Graph-Restricted Decompositions: A Further Generalization of Zeckendorf's Theorem

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Outline

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- Introduction to Zeckendorf Decompositions

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- Framework: Graph-Restricted Decompositions

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- Results on Decomposition Behavior

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- Framework: Graph-Restricted Decompositions
- Results on Decomposition Behavior
- Questions for Future Research

Introduction to Zeckendorf Decompositions

The Zeckendorf Decomposition

Definition (Zeckendorf Decomposition)

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Fibonacci numbers for reference:

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  - Repeat
- The largest number we can decompose using $\{F_1, \dots, F_{k-1}\}$ is less than F_k

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(Starting $1, 1$, or $0, 1$, would lose unique decomposition!)

Framework: Graph-Restricted Decompositions

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Let G be a graph on nodes indexed by $\mathbb{N}$, connected by edges.

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For example, the Zeckendorf decomposition rule corresponds to the graph G where adjacent vertices are connected.

The G-decomposition

Question: Does there always exist a good choice of sequence in which to G-decompose numbers?

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Given a graph G, the G-**sequence** is the sequence $\{a_k^G\}$ that satisfies

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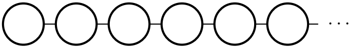
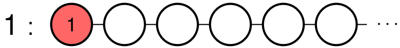
Example:

What is the G-sequence of this graph?

G : 

The G-sequence of the Zeckendorf graph



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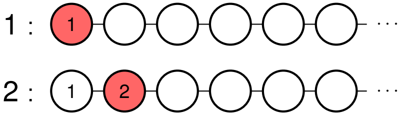

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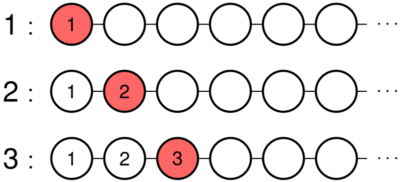

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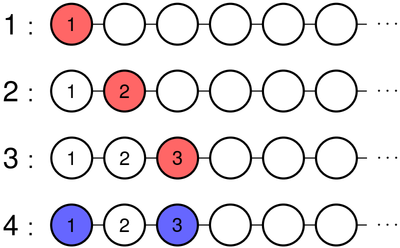

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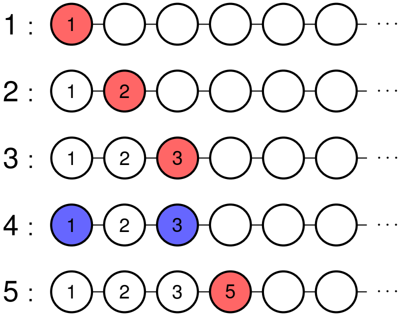

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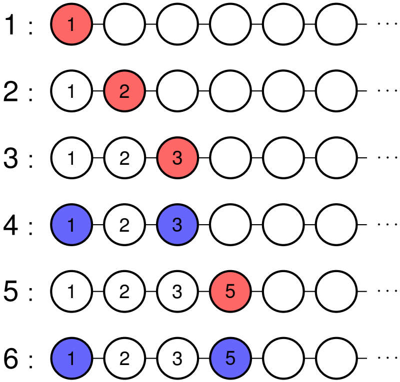

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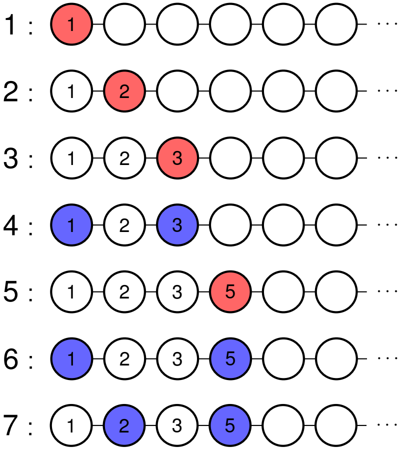

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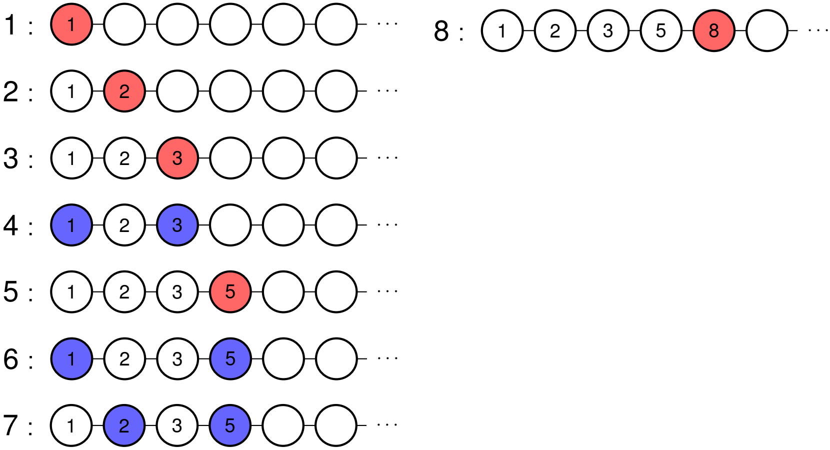

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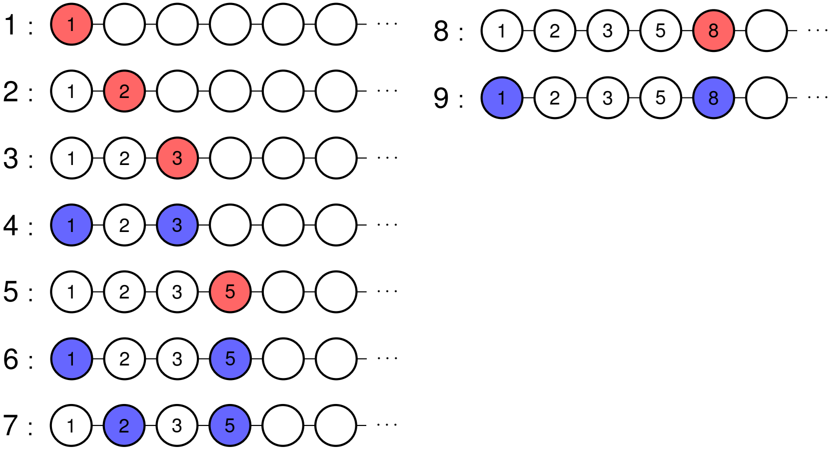

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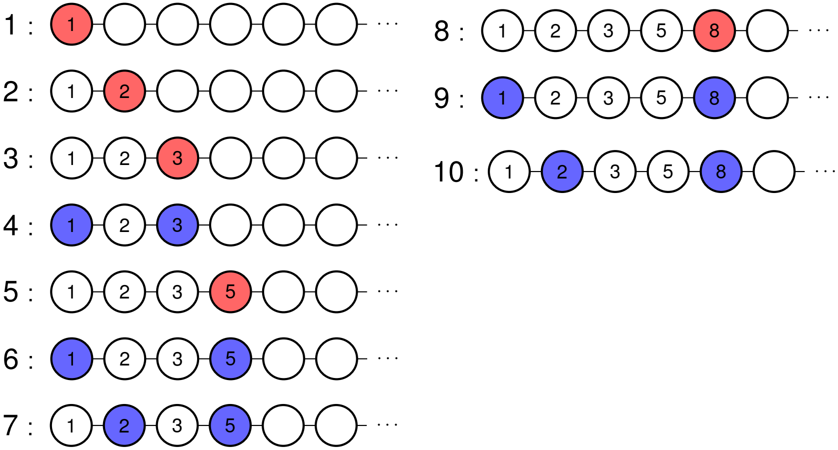

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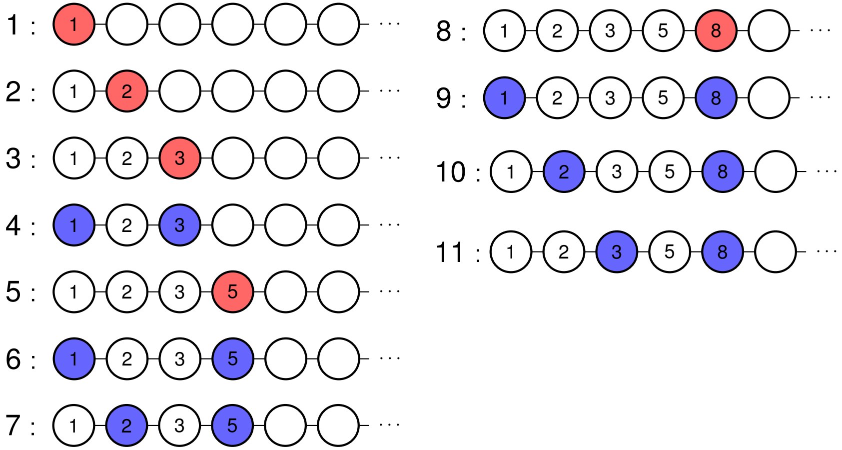

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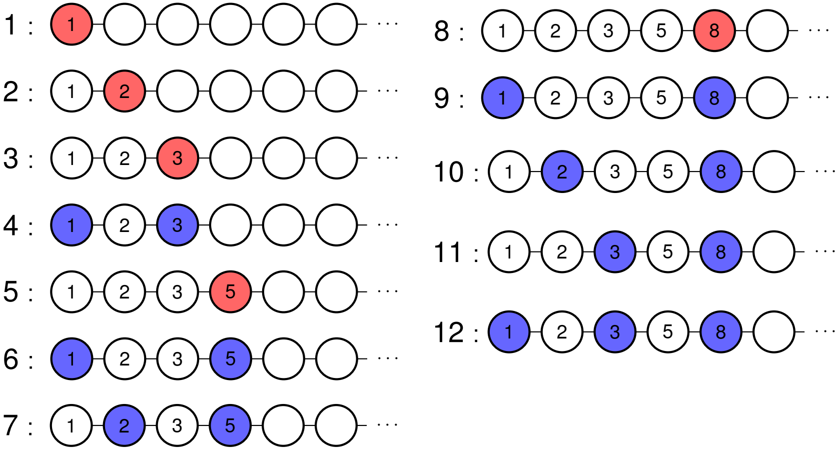

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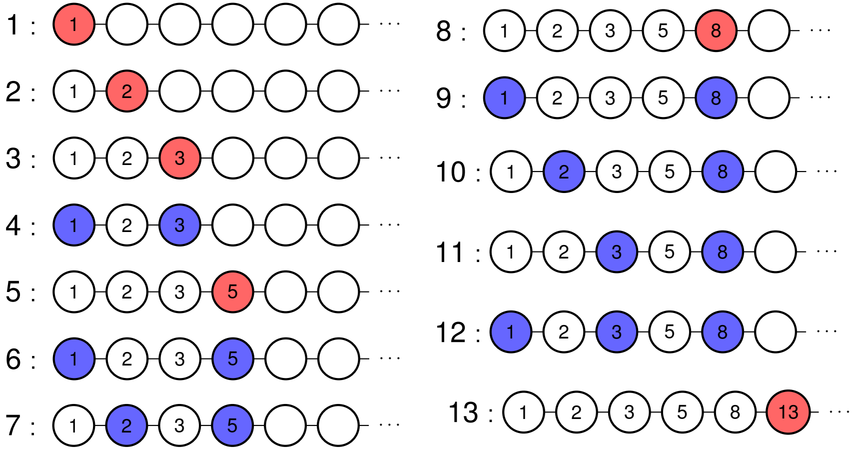

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Proposition

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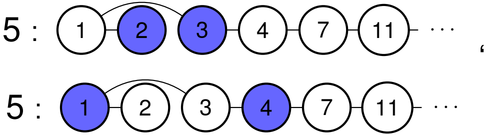

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Fix a graph G.

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Theorem (Special-ness of the G-sequence)

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From now on, when we say G-decomposition, we mean G-decomposition in the G-sequence.

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$G:$ (1)–(2)–(3)–(5)–(8)–(13)– $\cdots$

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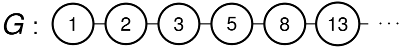

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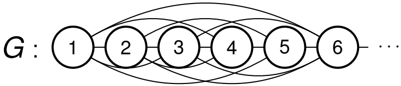

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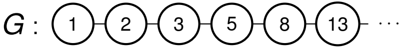

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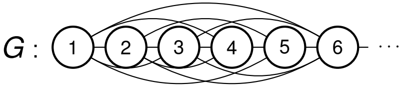

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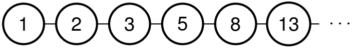
G : 

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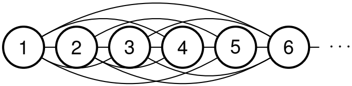
G : 

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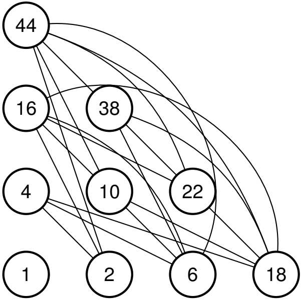
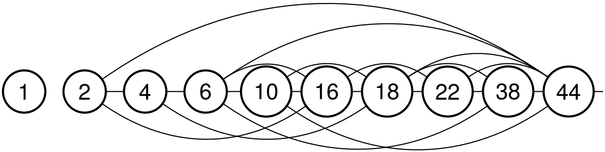
Graph for the Zeckendorf Lattice

The Zeckendorf Lattice

For each $n \in \mathbb{N}$, check if any downward/leftward path sums to the n. If not, add the number to the sequence so that it is added to the shortest unfilled diagonal moving from the bottom right to the top left.





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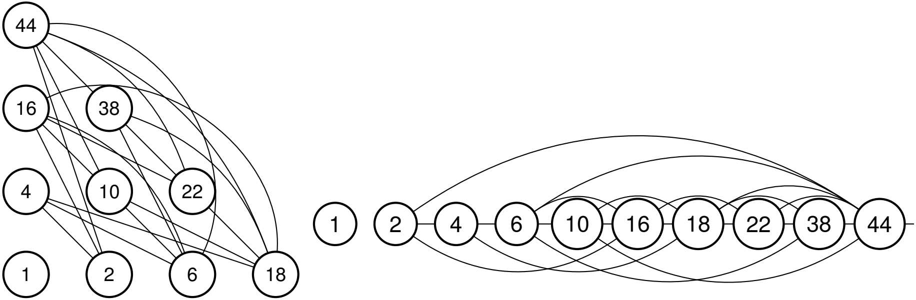

See Joshua Siktar's talk ~~tomorrow~~ yesterday *Gaussian Behavior in Zeckendorf Decompositions Arising From*

Results on Decomposition Behavior

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We produce a sufficient condition for uniqueness.
(It gives some additional nice properties.)

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Let $\{A_k^G\}$ be the sequence of sets

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Non-example:

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Recall that $\{a_k^G\}$ denotes the G-sequence.
Let $\{A_k^G\}$ be the sequence of sets

$$A_k^G \;=\; \{n \in \mathbb{N} \text{ with a } G\text{-decomp using only } a_1^G, \dots, a_k^G\}.$$

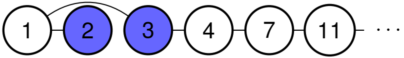
(Note A_k^G are strictly nested and $\bigcup A_k^G = \mathbb{N}$.)

Definition (Uniform graph)

We say G is **uniform** provided that for all $k \in \mathbb{N}$,

$$A_k^G \;=\; \{n < a_{k+1}^G\}$$

Non-example: $5 \in A_3^G$ but $5 > a_4^G = 4$



Structure of Uniform Graphs

We can completely characterize uniform graphs.

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Theorem

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This theorem shows that our uniformity condition is equivalent to the f-decompositions introduced by Demontigny, et al.

Our framework has helped justify their definition, and gives a new perspective from which to ask questions.

Uniformity implies Uniqueness

As promised, uniformity is a sufficient condition for uniqueness of decomposition.

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Corollary

If G is uniform, then

- G-decompositions are unique
- Greedy algorithm always finds **the** G-decomposition

Nice Properties of Uniform Graphs

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If G is uniform, then a_k^G is given by the recurrence

$$a_{k+1}^G = a_k^G + a_{i_k}^G \text{ for } k \in \mathbb{N}$$

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Corollary

If G, H are uniform graphs and H is a subgraph[a] of G then for all $k \in \mathbb{N}$, $a_k^H \geq a_k^G$.

[a] fewer edges, same vertices

Uniqueness Without Uniformity

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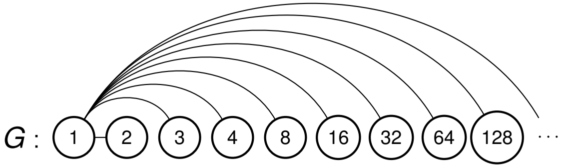

Distributions of Number of Summands

Counting Summands

We are interested in the *number of summands*.

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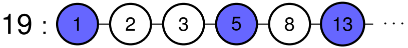

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Example: The Zeckendorf decomposition of 19

19 : 1 – 2 – 3 – 5 – 8 – 13 – $\cdots$

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Lekkerkerker's Theorem (1952)

The average number of summands in the Zeckendorf decomposition for integers in $[F_n, F_{n+1})$ tends to $\frac{n}{\varphi^2+1} \approx .276n$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden mean.

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We would also like to know what the distribution of the number of summands looks like.

Counting Summands

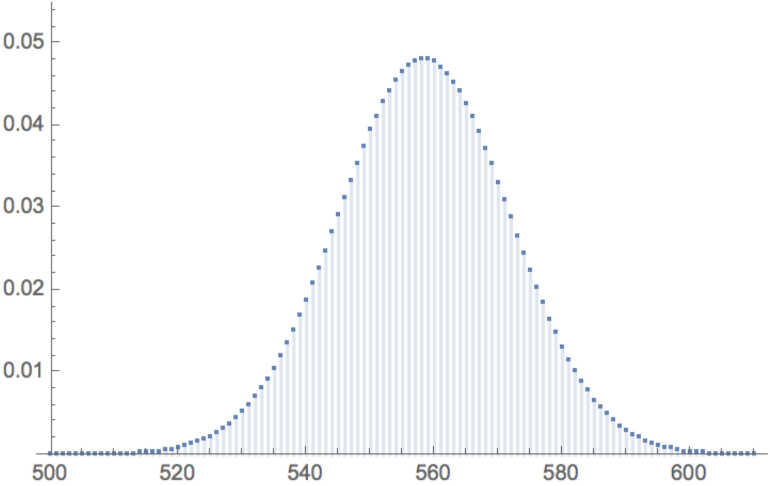

Figure: Number of summands in the Zeckendorf

Past Results

Theorem (KKMW 2010)

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We show Gaussianity for the following family of uniform graphs.

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If the only edges in G are between adjacent vertices (i.e. G is a subgraph of the Zeckendorf graph), then G is Gaussian.

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Example of a subgraph of the Zeckendorf Graph

Gaussianity

Based on past results, we expect Gaussianity to be the default behavior in many situations.

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Open Question: which graphs G do we expect to produce Gaussian distributions of summands?

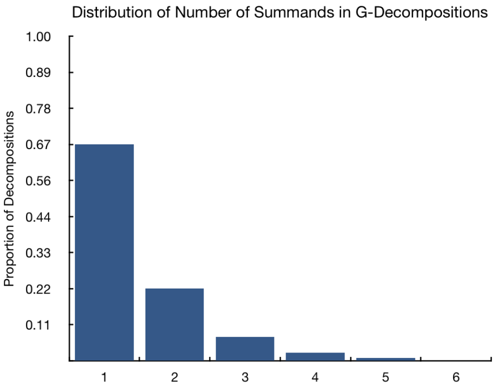
Non-Gaussianity

We exhibit a uniform graph with non-Gaussian distribution of number of summands.

Non-Gaussianity

We exhibit a uniform graph with non-Gaussian distribution of number of summands.

It has the following distribution of number of summands.



Connection to Growth Rate?

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We conjecture that if a uniform graph gives exponential growth rate, then it will produce a Gaussian distribution.

Questions for Future Research

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- How can you tell if a sequence is the G-sequence for some graph?

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Sequences to Graphs

- How can you tell if a sequence is the G-sequence for some graph?
- Is there an algorithm which takes in a sequence and spits out a graph G which generates it, if one exists?
- For which sequences is there a unique graph which generates them?

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- What growth rates can G-sequences have?

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Growth Rates

- What growth rates can G-sequences have? (Must be between linear w/ difference 1 and geometric w/ ratio 2.)
- Can we relate growth rate to edge density (or a different measure of how connected G is)?

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Gaussianity

- Which graphs G give Gaussian distributions of summands?
- Is this related to growth rate? E.g., does exponential growth of a_k^G imply Gaussianity?
- Can we find a non-Gaussian distribution whose mean goes to infinity?

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Questions?