Gaussian Behavior in Zeckendorf Decompositions Arising From Lattices

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Overview

- Introduction to Zeckendorf Decompositions
- Introduction to Main Result and Simulations
- Technical Lemmas
- Proof of Main Result
- Future Work

Definition (Fibonacci Numbers)

The **Fibonacci Numbers** are a sequence defined recursively with $F_n = F_{n-1} + F_{n-2}\ \forall n \geq 2$ where $F_0 = 1$ and $F_1 = 1$.

Beginning of sequence:
$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$

Definition (Zeckendorf Decompositions)

A **Zeckendorf Decomposition** is a way to write a natural number as the sum of non-adjacent Fibonacci Numbers.

Theorem (Zeckendorf's Theorem)

Every natural number has a unique Zeckendorf Decomposition.

Example (Greedy Algorithm):

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- $335 = 233 + 102$
- $335 = 233 + 89 + 13$

Definition (Simple Jump Paths)

A **simple jump path** is a path on the lattice grid where each movement on the lattice grid consists of at least one unit movement to the left and one unit movement downward.

- We count simple jump paths from (a, b) to $(0, 0)$, where $a, b. \in \mathbb{N}^+$.

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- Any simple jump path must include the use of (a, b) and $(0, 0)$.
- Let the number of simple jump paths from (a, b) to $(0, 0)$ with k steps be denoted $t_{a,b,k}$.

Simple Jump Paths

$$\begin{bmatrix} 50 & \cdots & \cdots & \cdots & \cdots & \cdots \\ 28 & 48 & \cdots & \cdots & \cdots & \cdots \\ 14 & 24 & 40 & \cdots & \cdots & \cdots \\ 7 & 12 & 20 & 33 & \cdots & \cdots \\ 3 & 5 & 9 & 17 & 30 & \cdots \\ 1 & 2 & 4 & 8 & 16 & 29 \end{bmatrix}$$

- Our goal is to enumerate how many paths are required for a linear search of a Zeckendorf decomposition from a certain starting point in the lattice.

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- For each $n \in \mathbb{N}^+$, check if any downward/leftward path sums to the number. If not, add the number to the sequence so that it is added to the shortest unfilled diagonal moving from the bottom right to the top left.
- Each simple jump path on this lattice represents a Zeckendorf Decomposition.

General useful formulas for random variables:

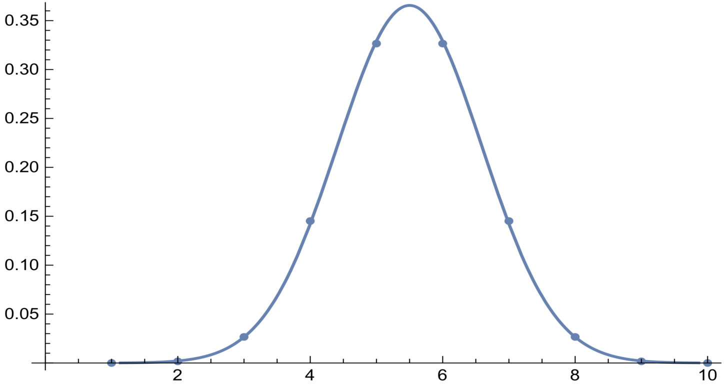
- Gaussian (continuous): Random variable with density $(2\pi\sigma^2)^{-1/2}\exp(-(x-\mu)^2/2\sigma^2)$, mean μ, variance σ^2.

- Central Limit Theorem: Let $X_1, ..., X_N$ be i.i.d. random variables with finite moments, mean μ and standard deviation σ. Also denote $\overline{X}_N := \frac{\sum_{i=1}^N X_i}{N}$. Then the distribution of $Z_N := \frac{\overline{X}_{N-\mu}}{\frac{\sigma}{\sqrt{N}}}$ converges to a Gaussian.

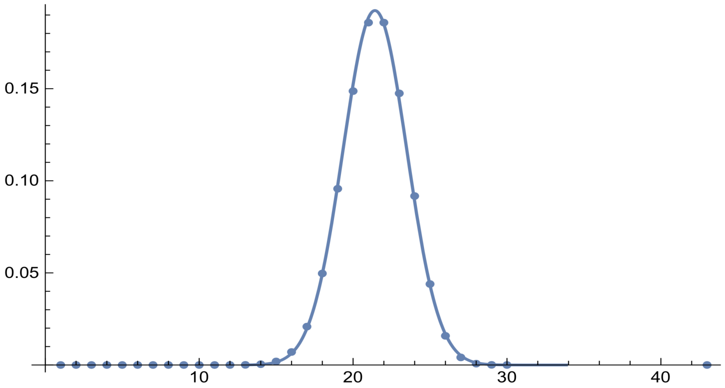
Theorem (Gaussianity on a Square Lattice)

Let n be a positive integer, and consider the distribution of the number of summands among all simple jump paths with starting point (i, j) wjere $1 \leq i, j \leq n$, and each distribution represents a (not necessarily unique) decomposition of some positive number. This distribution converges to a Gaussian as $n \to \infty$.

Simulations and Explanation of Main Result Statement



- Represents $\{t_{10,10,k}\}_{k=1}^{10}$
- Special case: simple jump paths over a square lattice for $n = 10$, starting point $(10, 10)$



- Represents $\{t_{30,70,k}\}_{k=1}^{30}$
- Simple jump paths over a rectangular lattice with starting point $(70, 30)$

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- Simple jump paths: $k \in \{1, 2, ..., n\}$

Lemma (Simple Jump Path Partition Lemma)

$\forall a, b \in \mathbb{N}, s_{a,b} = \sum_{k=1}^{\min\{a,b\}} t_{a,b,k}.$

Lemma (The Cookie Problem)

The number of ways of dividing C identical cookies among P distinct people is $\binom{C+P-1}{P-1}$.

- Line up $C + P - 1$ identical cookies

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- Line up $C + P - 1$ identical cookies
- Choose $P - 1$ cookies to hide and place dividers in those positions

Lemma (Enumerating Simple Jump Paths)

$\forall a, b \in \mathbb{N}, k \in \min\{a, b\}, t_{a,b,k} = \binom{a-1}{k-1}\binom{b-1}{k-1}.$

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- Second factor is number of ways to group b objects into k nonempty groups
- Groupings are independently determined, use Cookie Problem lemma

General useful formulas:

- $p(x_k)$: probability of event x_k occurring, one of finitely many values (events)
- Density function: $f_n(k+1) := \frac{t_{n+1,n+1,k+1}}{s_{n+1,n+1}} = \frac{\binom{n}{k}^2}{\binom{2n}{n}}$
- Mean (discrete): $\mu = \sum x_k p(x_k)$
- Variance (discrete): $\sigma^2 = \sum (x_n - \mu)^2 p(x_n)$

General useful formulas (continued):

- Gaussian (continuous): Density $(2\pi\sigma^2)^{-1/2}\exp(-(x-\mu)^2/2\sigma^2)$
- Taylor Approximation of $\log(1+x)$: $\log(1+x) \ = \ x - \frac{x^2}{2} + \frac{x^3}{3} + O(x^4)$
- Taylor Approximation of $\log(1-x)$: $\log(1-x) \ = \ -x - \frac{x^2}{2} - \frac{x^3}{3} + O(x^4)$

Theorem (Mean on Square Lattice)

$\forall n \in \mathbb{N}^+$, $\mu_{n+1,n+1} \;=\; \frac{1}{2}n + 1 \;\sim\; \frac{n}{2}$.

- Calculate using definition of first moment (mean)

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- Calculate using definition of first moment (mean)
- Use index shift $\sum_{k=1}^{n+1} k \binom{n}{k-1}^2 = \sum_{k=0}^{n} k \binom{n}{k}^2 + \sum_{k=0}^{n} \binom{n}{k}^2$

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- Use standard techniques for evaluating binomial coefficients

Theorem (Standard Deviation on Square Lattice)

$$\forall n \in \mathbb{N}^+, \sigma_{n+1,n+1} = \frac{n}{2\sqrt{2(n-1)}} \sim \frac{\sqrt{n}}{2\sqrt{2}}.$$

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- Calculate using definition of second standardized moment (standard deviation)
- Use index shift $\sum_{k=1}^{n+1} \left(k - \left(\frac{1}{2}n + 1\right)\right)^2 \binom{n}{k-1}^2 = \sum_{k=0}^{n}(k + 1 - (\frac{1}{2}n + 1))^2 \binom{n}{k}^2$

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- Density function: $f_n(k+1) \;:=\; \frac{t_{n+1,n+1,k+1}}{s_{n+1,n+1}} \;=\; \frac{\binom{n}{k}^2}{\binom{2n}{n}}$

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- Use Stirling's Approximation on each factor:
  $m! \ \sim \ m^m e^{-m}\sqrt{2\pi m}$

- End result of Stirling expansion is
$f_n(k+1) \sim \frac{n^{2n}}{k^{2k} \cdot (n-k)^{2n-2k} \cdot 2^{2n} \cdot \frac{1}{4} \cdot \sqrt{4\pi n}}$

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- Let $k := \mu_{n+1,n+1} + x\sigma_{n+1,n+1}$, then $f_n(k+1)dk = f_n(\mu_n + x\sigma_n + 1)\sigma_n dx \sim f_n(\mu_n + x\sigma_n + 1)\frac{\sqrt{n}}{2}dx$

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  $f_n(k+1)dk = f_n(\mu_n + x\sigma_n + 1)\sigma_n dx \sim f_n(\mu_n + x\sigma_n + 1)\frac{\sqrt{n}}{2}dx$
- x quantifies number of standard deviations from mean

Apply logarithm to $P_n(k+1) = \frac{n^n}{k^k(n-k)^{n-k}2^n}$:

$$\log P_n(k+1) = n\log(n) - k\log(k) - (n-k)\log(n-k) - n\log(2)$$

Rewrite $k = \frac{n}{2} + \frac{x\sqrt{n}}{2\sqrt{2}} = \frac{n}{2}\left(1 + \frac{x}{\sqrt{2n}}\right)$ to expand $\log(k)$ and $\log(n-k)$:

$$\log(k) = \log\left(\frac{n}{2}\left(1 - \frac{x}{\sqrt{2n}}\right)\right) \approx \log(n) - \log(2) + \log\left(1 - \frac{x}{\sqrt{2n}}\right)$$

$$\log(n-k) = \log\left(\frac{n}{2}\left(1 + \frac{x}{\sqrt{2n}}\right)\right) \approx \log(n) - \log(2) + \log\left(1 + \frac{x}{\sqrt{2n}}\right)$$

Substitute logarithm expansions and approximate $\log\left(1+\frac{x}{\sqrt{2n}}\right)$ and $\log\left(1-\frac{x}{\sqrt{2n}}\right)$ to second order to conclude

$$\log P_n(k+1) \sim -\frac{n}{2}\log\left(1-\frac{x^2}{2n}\right)-\frac{x\sqrt{n}}{2}\left(\frac{x}{\sqrt{n}}+O\left(\frac{1}{n^{\frac{3}{2}}}\right)\right)$$

Approximate $\log\left(1-\frac{x^2}{2n}\right)$ up to second order:

$$-\frac{n}{2}\left(-\frac{x^2}{2n}+O\left(\frac{1}{n^2}\right)\right)-\frac{x\sqrt{n}}{2}\left(\frac{x}{\sqrt{n}}+O\left(\frac{1}{n^{\frac{3}{2}}}\right)\right) \sim -\frac{x^2}{4}$$

It follows that

$$P_n(k+1) \sim e^{-\frac{x^2}{4}} \Rightarrow P_n(k+1)^2 \sim e^{-\frac{x^2}{2}} \Rightarrow$$

$$f_n(k+1) \sim \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$$

- Normal distribution, mean 0, standard deviation 1.

- Find closed formulas for enumerating compound jump paths
- Generalize Gaussianity result to compound jump paths
- Generalize methodology to general positive linear recurrences

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Thank You

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