Biases in Second Moments of Elliptic Curves

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Outline

- Moments of Elliptic Curves
- Bias Conjecture
- Explicit Formulas
- Second Moments of $\mathcal{F} : y^2 = x^3 + x + t^3$
- First Moments of $\mathcal{F} : y^2 = x^3 + x + t^3$

A *one-parameter family* of elliptic curves is given by

$$\mathcal{E} : y^2 \;=\; x^3 + A(T)x + B(T),$$

where $A(T)$, $B(T)$ are polynomials in $\mathbb{Z}[T]$.
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Moments of a family of elliptic curves

The r^{th} moment (note we do not normalize by $1/p$) is

$$\mathcal{A}_{r,\mathcal{E}}(p) \ = \ \sum_{t(p)} a_{E_t}(p)^r,$$

where $a_{\mathcal{E}(t)}(p) = p + 1 - \#(\text{solutions to } E_t \bmod p)$ is the Frobenius trace of E_t.

The first moment is related to the rank of the elliptic curve family:

$\mathcal{A}_{1,\mathcal{E}}(p)$ and Family Rank (Nagao, Rosen-Silverman, 1998)

Given certain technical assumptions (Tate's Conjecture) hold for $\mathcal{E}$, then

$$\lim_{X\to\infty} \frac{1}{X} \sum_{p\le X} \mathcal{A}_{1,\mathcal{E}}(p) \frac{\log p}{p} \;=\; -\operatorname{rank} \mathcal{E}(\mathbb{Q}(t)).$$

Negative Bias in the First Moment

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- The "rank" of the family means that except for finitely many t, the elliptic curve E_t has rank greater or equal to r.

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The $j(T)$-invariant is $j(T) = 1728\frac{4A(T)^3}{4A(T)^3+27B(T)^2}$.

Second Moment Asymptotic (Michel, 1995)

For an elliptic surface $\mathcal{E}$ with $j(T)$-invariant non-constant, the second moment is

$$A_{2,\mathcal{E}} \;=\; p^2 + O(p^{3/2}),$$

with lower-order terms of size $p^{3/2}$, p, $p^{1/2}$, and 1.

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Strong and Weak Bias Conjectures (Miller)

- **Weak:** The largest lower term in the second moment expansion which does not average to 0 is on average **negative**.
- **Strong:** The largest lower term in the second moment expansion which does not average to 0 is **negative except for finitely many** p**.**

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- However, lower order negative biases increases bound only by a small amount, which is not enough to explain observed excess rank.

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For a specialization of the one-parameter family $\mathcal{E} : y^2 = x^3 + A(T)x + B(T)$, we may write

$$a_{E_t}(p) = -\sum_{x(p)} \left(\frac{x^3 + A(t)x + B(t)}{p} \right)$$

where $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol mod p given by

$$\left(\frac{x}{p}\right) = \begin{cases} 1 & x \text{ a non-zero square modulo } p, \\ 0 & x \equiv 0 \mod p, \\ -1 & \text{otherwise.} \end{cases}$$

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Observe that $\left(\dfrac{x}{p}\right) + 1$ is precisely the number of solutions to $x = y^2 \pmod{p}$.

Linear and quadratic Legendre sums

We have the following

$$\sum_{x(p)} \left(\frac{ax+b}{p}\right) = 0 \qquad p \nmid a,$$

$$\sum_{x(p)} \left(\frac{ax^2+bx+c}{p}\right) = \begin{cases} -\left(\frac{a}{p}\right) & p \nmid b^2-4ac, \\ (p-1)\left(\frac{a}{p}\right) & p \mid b^2-4ac. \end{cases}$$

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Average values of Legendre symbols

Taking the limit of the average of the Legendre symbol over all primes gives

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\le x} \left(\frac{x}{p}\right) = \begin{cases} 1 & x \text{ a non-zero square}, \\ 0 & \text{otherwise}. \end{cases}$$

- The moments become intractible when $A(T)$ and $B(T)$ have high degree.
- For the following special families, the following is known:

Family	$A_{1,\mathcal{E}}(p)$	$A_{2,\mathcal{E}}(p)$
$y^2 = x^3 + 2^4(-3)^3(9T+1)^2$	0	$\begin{cases} 2p^2-2p & p \equiv 2 \bmod 3 \\ 0 & p \equiv 1 \bmod 3 \end{cases}$
$y^2 = x^3 \pm 4(4T+2)x$	0	$\begin{cases} 2p^2-2p & p \equiv 1 \bmod 4 \\ 0 & p \equiv 3 \bmod 4 \end{cases}$
$y^2 = x^3 + (T+1)x^2 + Tx$	0	$p^2 - 2p - 1$
$y^2 = x^3 + x^2 + 2T + 1$	0	$p^2 - 2p - 3$
$y^2 = x^3 + Tx^2 + 1$	$-p$	$p^2 - n_{3,2,p}p - 1 + c_{3/2}(p)$
$y^2 = x^3 - T^2x + T^2$	$-2p$	$p^2 - p - c_1(p) - c_0(p)$
$y^2 = x^3 - T^2x + T^4$	$-2p$	$p^2 - p - c_1(p) - c_0(p)$
$y^2 = x^3 + Tx^2 - (T+3)x + 1$	$-2c_{p,1;4}p$	$p^2 - 4c_{p,1;6}p - 1$

where $c_{p,a;m} = 1$ if $p \equiv a \bmod m$ and 0 otherwise; $n_{3,2,p}$ is the number of cubes root of $2 \bmod p$; $c_\alpha(p)$ are certain Legendre sums multiplied by p.

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- We computationally evaluated second moments of various families of elliptic curves.
- By Michel's theorem, we assume that

$$\mathcal{A}_{2,\mathcal{E}}(p) = p^2 + \alpha(p)p^{3/2} + \beta(p)p + O(p^{1/2})$$

where $\alpha(p)$ and $\beta(p)$ are $O(1)$. To investigate the $\alpha(p)$ coefficient, we graphed the *bias* of the second moment.

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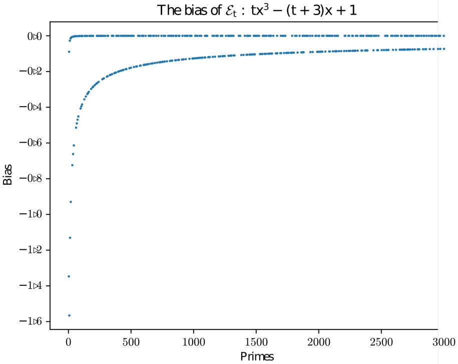
Bias

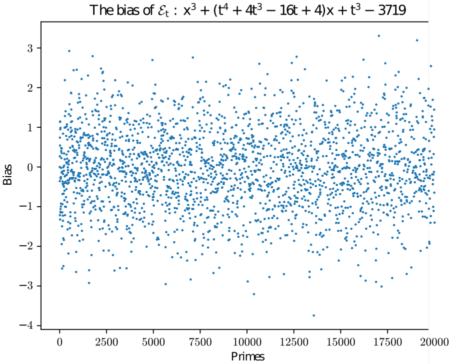
We compute the *bias* of $\mathcal{A}_{2,\mathcal{E}}$ defined by

$$\mathcal{B}_{\mathcal{E}}(p) \;=\; \frac{\mathcal{A}_{2,\mathcal{E}} - p^2}{p^{3/2}}.$$

Graphs of Biases

Here are two examples for the graph of the biases, one for a tractable family, and one for not





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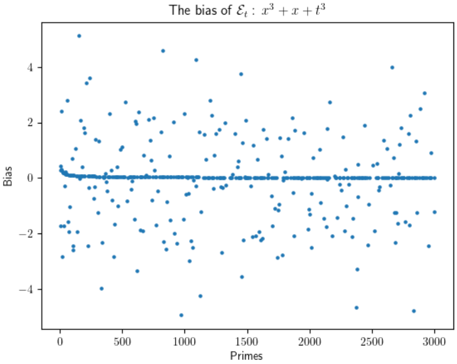

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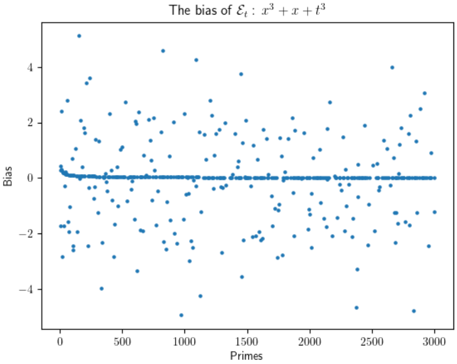

The graph indicates a clear line where the bias is positive, compared to the graphs in the previous slides.

Consider the family

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$$\begin{aligned}
\mathcal{A}_{2,\mathcal{F}}(p) &= \sum_{t(p)} \sum_{x,y(p)} \left(\frac{x^3+x+t^3}{p}\right)\left(\frac{y^3+y+t^3}{p}\right) \\
&= \sum_{t(p)} \sum_{x,y(p)} \left(\frac{x^3+x+t}{p}\right)\left(\frac{y^3+y+t}{p}\right) \\
&= \mathcal{A}_{2,\widetilde{\mathcal{F}}}(p) \;=\; p^2 - \left(\frac{-3}{p}\right)p \;=\; p^2 + p.
\end{aligned}$$

Bias Revisited

We graph the *bias* of $\mathcal{A}_{2,\mathcal{E}}$, for calculated values, defined by

$$\mathcal{B}_{\mathcal{E}}(p) \;=\; \frac{\mathcal{A}_{2,\mathcal{E}} - p^2}{p^{3/2}}.$$

Recall by Michel's theorem, we have

$$\mathcal{A}_{2,\mathcal{E}}(p) = p^2 + \alpha(p)p^{3/2} + \beta(p)p + O(p^{1/2})$$

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- Show that $\beta(p)$ averages to a positive number.

By the prime number theorem, one shows

$$\frac{1}{\pi(x)}\sum_{p\leq x}\alpha(p) \;=\; \frac{1}{\pi(x)}\sum_{p\leq x}\mathcal{B}_{\mathcal{E}}(p)+O(x^{-1/2}\log x)$$

Problem: The constant in the big O term might dominate.

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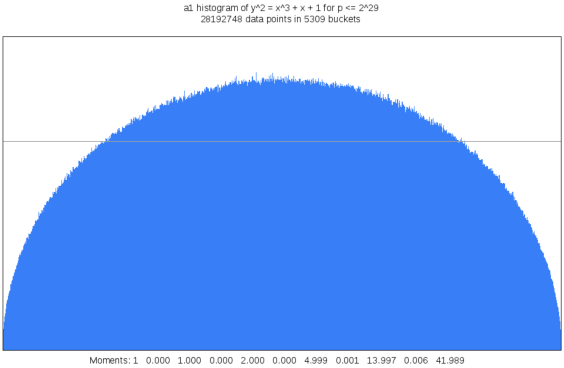
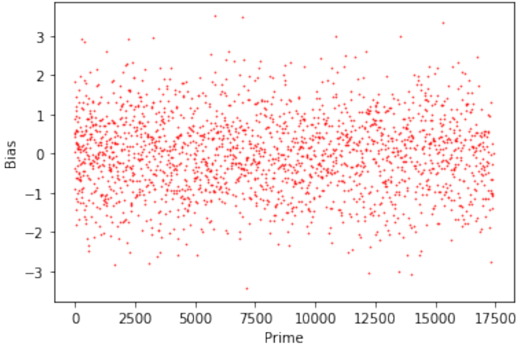
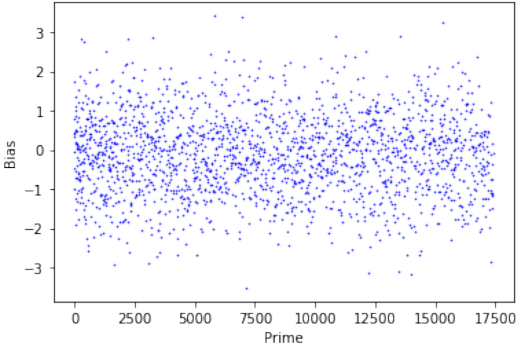
Solution: Randomly simulate elliptic moments using the *Sato-Tate distribution*.

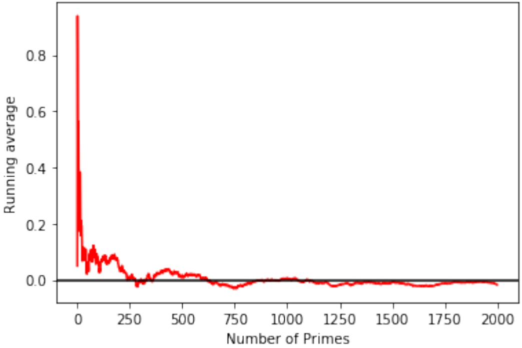


The following are two graphs which randomly simulate the bias.
One graph has coefficient $\alpha(p) = -.1$ and the other has $\alpha(p) = 0$.
Can you guess which is which?





Taking the running average of the biases, it is clear there is a bias:



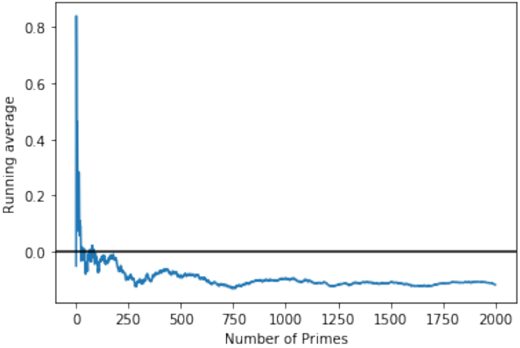
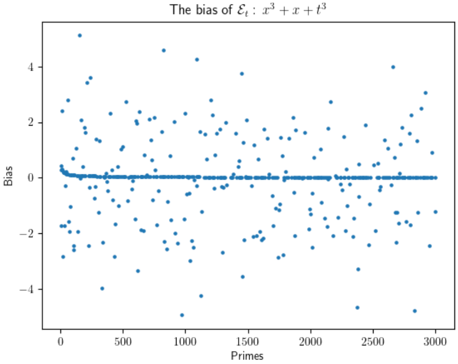
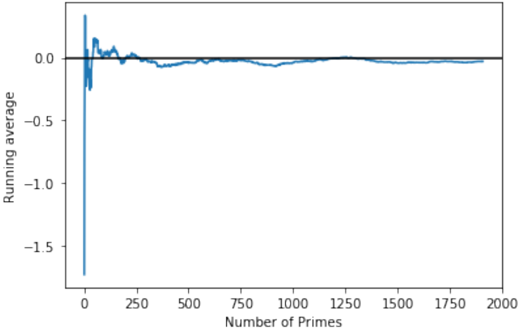


Figure: Unbiased Running Averages (Red) versus Biased Running Averages (Blue) for a random simulation

Computational Success!

Doing the same with our family of interest, that is, $y^2 = x^3 + x + t^3$, we get





So we have strong computational evidence the largest term averages to 0.

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Conjecture (First Moment of $y^2 = x^3 + x + t^3$)

The first moment $\mathcal{A}_{1,p}$ satisfies

$$|\mathcal{A}_{1,p}| = \begin{cases} 4p & p \text{ is of the form } a^2 + 36b^2, \\ 0 & \text{otherwise.} \end{cases}$$

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For a prime $p \not\equiv 1(12)$, the Chinese remainder theorem in conjunction with the changes of variable yields

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Using binary quadratic forms, $\mathcal{A}_{1,p} \neq 0$ forces p to be of the form

$$p = a^2 + 36b^2 \quad \text{or} \quad p = 4a^2 + 9b^2.$$

We have evidence for $|\mathcal{A}_{1,p}| = 4p$ in the former and $\mathcal{A}_{1,p} = 0$ in the latter.

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