_n, \underbrace{\infty, \cdots, \infty}_{t-1}]$ is a valid **generalized Zeckendorf representation** (GZR) if it is made up of valid blocks.

Example. Given the signature $(1, 1)$ and ideal initial condition, we get the sequence

$$0, 1, 1, 2, 3, 5, 8, \ldots,$$

i.e., the Fibonacci sequence. The only valid blocks are $[0]$ and $[1, 0]$. The valid generalized Zeckendorf representation of 45 is

$$[1, 0, 0, 1, 0, 1, 0, 0, 0, \infty].$$

Example. Given the signature $(1, 2, 3)$ and ideal initial condition, we get the sequence

$$0, 0, 1, 1, 3, 8, 17, 42, \ldots.$$

The valid blocks are: $[0]$, $[1, 0]$, $[1, 1]$, $[1, 2, 0]$, $[1, 2, 1]$, $[1, 2, 2]$.

Results

Theorem

Consider a non-negative recurrence sequence with signature $(c_1, \ldots, c_t)$. Then, the recurrence is summand minimal **if and only if** its signature is weakly decreasing, i.e. $c_1 \geq c_2 \geq \cdots \geq c_t$.

Acknowledgement

We thank the SMALL Program, Williams College, Professor Amanda Folsom, and NSF Grants DMS1265673, DMS1561945, DMS1347804, and DMS1449679 for support.

Main Ideas

We develop an **algorithm** to find the GZR of a non-negative integer starting from any other representation written in the form $[d_1, \ldots, d_r, \infty]$ where we have abbreviated ∞ to stand for the block $[\underbrace{\infty, \ldots, \infty}_{t-1}]$ (see Definition 3).

1. Reading from left to right, find the first block that is invalid. This means either the last digit of that block is negative or too large.
2. If the digit in question is negative, "borrow" from earlier positive digits to make it non-negative. Otherwise, we can either "carry" right away to make the block valid or "borrow" until we can carry. (Ask us more about this :))
3. Repeat.

Example. We consider the signature $(4, 3, 2)$.

		4	5	0	5	4	2	5	0	∞	∞
B			-1	4	3	2					
C	1	-4	-3	-2							
	1	0	1	2	8	6	2	5	0	∞	∞
C				1	-4	-3	-2				
	1	0	1	3	4	3	0	5	0	∞	∞
B								-1	4	3	2
C							1	-4	-3	-2	
	1	0	1	3	4	3	1	0	1	∞	∞

Idea of Proof for Sufficiency We start with any non-negative representation and perform the algorithm. Note that every time we borrow, we increase the number of summands and every time we carry, we decrease the number of summands. Because we start with a non-negative representation, if any block is invalid, the last digit must be too large, and hence we can "borrow" until we are able to carry. With weakly decreasing signature, we only need to borrow once before we can carry. So overall, the number of summands can only decrease.

Idea for Proof of Necessity To prove sufficiency, for $c_1 > 1$, we give examples of a non-GZR representation that has fewer summands than the GZR by performing the same above-mentioned algorithm (most of the examples are of the form $[c_1 + 1, 0, \ldots, 0, \infty]$). When $c_1 = 1$, we non-constructively prove the existence of a counterexample by utilizing the irreducibility of a certain family of polynomials together with growth rate arguments.