From Nuclear Physics to Number Theory

How the Manhattan Project Helps us Understand Primes and Elliptic Curves

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Fundamental Problem: Spacing Between Events

General Formulation: Studying system, observe values at t_1, $t_2, t_3, \dots$.

Question: what rules govern the spacings between the t_i?

Examples:

- Spacings between Primes.
- Spacings between Energy Levels of Nuclei.
- Spacings between Eigenvalues of Matrices.
- Spacings between Zeros of Functions.

Goals of the Talk

- Determine correct scale to study spacings.

- See similar behavior in different systems.

- Discuss tools / techniques needed to prove the results.

- Predictive power of Random Matrix Theory: suggests answers for questions in Number Theory.

PART I

NORMALIZED SPACINGS

Example: Fractional Parts

- For $\alpha \notin \mathbb{Q}$, set $x_n = n^2\alpha \bmod 1$.

- Order $x_1, \dots, x_N$: $0 \le y_1 \le \cdots \le y_N \le 1$.

- Expect spacings between adjacent y's of size $\frac{1}{N}$.

- Should study $\frac{y_{n+1}-y_n}{1/N}$.

- Poissonian behavior for most α: behave like N numbers chosen uniformly in $[0, 1]$.

Example: Primes

- $\pi(x) = \#\{p : p \text{ prime}, p \leq x\} \sim \frac{x}{\log x}$

- Average spacing between primes at most x is $\frac{x}{\pi(x)} \sim \log x$.

- If $p_n, p_{n+1} \sim x$, study $\frac{p_{n+1}}{\log p_{n+1}} - \frac{p_n}{\log p_n}$.

- One reason why twin primes are so hard: $\frac{2}{\log x} \to 0$.

  Aside: (Nicely 1994) Twin primes led to finding the Pentium bug!

$$\sum_{\substack{p \\ p,p+2 \text{ prime}}} \left(\frac{1}{p} + \frac{1}{p+2}\right) \approx 1.90216058.$$

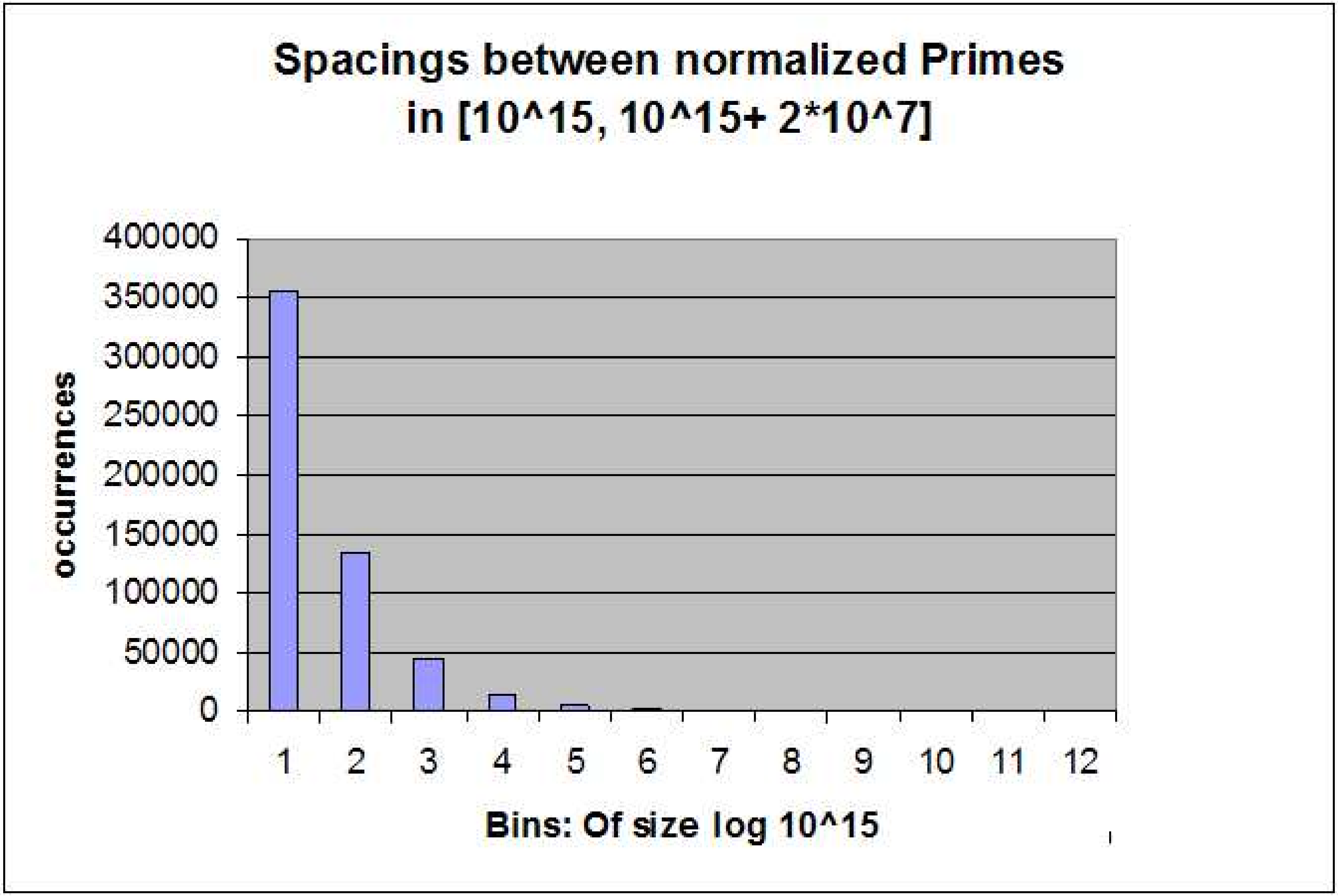
Poisson Process: Spacings between Primes

Observation: See similar behavior in spacings between normalized fractional parts $n^2\alpha \bmod 1$ and normalized primes.

Study spacings between normalized adjacent primes in $I = [10^{15}, 10^{15} + 2 \cdot 10^7]$:

Numerically observe and conjecture that as $|I| \to \infty$

$$\frac{\#\left\{p_i \in I : \frac{p_{i+1}}{\log p_{i+1}} - \frac{p_i}{\log p_i} \in [\alpha, \beta]\right\}}{\#\{p_i \in I\}} \longrightarrow \int_\alpha^\beta e^{-x} dx.$$


Spacings between normalized Primes
in [10^15, 10^15+ 2*10^7]
occurrences
400000
350000
300000
250000
200000
150000
100000
50000
0
1
2
3
4
5
6
7
8
9
10
11
12
Bins: Of size log 10^15

PART II

RANDOM MATRIX THEORY

Origins of Random Matrix Theory

Classical Mechanics: 3-Body Problem Intractable.

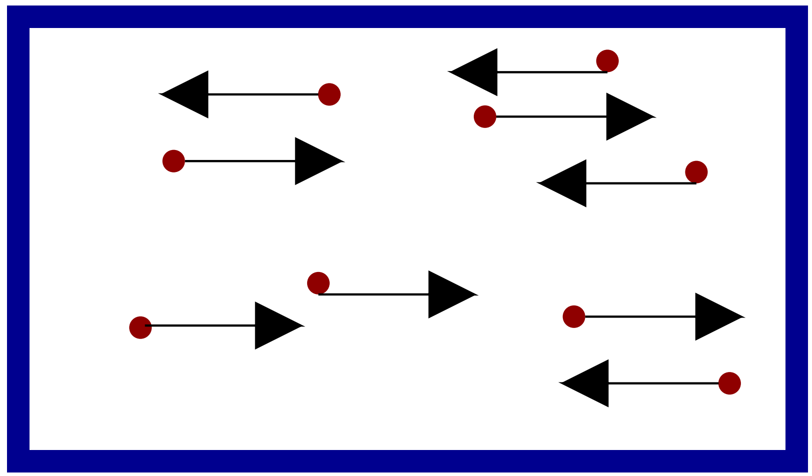
Heavy nuclei like Uranium ($200+$ protons / neutrons) even worse!

Get some info by shooting high-energy neutrons into nucleus, see what comes out.

Fundamental Equation: $H\psi_n = E_n\psi_n$

H : matrix, entries depend on system
E_n : energy levels
ψ_n : energy eigenfunctions

Origins of Random Matrix Theory (continued)



Statistical Mechanics: for each configuration, calculate quantity (say pressure).

Average over all configurations – most configurations close to system average.

Nuclear physics: choose matrix at random, calculate eigenvalues, average over matrices.

Look at: Real Symmetric ($A^T = A$), Complex Hermitian ($\overline{A}^T = A$), Classical Compact groups (unitary, orthogonal).

Random Matrix Ensembles

Real Symmetric Matrices:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & a_{N3} & \cdots & a_{NN} \end{pmatrix} = A^T, \quad a_{ij} = a_{ji}, \ \lambda_i \in \mathbb{R}.$$

Define

$$\text{Prob}\left(A : a_{ij} \in [\alpha_{ij}, \beta_{ij}]\right) = \prod_{1 \le i \le j \le N} \int_{\alpha_{ij}}^{\beta_{ij}} p(x_{ij}) dx_{ij}.$$

Want to understand eigenvalues of randomly chosen A.

MAIN TOOL: Eigenvalue Trace Lemma

$$\mathrm{Trace}(A) \;=\; a_{11} + a_{22} + \cdots + a_{NN}.$$

Eigenvalue Trace Lemma: $\mathrm{Trace}(A^k) \;=\; \sum_{i=1}^{N} \lambda_i(A)^k.$

- Will give correct normalization for zeros;
- Allows us to pass from knowledge of matrix entries to knowledge of eigenvalues.

Correct Scale for Eigenvalues of Real Symmetric Matrices
Entries chosen from Mean 0, Variance 1 Density

$$\text{Trace}(A^2) \;=\; \sum_{i=1}^{N} \lambda_i(A)^2.$$

By the Central Limit Theorem:

$$\text{Trace}(A^2) \;=\; \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}a_{ji} \;=\; \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^2 \;\sim\; N^2$$

$$\sum_{i=1}^{N} \lambda_i(A)^2 \;\sim\; N^2$$

Gives NAverage$(\lambda_i(A)^2) \sim N^2$ or Average$(\lambda_i(A)) \sim \sqrt{N}$.

Eigenvalue Distribution

$\delta(x - x_0)$ is a unit point mass at x_0.

For each $N \times N$ matrix A, attach a probability measure:

$$\mu_{A,N}(x) \;=\; \frac{1}{N}\sum_{i=1}^{N}\delta\left(x - \frac{\lambda_i(A)}{2\sqrt{N}}\right).$$

Equivalently,

$$\int_\alpha^\beta \mu_{A,N}(x)dx \;=\; \frac{\#\left\{\lambda_i : \frac{\lambda_i(A)}{2\sqrt{N}} \;\in\; [\alpha,\beta]\right\}}{N}.$$

$$k^{\text{th}}\text{ Moment of }\mu_{A,N} \;=\; \frac{1}{N}\sum_{i=1}^{N}\frac{\lambda_i(A)^k}{(2\sqrt{N})^k} \;=\; \frac{\text{Trace}(A^k)}{2^k N^{\frac{k}{2}+1}}.$$

Wigner's Semi-Circle Law

$N \times N$ real symmetric matrices, upper triangular entries independently chosen from a fixed probability density p on $\mathbb{R}$.

$$\int_\alpha^\beta \mu_{A,N}(x)dx \ = \ \frac{\#\left\{\lambda_i : \frac{\lambda_i(A)}{2\sqrt{N}} \ \in \ [\alpha,\beta]\right\}}{N}.$$

THEOREM: Wigner's Semi-Circle Law: Assume p has mean 0, variance 1, other moments finite. As $N \to \infty$, almost all A have $\mu_{A,N}$ close to the Semi-Circle density

$$S(x) \ = \ \begin{cases} \frac{2}{\pi}\sqrt{1-x^2} & \text{if } |x| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Technical: As $N \to \infty$ with probability one the Kolmogorov-Smirnov discrepency between $\mu_{A,N}$ and S tends to zero.

$$\text{Disc}(\mu_{A,N}, S) \ = \ \sup_x \left| \int_{-\infty}^x \mu_{A,N}(t)dt - \int_{-\infty}^x S(t)dt \right|$$

Proof of Wigner's Semi-Circle Law

1. Eigenvalue Trace Lemma $\left(\text{Trace}(A^k) = \sum_i \lambda_i(A)^k\right)$ converts sums over eigenvalues to sums over entries of A.

2. Expected value of k^{th}-moment of $\mu_{A,N}(x)$ is

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{\text{Trace}(A^k)}{2^k N^{\frac{k}{2}+1}} \prod_{i \leq j} p(a_{ij}) da_{ij}.$$

3. Show the expected value of k^{th}-moment of $\mu_{A,N}(x)$ equals the k^{th}-moment of the Semi-Circle.

Random Matrix Theory: Semi-Circle Experiments

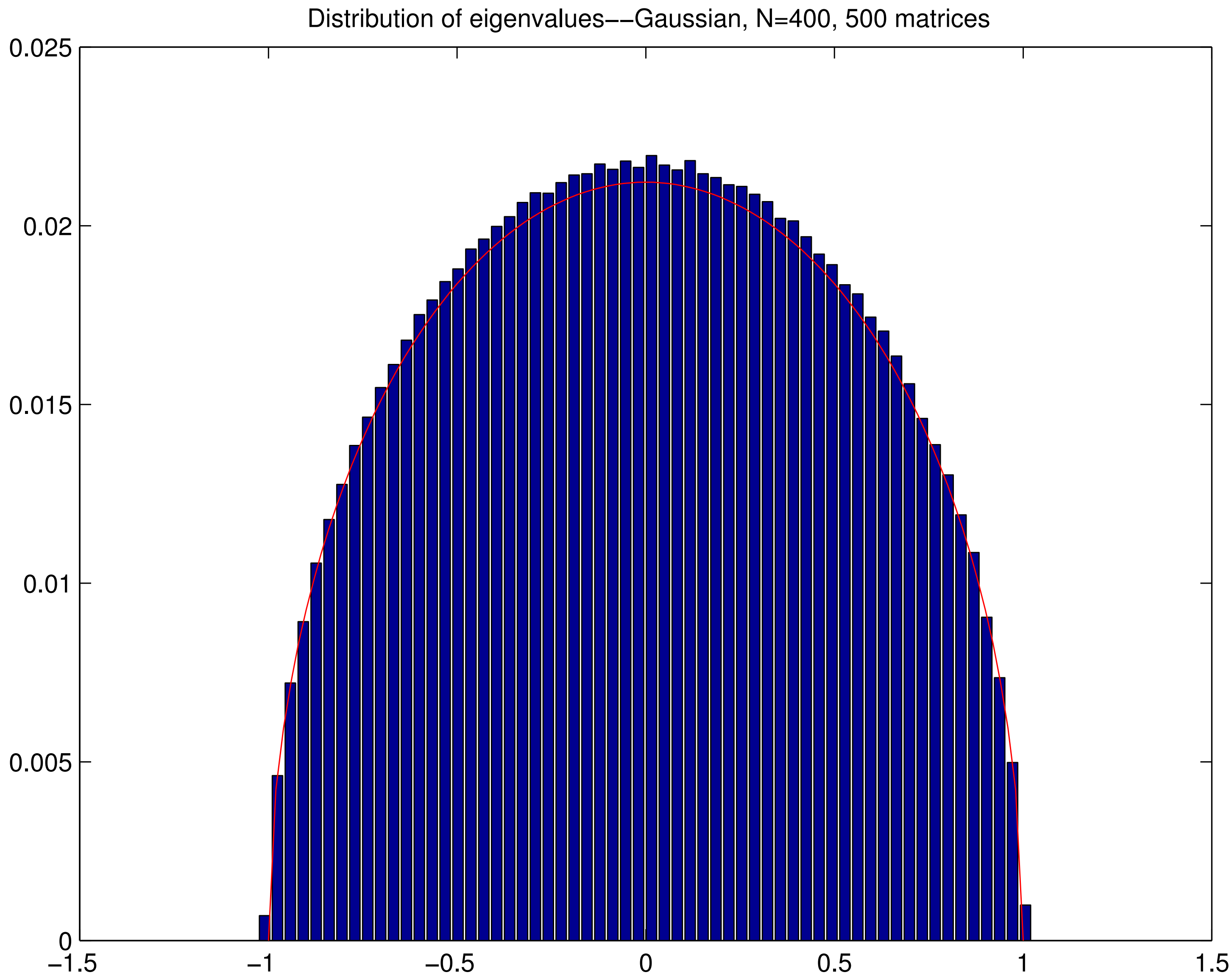
1. Choose a probability distribution p on $\mathbb{R}$.

2. Choose a large N (size of matrices).

3. Look at a large number of $N \times N$ real symmetric matrices, each matrix has its upper triangular entries independently chosen from p.

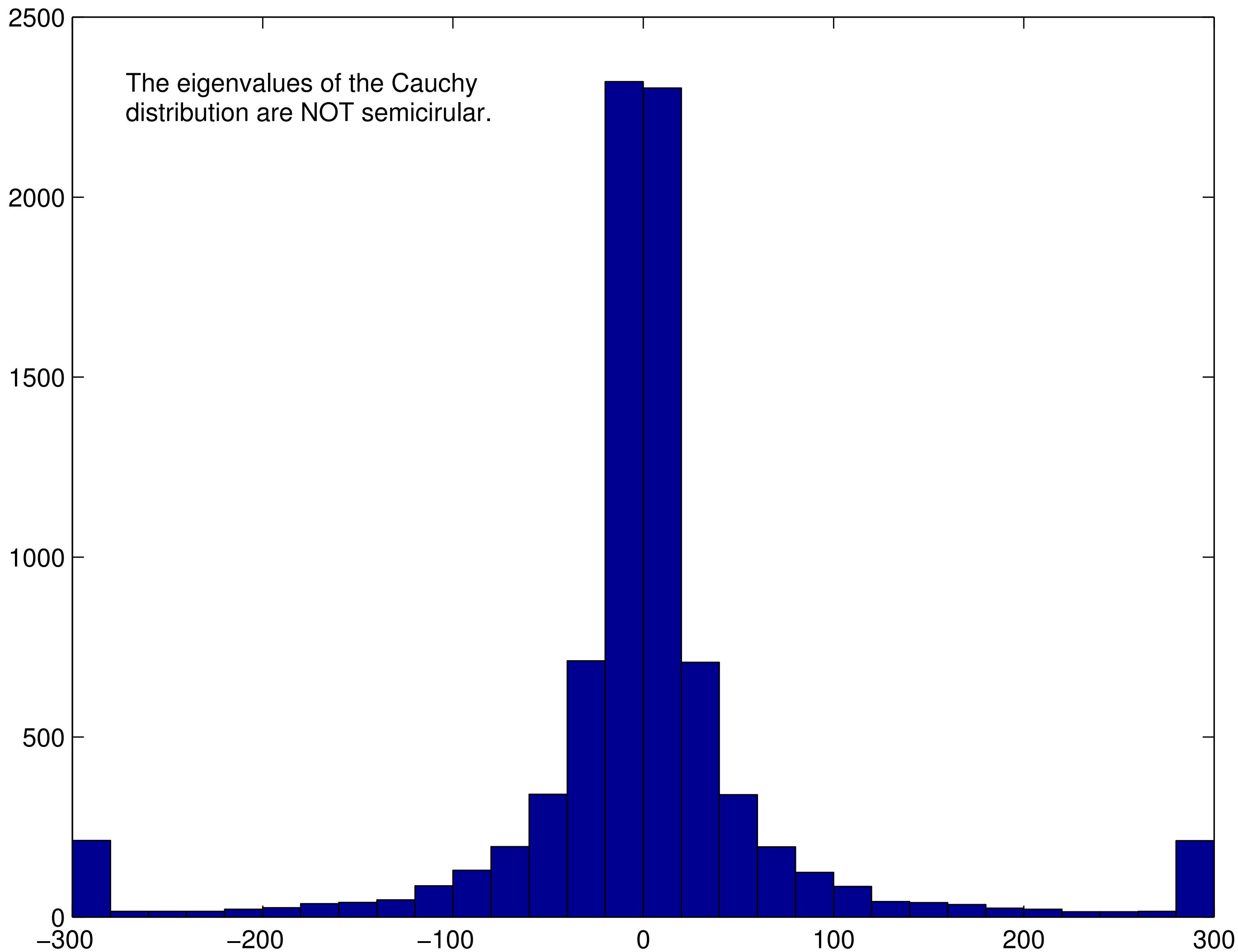
4. Calculate the normalized eigenvalues $\frac{\lambda_i(A)}{2\sqrt{N}}$, investigate their properties.

Random Matrix Theory: Semi-Circle Law



500 **Matrices: Gaussian** 400×400 $p(x) \;=\; \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}$

Random Matrix Theory: Semi-Circle Law



Cauchy Distribution: $p(x) = \frac{1}{\pi(1+x^2)}$, $x \in \mathbb{R}$

Infinite Variance

Gaussian Orthogonal Ensemble (GOE) Conjecture

Let $\mathrm{GOE}(x) \ \approx \ \frac{\pi}{2}\, x e^{-\frac{\pi}{4}x^2}$.

Let p be a probability distribution on $\mathbb{R}$ with mean zero, variance one and finite higher moments.

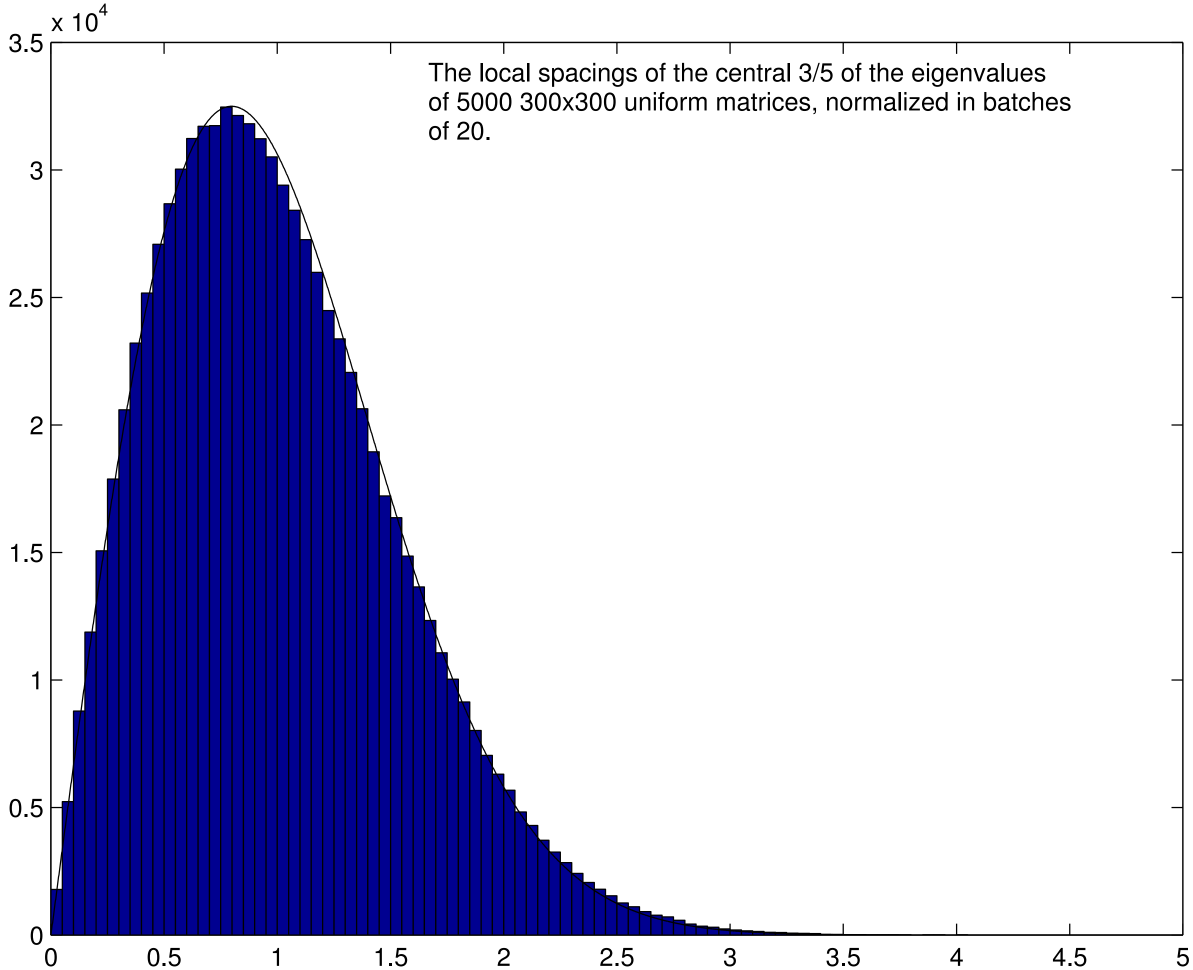
GOE Conjecture: As $N \to \infty$, for most A the probability density of the spacing between consecutive normalized eigenvalues converges to $\mathrm{GOE}(x)$:

$$\frac{\#\left\{i : \frac{\lambda_{i+1}(A)}{2\sqrt{N}} - \frac{\lambda_i(A)}{2\sqrt{N}} \in [\alpha, \beta]\right\}}{N} \longrightarrow \int_\alpha^\beta \mathrm{GOE}(x)dx.$$

Only proved for p a Gaussian with mean 0 and variance 1.

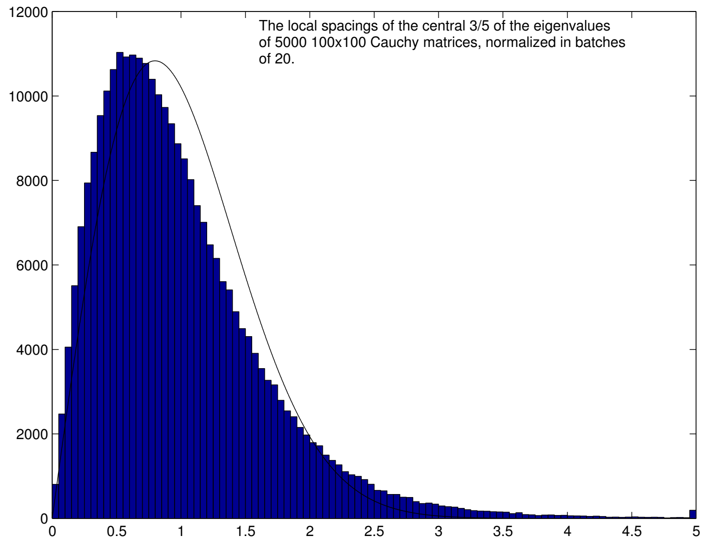
What about numerical data in other cases?

Uniform Distribution: $p(x) = \frac{1}{2}$ for $|x| \leq 1$

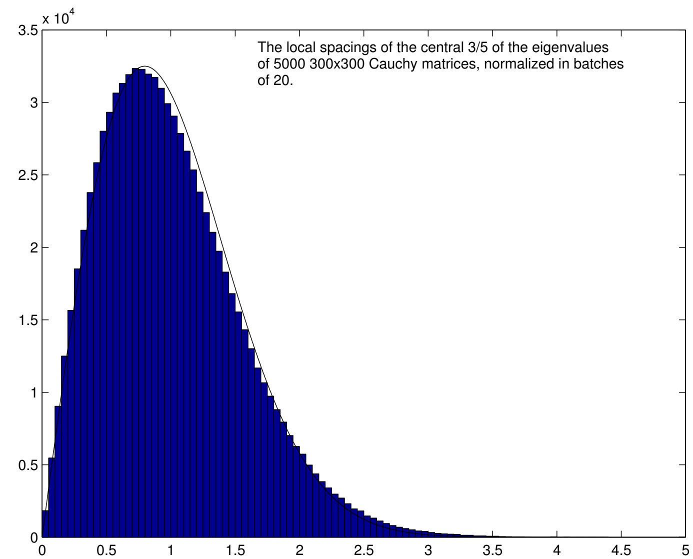


5000: 300×300 uniform on $[-1, 1]$

Cauchy Distribution: $p(x) = \frac{1}{\pi(1+x^2)}$, $x \in \mathbb{R}$



5000: 100×100 Cauchy



5000: 300×300 Cauchy

PART III

NUMBER THEORY

IDEA:

Zeros of Random Matrices Provide a Good Model for Zeros of Number Theoretic Functions

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Geometric Series (and Extending Functions): If $|u| < 1$,

$$\sum_{k=0}^{\infty} u^k = 1 + u + u^2 + u^3 + \cdots = \frac{1}{1-u}$$

Unique Factorization: $n = p_1^{r_1} \cdots p_m^{r_m}$.

$$\prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \left[1 + \frac{1}{2^s} + \left(\frac{1}{2^s}\right)^2 + \cdots\right]\left[1 + \frac{1}{3^s} + \left(\frac{1}{3^s}\right)^2 + \cdots\right]\cdots$$

$$= \sum_n \frac{1}{n^s}.$$

Riemann Zeta Function (continued):

$$\begin{aligned}\zeta(s) &= \sum_n \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1 \\ \pi(x) &= \#\{p : p \text{ is prime}, p \leq x\}\end{aligned}$$

Properties of $\zeta(s)$ and Primes: Proofs that $\pi(x) \to \infty$.

- $\lim_{s\to 1^+} \zeta(s) = \infty$.
- $\zeta(2) = \frac{\pi^2}{6}$.
- $\zeta(1+it) \neq 0$ for $t \in \mathbb{R}$ implies $\pi(x) \sim \frac{x}{\log x}$.

Riemann Zeta Function (continued):

$$\zeta(s) \ = \ \sum_{n=1}^{\infty} \frac{1}{n^s} \ = \ \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1.$$

Functional Equation:

$$\xi(s) \ = \ \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) \ = \ \xi(1-s).$$

Riemann Hypothesis:

- All zeros have $\text{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma, \gamma \in \mathbb{R}$.
- Number of zeros with $0 \leq \gamma \leq T$ is about $T \log T$.

Observation:

- Spacings between normalized zeros appear same as between normalized eigenvalues of Complex Hermitian matrices ($\overline{A}^T = A$).

Normalized Zeros of Riemann Zeta Function
Zeros $\frac{1}{2} + i\gamma, \gamma \in \mathbb{R}$

Know $\#\{\gamma : 0 \le \gamma \le T\}$ is about $T \log T$.

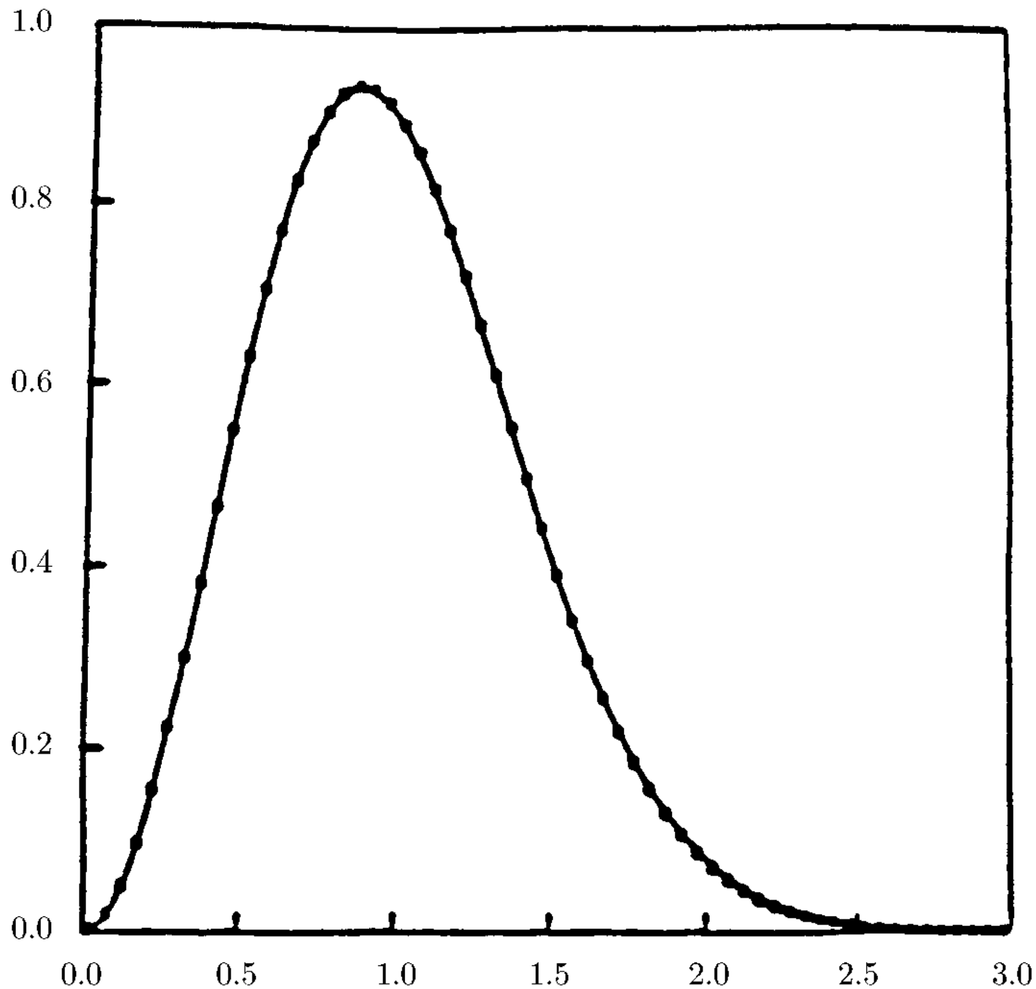
Average spacing of zeros with $\gamma \sim T$ is $\frac{T}{T \log T} = \frac{1}{\log T}$.

Contrast with primes: Average spacing of primes $p \sim x$ is $\frac{x}{x/\log x} = \log x$.

Normalized zeros: study $\gamma_{n+1} \log \gamma_{n+1} - \gamma_n \log \gamma_n$.

Contrast with primes, where we studied $\frac{p_{n+1}}{\log p_{n+1}} - \frac{p_n}{\log p_n}$.

Zeros of $\zeta(s)$ vs. GUE(x):



70 million spacings between adjacent normalized zeros of $\zeta(s)$, starting at the $10^{20\text{th}}$ zero (from Odlyzko)

General L-Functions

- **Euler Product:**

$$L(s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_p \frac{1}{L_p(p^{-s})}, \quad \mathrm{Re}(s) \gg 0, \; L_p(x) = \text{polynomial}.$$

- **Functional Equation:**

$$\begin{aligned} \Lambda(s) &:= (\Gamma - \text{Factors}) \cdot L(s) \\ \Lambda(s) &= \frac{\varepsilon(s)}{C^s} \overline{\Lambda(1-\overline{s})}, \; C > 0 \text{ is called the Conductor} \end{aligned}$$

- **Riemann Hypothesis:**

  All zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma, \gamma \in \mathbb{R}$.

- **Number of Zeros:**

  Number of zeros with $\gamma \sim T$ is like $T \log T$

  Zeros near $s = \dfrac{1}{2}$ have $\gamma \sim \dfrac{1}{\log C}$

(Heuristic): Knowledge of Zeros

$P(x)$ polynomial, zeros $r_1, \ldots, r_n$. Then

$$\begin{aligned} P(x) &= (x - r_1)(x - r_2)\cdots(x - r_n) \\ &= x^n + a_{n-1}(r_1, \ldots, r_n)x^{n-1} + \cdots + a_0(r_1, \ldots, r_n) \end{aligned}$$

where

$$\begin{aligned} a_{n-1}(r_1, \ldots, r_n) &= -(r_1 + \cdots + r_n) \\ &\vdots \\ a_0(r_1, \ldots, r_n) &= r_1 r_2 \cdots r_n. \end{aligned}$$

Knowledge of zeros gives info on coefficients of $P(x)$.

Explicit Formula / Contour Integration: Analogue of Eigenvalue Trace Lemma

$$
\begin{aligned}
-\frac{\zeta'(s)}{\zeta(s)} &= -\frac{\mathrm{d}}{\mathrm{d}s}\log\zeta(s) \\
&= \frac{\mathrm{d}}{\mathrm{d}s}\sum_p \log\left(1-p^{-s}\right) \\
&= \sum_p \frac{\log p \ \cdot \ p^{-s}}{1-p^{-s}} \\
&= \sum_p \frac{\log p}{p^s} \ + \ \mathrm{Good}(s).
\end{aligned}
$$

Contour Integration:

$$
\int -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds \quad \text{vs} \quad \sum_p \log p \int \left(\frac{x}{p}\right)^s \frac{ds}{s}.
$$

Knowledge of zeros gives info on the L-function coefficients.

Interesting L-Functions

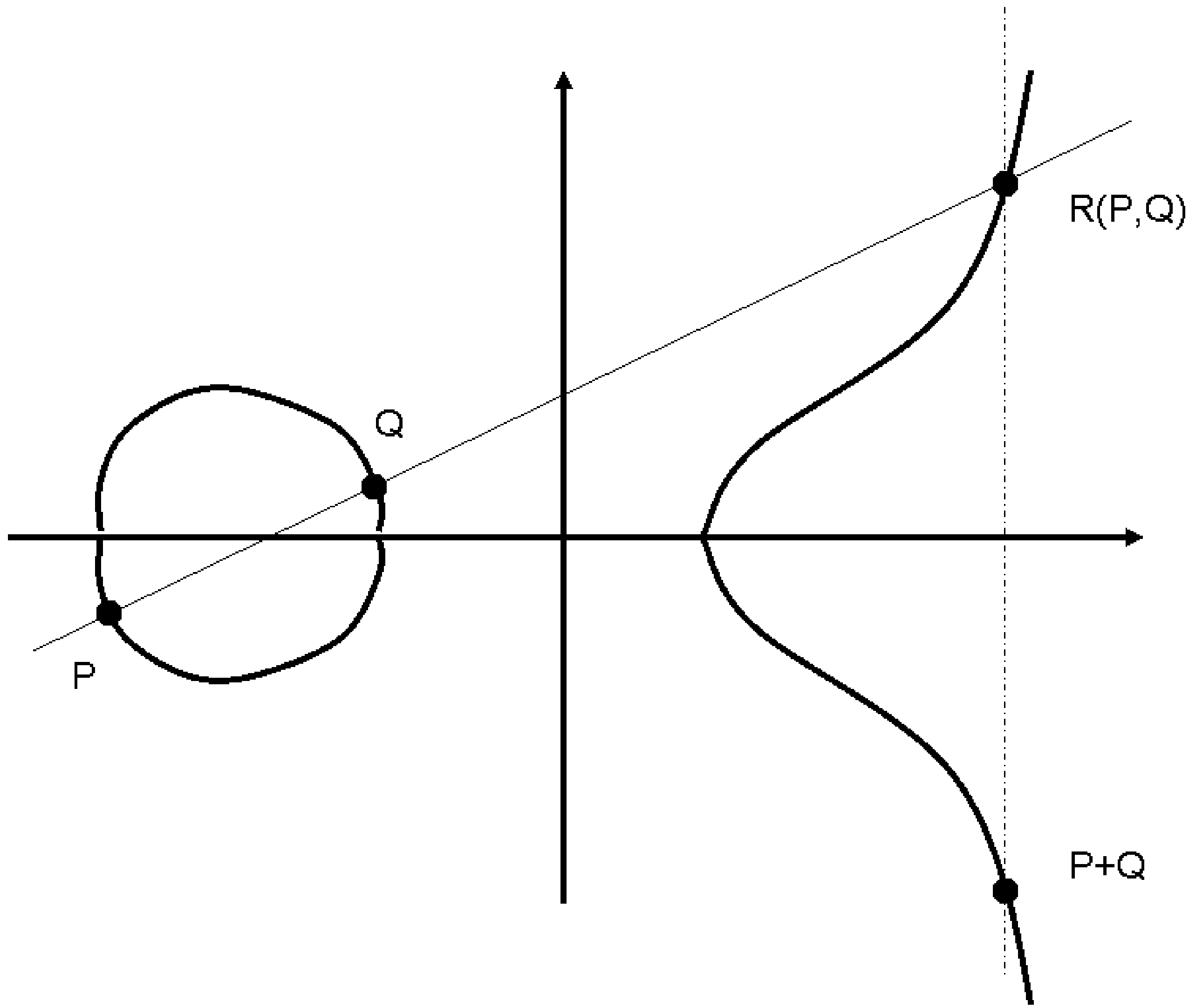
What makes an L-Function interesting?

- Coefficients a_n of arithmetic significance.
- Look for L-Functions with multiple zeros:
  - Conjectured that all zeros are simple except for deep reasons;
  - Do multiple zeros attract or repel nearby zeros?

Will see L-Functions of Elliptic Curves are interesting.

- Many have multiple zeros at $s = \frac{1}{2}$.
- Can investigate if these zeros attract or repel.

Elliptic Curves: E: $y^2 = x^3 + Ax + B$



Elliptic Curves: Group of Rational Solutions $E(\mathbb{Q})$

Studying E: $y^2 = x^3 + Ax + B$

Mordell-Weil Theorem: Rational solutions: $E(\mathbb{Q}) = \mathbb{Z}^r \bigoplus$ Finite Group.

Question: how does r depend on E?

Attach an L-Function to E: As $\zeta(s)$ gives us information on primes, expect L-Function gives us information on E.

Review: Legendre Symbol: $\left(\frac{0}{p}\right) = 0$ and

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } x^2 \equiv a \bmod p \text{ has two solutions} \\ -1 & \text{if } x^2 \equiv a \bmod p \text{ has no solutions.} \end{cases}$$

Note $1 + \left(\frac{a}{p}\right)$ is the number of solutions to $x^2 \equiv a \bmod p$.

L-Function of an Elliptic Curve $E : y^2 = x^3 + Ax + B$

Let N_p be the number of solutions mod p:

$$N_p \ := \sum_{x \bmod p} \left[1 + \left(\frac{x^3 + Ax + B}{p}\right)\right] = p + \sum_{x \bmod p} \left(\frac{x^3 + Ax + B}{p}\right)$$

Local data: $a_p = p - N_p$. Use to build the L-function:

$$L(E, s) \ := \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

Local to Global: $\{a_p\}_{p \text{ prime}} \longleftrightarrow E(\mathbb{Q}) = \mathbb{Z}^r \bigoplus$ Finite Group.

Birch and Swinnerton-Dyer Conjecture: Geometric rank r equals number of zeros of $L(E, s)$ at $s = \frac{1}{2}$. **Possibility of repulsion / attraction from zeros at $s = \frac{1}{2}$!**

Families of Elliptic Curves:

$$\mathcal{E} : y^2 \;=\; x^3 + A(T)x + B(T), \;\; A(T), B(T) \in \mathbb{Z}[T].$$

Have a FAMILY of L-Functions:

- $t \in \mathbb{Z}$ gives an elliptic curve E_t with conductor C_t.
- $t \in \mathbb{Z}$ gives a family of L-functions $L(E_t, s)$.
- C_t is typically growing polynomially in t.

Similar to RMT where we had many $N \times N$ matrices

- Do E_t from the same family have similar properties?
- $N \to \infty$ becomes $C_t \to \infty$.

Families of Elliptic Curves

Mordell-Weil Theorem for Families:

- $\mathcal{E}$: $y^2 = x^3 + A(T)x + B(T)$, $A(T), B(T) \in \mathbb{Z}[T]$.

- Group of Rational Function Solutions:
  $P(T) = (x(T), y(T))$.
  $\mathcal{E}(\mathbb{Q}(T)) = \mathbb{Z}^{r(\mathcal{E})} \bigoplus$ Finite Group.

- Specialization Theorem: For all $t \in \mathbb{Z}$ sufficiently large: $r(E_t) \geq r(\mathcal{E})$.

Questions:

- How does $r(E_t)$ vary in the family?
- How do the zeros of $L(s, E_t)$ vary in the family?

Example of a Family of Elliptic Curves

Consider $\mathcal{E} : y^2 \ = \ x^3 - T^2x + T^4$.

- **Can find some rational solutions by inspection:**
  - $P_1(T) = (T, T^2)$
  - $P_2(T) = (T^2, T^3)$
  - These points generate infinite part of $\mathcal{E}(\mathbb{Q}(T))$.
  - This means the rank of $\mathcal{E}(\mathbb{Q}(T))$ is 2.

- **What is $r(E_t)$ for $t \in \mathbb{Z}$**
  - $r(E_1) = 1$. Not surprising as $P_1(1) = P_2(1)$!
  - $r(E_2) = 2$.
  - $r(E_{42}) = 4$!

L-Functions and Random Matrix Theory

- Spacings between normalized zeros of all "nice" L-functions look like spacings between normalized eigenvalues of complex Hermitian matrices.
  - (GUE Model) Limit as $N \to \infty$ of $N \times N$ complex Hermitian matrices with upper triangular entries independently chosen from the standard normal.
  - Zeros with $\gamma \sim T$ well-modelled by $N \times N$ matrices with $N \sim \log T$.

- For a while, seemed this was the only set of matrices relevant to number theory.

L-Functions and Random Matrix Theory

- GUE Model cannot be enough for all of Number Theory:
  - Appears to be good for describing behavior of zeros far from $s = \frac{1}{2}$.
  - Insensitive to finitely many zeros.

- Often zeros near $s = \frac{1}{2}$ are interesting:
  - Different types of of L-functions have different behavior of zeros near $s = \frac{1}{2}$.
  - Possibility of multiple zeros at $s = \frac{1}{2}$ and attraction or repulsion.
  - Need a theory for these low zeros.

Measures of Spacings: 1-Level Density and Families

Let ϕ be an even Schwartz function whose Fourier Transform is compactly supported. Let $L(s)$ be an L-function with zeros $\frac{1}{2} + i\gamma$ ($\gamma \in \mathbb{R}$) and conductor C. Define the 1-level density by

$$\sum_n \phi\left(\gamma_n \log C\right)$$

- Individual zeros γ_n contribute in limit
- Most of contribution is from low zeros
- Average over similar L-functions (family)

To any geometric family, Katz-Sarnak predict the 1-level density depends only on a symmetry group (a classical compact group) attached to the family.

Random Matrix Ensembles and Number Theory

- **Zeros far away from $s = \frac{1}{2}$ well-modelled by GUE.**
  - Choose one L-Function, look at high zeros.
  - One L-function has enough freedom to average.

- **Story different for zeros near $s = \frac{1}{2}$.**
  - One L-function no longer suffices for averaging.
  - Look at many similar L-functions.
  - Hope L-functions' zeros near $s = \frac{1}{2}$ behave similarly.

- **Analogy with Random Matrix Theory:**
  - RMT: pick many $N \times N$ matrices at random, average, study limit as $N \to \infty$.
  - NT: pick many L-functions in a family with conductors about the same size, average, study what happens as conductors go to infinity.

Random Matrix Ensembles

Real Symmetric, Complex Hermitian Matrices:

1. $\lambda \in \mathbb{R}$.
2. Randomness: upper triangular entries independently chosen from p; freedom to choose p.

Classical Compact Groups:

1. $\lambda = e^{i\theta}$, $\theta \in (-\pi, \pi] \subset \mathbb{R}$.
2. Randomness: Haar measure; canonical choice.
3. Subgroups: Orthogonal Matrices ($Q^T Q = I$):

$$\begin{array}{rl} \text{SO(even)} : e^{i\theta} : & \cdots \le -\theta_2 \le -\theta_1 \le \quad 0 \quad \le \theta_1 \le \theta_2 \le \cdots \\ \text{SO(odd)} : e^{i\theta} : & \cdots \le -\theta_2 \le -\theta_1 \le \theta_0 = 0 \le \theta_1 \le \theta_2 \le \cdots \end{array}$$

Theorems for Families of Elliptic Curves
Family $\mathcal{E} : y^2 = x^3 + A(T)x + B(T)$, specialized curves E_t

If family $\mathcal{E}$ has rank $r(\mathcal{E})$: As conductors go to infinity:

- Results suggest E_t has at least $r(\mathcal{E})$ zeros at $s = \frac{1}{2}$;
- Behavior of remaining zeros near $s = \frac{1}{2}$ agree with eigenvalues near 1 of orthogonal groups.

Method of Proof

- Explicit Formula: *(Analogue of Eigenvalue Trace Lemma)*
  - Relates Sums over Zeros to Sums over Primes.
  - Gives correct scale.
- Summation Formulas: *(Analogue of Integrating Matrix Entries)*
  - Sums of Legendre Symbols.
  - Like Central Limit Theorem, just first two moments matter:

    $\sum_{t \bmod p} a_{p,t}$ and $\sum_{t \bmod p} a_{p,t}^2$.

Predictions from Random Matrix Theory
Family $\mathcal{E}$ of Elliptic Curves with rank $r(\mathcal{E})$

Families of Elliptic Curves well-modelled by Orthogonal Groups: zeros near $s = \frac{1}{2}$ look like eigenvalues near 1.

As conductor C_t goes to infinity, expect half the elliptic curves E_t to have rank $r(\mathcal{E})$, half to have rank $r(\mathcal{E}) + 1$.

As conductor C_t goes to infinity, for each E_t expect the $r(\mathcal{E})$ family zeros to be independent of the other zeros of E_t near $s = \frac{1}{2}$.

In particular, the distribution of the first zero above $s = \frac{1}{2}$ should be independent of $r(\mathcal{E})$.

Excess Rank

One-parameter family, rank $r(\mathcal{E})$ over $\mathbb{Q}(T)$.

For each $t \in \mathbb{Z}$, consider curves E_t.

RMT $\Longrightarrow$ 50% rank $r(\mathcal{E})$, 50% rank $r(\mathcal{E}) + 1$.

For many families, observe

Percent with rank $r(\mathcal{E})$	$= 32\%$	Percent with rank $r(\mathcal{E}) + 1$	$= 48\%$
Percent with rank $r(\mathcal{E}) + 2$	$= 18\%$	Percent with rank $r(\mathcal{E}) + 3$	$= 2\%$

Problem: small data sets, sub-families, convergence rate $\log(\text{conductor})$?

Interval	Primes	Twin Primes Pairs
$[1, 10]$	$2, 3, 5, 7$ (40%)	$(3, 5), (5, 7)$ (20%)
$[11, 20]$	$11, 13, 17, 19$ (40%)	$(11, 13), (17, 19)$ (20%)

Small data can be misleading! Remember $\sum_{n \le N} \frac{1}{n} \sim \log N$.

Data on Excess Rank

$$y^2 \;=\; x^3 + 16Tx + 32$$

Each data set runs over 2000 consecutive t-values.

t-Start	Rk 0	Rk 1	Rk 2	Rk 3	Time (hrs)
-1000	39.4	47.8	12.3	0.6	<1
1000	38.4	47.3	13.6	0.6	<1
4000	37.4	47.8	13.7	1.1	1
8000	37.3	48.8	12.9	1.0	2.5
24000	35.1	50.1	13.9	0.8	6.8
50000	36.7	48.3	13.8	1.2	51.8

Last set has conductors of size 10^{11}, but on logarithmic scale still small.

SO(even) Random Matrix Models

RMT: $2N$ eigenvalues, in pairs $e^{\pm i\theta_j}$, probability measure on $[0,\pi]^N$:

$$\prod_{j<k}(\cos\theta_k - \cos\theta_j)^2 \prod_j d\theta_j$$

Independent Model: $2r$ Eigenvalues at 1

$$\left\{ \begin{pmatrix} g & \\ & I_{2r} \end{pmatrix} : g \in SO(2N-2r) \right\}$$

Interaction Model: $2r$ Eigenvalues at 1

Sub-ensemble of SO$(2N)$ with $2r$ eigenvalues forced to be $+1$:

$$\prod_{j<k}(\cos\theta_k - \cos\theta_j)^2 \prod_j (1-\cos\theta_j)^{2r} \prod_j d\theta_j,$$

with $1 \le j,k \le N-r$.

Comparing the two Random Matrix Models

For small support, 1-level densities for Elliptic Curves agree with Independent Model.

Elliptic Curve E, conductor C, expect first zero above $s = \frac{1}{2}$ to be $\frac{1}{2} + i\gamma$ with $\gamma \sim \frac{1}{\log C}$.

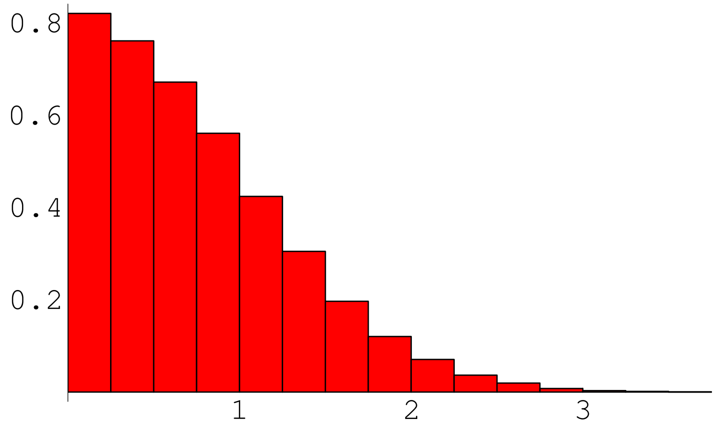
If r zeros at central point, if repulsion of zeros is of size $\frac{c_r}{\log C}$, would detect in zeros near central point:

$$\sum_n \phi\left(\gamma_n \log C\right).$$

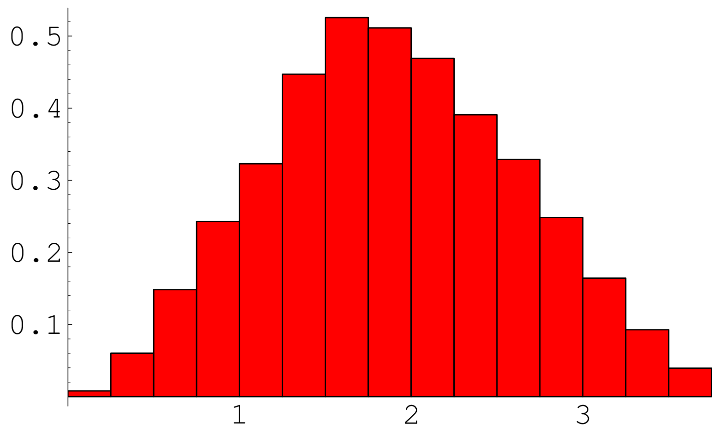
Corrections of size

$$\phi\left(x + c_r\right) - \phi(x) \;\approx\; \phi'\left(x\right) \cdot c_r.$$

Theoretical Distribution of First Normalized Zero



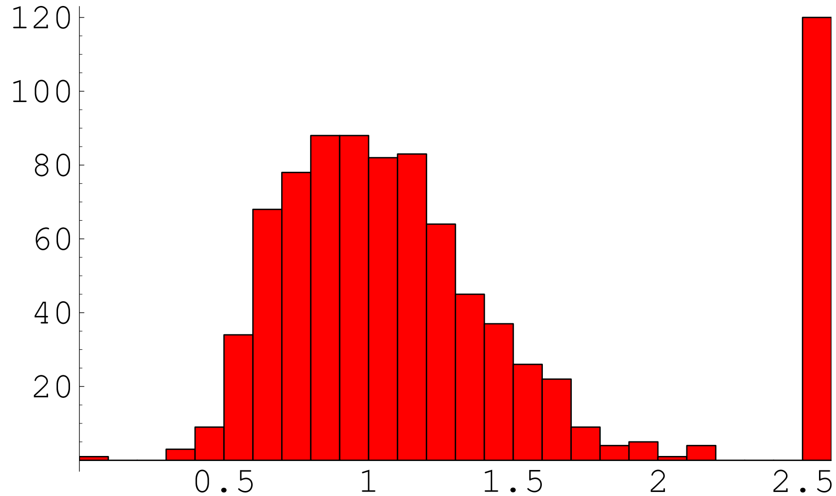
First normalized eigenvalue: 230,400 from $SO(6)$ with Haar Measure



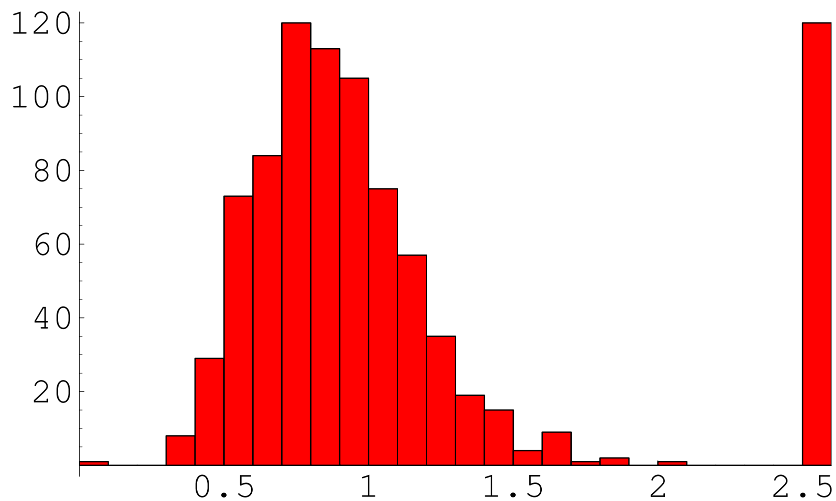
First normalized eigenvalue: 322,560 from $SO(7)$ with Haar Measure

Rank 0 Curves: 1st Normalized Zero

(Far left and right bins just for formatting)

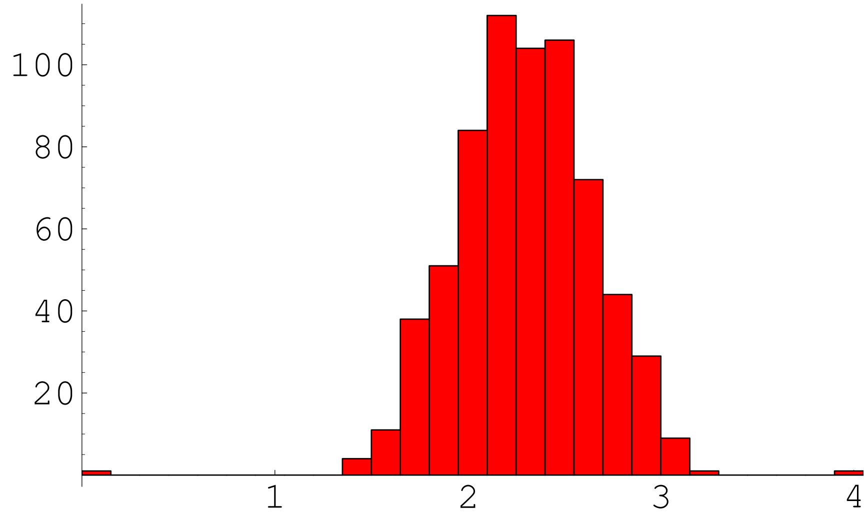


750 **curves**, $\log(\textbf{cond}) \in [3.2, 12.6]$; **mean** $= 1.04$

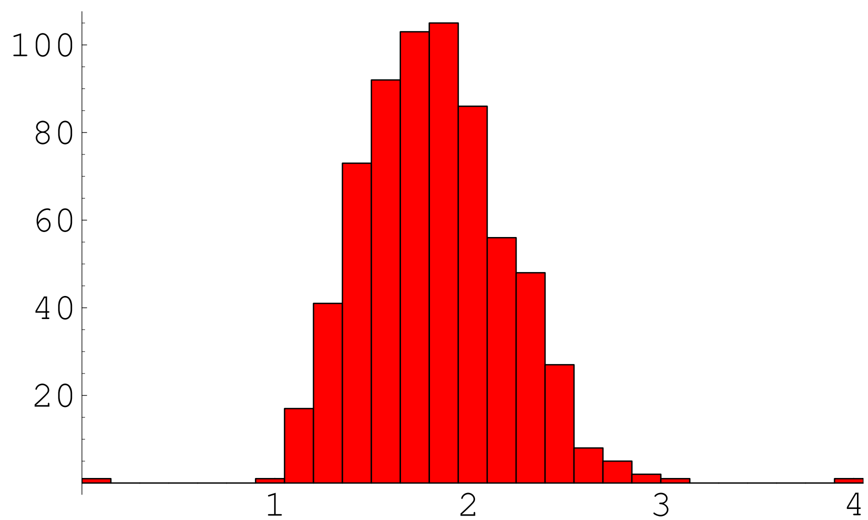


750 **curves**, $\log(\textbf{cond}) \in [12.6, 14.9]$; **mean** $= .88$

Rank 2 Curves: 1st Normalized Zero

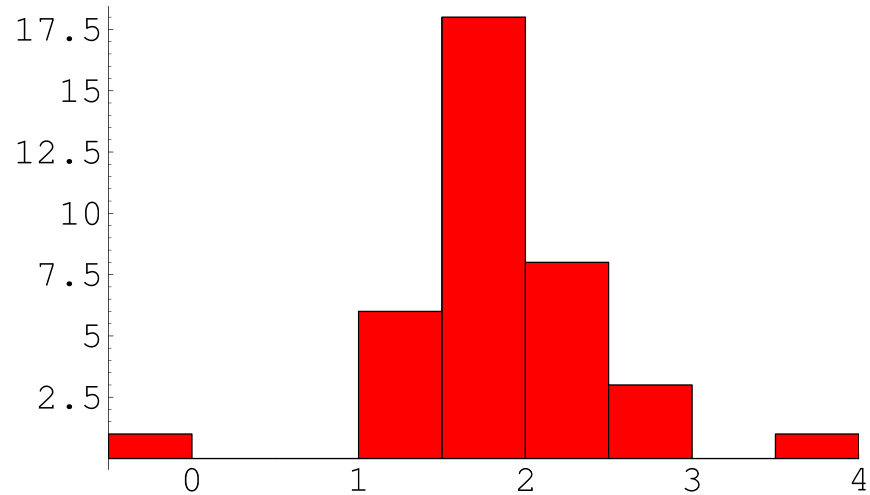


665 **curves**, $\log(\textbf{cond}) \in [10, 10.3125]$; **mean** $= 2.30$

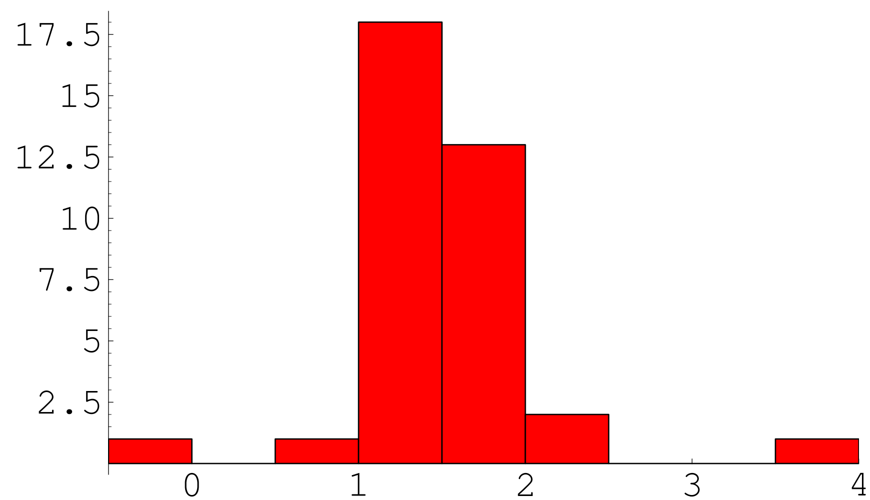


665 **curves**, $\log(\textbf{cond}) \in [16, 16.5]$; **mean** $= 1.82$

Rank 2 Curves: $y^2 = x^3 - T^2x + T^2$: 1st Normalized Zero



35 **curves**, $\log(\textbf{cond}) \in [7.8, 16.1]$; **mean** $= 2.24$



34 **curves**, $\log(\textbf{cond}) \in [16.2, 23.3]$; **mean** $= 2.00$

PART VI

CONCLUSIONS

Correspondences

Similarities between Nuclei and Primes:

Energy Levels $\longleftrightarrow$ Zeros of L-Functions

Neutron Energy $\longleftrightarrow$ Support of Test Functions

Different Elements: U, Pu, ... $\longleftrightarrow$ Different L-Functions

Summary

- Find correct scale to compare different systems.
- Similar behavior in different systems.
- Need a Trace Lemma.
- Average over similar elements.
- Need more data.

Open Problems (NT)

Open Problems (NT)

Identifying Classical Compact Group:

Given a reasonable family of L-functions, determine the corresponding symmetry group.

Montgomery-Odlyzko Law:

Show that zeros of L-functions at height $T \to \infty$ behave like eigenvalues of $N \times N$ matrices with $N \sim \log \frac{T}{2\pi}$.

Finite Height / Finite Family Size:

Know correct model for high zeros ($N = \log \frac{T}{2\pi}$); what is the correct model for zeros near the central point as we move through the family (ordered by conductor)?

APPENDIX I:
Dirichlet L-Functions

Dirichlet Characters: m Prime

$(\mathbb{Z}/m\mathbb{Z})^*$ is cyclic of order $m-1$ with generator g.

Let $\zeta_{m-1} = e^{2\pi i/(m-1)}$.

The principal character χ_0 is given by

$$\chi_0(k) = \begin{cases} 1 & (k,m) = 1 \\ 0 & (k,m) > 1. \end{cases}$$

The $m-2$ primitive characters are determined (by multiplicativity) by action on g.

As each $\chi : (\mathbb{Z}/m\mathbb{Z})^* \to \mathbb{C}^*$, for each χ there exists an l such that $\chi(g) = \zeta_{m-1}^l$. Hence for each $l, 1 \leq l \leq m-2$ we have

$$\chi_l(k) \;=\; \begin{cases} \zeta_{m-1}^{la} & k \equiv g^a(m) \\ 0 & (k,m) > 0 \end{cases}$$

Dirichlet L-Functions

Let χ be a primitive character mod m. Let

$$c(m,\chi) = \sum_{k=0}^{m-1} \chi(k) e^{2\pi i k/m}.$$

$c(m,\chi)$ is a Gauss sum of modulus $\sqrt{m}$.

$$\begin{aligned} L(s,\chi) &= \prod_p (1-\chi(p)p^{-s})^{-1} \\ \Lambda(s,\chi) &= \pi^{-\frac{1}{2}(s+\epsilon)}\Gamma\left(\frac{s+\epsilon}{2}\right) m^{\frac{1}{2}(s+\epsilon)} L(s,\chi), \end{aligned}$$

where

$$\begin{aligned} \epsilon &= \begin{cases} 0 & \text{if } \chi(-1) = 1 \\ 1 & \text{if } \chi(-1) = -1 \end{cases} \\ \Lambda(s,\chi) &= (-i)^{\epsilon} \frac{c(m,\chi)}{\sqrt{m}} \Lambda(1-s,\bar{\chi}). \end{aligned}$$

Explicit Formula

Let ϕ be an even Schwartz function with compact support in $(-\sigma, \sigma)$.

Let χ be a non-trivial primitive Dirichlet character of conductor m.

$$\begin{aligned}
\sum \phi\left(\gamma \frac{\log(\frac{m}{\pi})}{2\pi}\right) &= \int_{-\infty}^{\infty} \phi(y)dy \\
&\quad - \sum_p \frac{\log p}{\log(m/\pi)} \widehat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right) [\chi(p) + \overline{\chi}(p)] p^{-\frac{1}{2}} \\
&\quad - \sum_p \frac{\log p}{\log(m/\pi)} \widehat{\phi}\left(2\frac{\log p}{\log(m/\pi)}\right) [\chi^2(p) + \overline{\chi}^2(p)] p^{-1} \\
&\quad + O\left(\frac{1}{\log m}\right).
\end{aligned}$$

Expansion

$\{\chi_0\} \cup \{\chi_l\}_{1 \leq l \leq m-2}$ are all the characters mod m.

Consider the family of primitive characters mod a prime m ($m-2$ characters):

$$\begin{aligned}
&\int_{-\infty}^{\infty} \phi(y)dy \\
-&\ \frac{1}{m-2} \sum_{\chi \neq \chi_0} \sum_p \frac{\log p}{\log(m/\pi)} \widehat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right) [\chi(p) + \overline{\chi}(p)] p^{-\frac{1}{2}} \\
-&\ \frac{1}{m-2} \sum_{\chi \neq \chi_0} \sum_p \frac{\log p}{\log(m/\pi)} \widehat{\phi}\left(2\frac{\log p}{\log(m/\pi)}\right) [\chi^2(p) + \overline{\chi}^2(p)] p^{-1} \\
+&\ O\left(\frac{1}{\log m}\right).
\end{aligned}$$

Note can pass Character Sum through Test Function.

Character Sums

$$\sum_{\chi} \chi(k) = \begin{cases} m-1 & k \equiv 1(m) \\ 0 & \text{otherwise} \end{cases}$$

For any prime $p \neq m$

$$\sum_{\chi \neq \chi_0} \chi(p) \ = \ \begin{cases} m-1-1 & p \equiv 1(m) \\ -1 & \text{otherwise} \end{cases}$$

Substitute into

$$\frac{1}{m-2} \sum_{\chi \neq \chi_0} \sum_{p} \frac{\log p}{\log(m/\pi)} \widehat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right) [\chi(p) + \overline{\chi}(p)] p^{-\frac{1}{2}}$$

First Sum

$$
\begin{aligned}
& \frac{-2}{m-2}\sum_{p}^{m^\sigma}\frac{\log p}{\log(m/\pi)}\widehat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right)p^{-\frac{1}{2}} \\
+ \quad & 2\frac{m-1}{m-2}\sum_{p\equiv 1(m)}^{m^\sigma}\frac{\log p}{\log(m/\pi)}\widehat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right)p^{-\frac{1}{2}} \\
\ll \quad & \frac{1}{m}\sum_{p}^{m^\sigma}p^{-\frac{1}{2}} + \sum_{p\equiv 1(m)}^{m^\sigma}p^{-\frac{1}{2}} \\
\ll \quad & \frac{1}{m}\sum_{k}^{m^\sigma}k^{-\frac{1}{2}} + \sum_{\substack{k\equiv 1(m)\\ k\geq m+1}}^{m^\sigma}k^{-\frac{1}{2}} \\
\ll \quad & \frac{1}{m}\sum_{k}^{m^\sigma}k^{-\frac{1}{2}} + \frac{1}{m}\sum_{k}^{m^\sigma}k^{-\frac{1}{2}} \\
\ll \quad & \frac{1}{m}m^{\sigma/2}.
\end{aligned}
$$

No contribution if $\sigma < 2$.

Second Sum

$$\frac{1}{m-2}\sum_{\chi\neq\chi_0}\sum_{p}\frac{\log p}{\log(m/\pi)}\widehat{\phi}\left(2\frac{\log p}{\log(m/\pi)}\right)\frac{\chi^2(p)+\overline{\chi}^2(p)}{p}.$$

$$\sum_{\chi\neq\chi_0}[\chi^2(p)+\overline{\chi}^2(p)] = \begin{cases}2(m-2) & p\equiv\pm1(m)\\ -2 & p\not\equiv\pm1(m)\end{cases}$$

Up to $O\left(\frac{1}{\log m}\right)$ we find that

$$\begin{aligned}
&\ll \frac{1}{m-2}\sum_{p}^{m^{\sigma/2}}p^{-1} + \frac{2m-2}{m-2}\sum_{p\equiv\pm1(m)}^{m^{\sigma/2}}p^{-1}\\
&\ll \frac{1}{m-2}\sum_{k}^{m^{\sigma/2}}k^{-1} + \sum_{\substack{k\equiv1(m)\\k\geq m+1}}^{m^{\sigma/2}}k^{-1} + \sum_{\substack{k\equiv-1(m)\\k\geq m-1}}^{m^{\sigma/2}}k^{-1}\\
&\ll \frac{\log(m^{\sigma/2})}{m-2} + \frac{1}{m}\sum_{k}^{m^{\sigma/2}}k^{-1} + \frac{1}{m}\sum_{k}^{m^{\sigma/2}}k^{-1} + O\left(\frac{1}{m}\right)\\
&\ll \frac{\log m}{m} + \frac{\log m}{m} + \frac{\log m}{m}.
\end{aligned}$$

Dirichlet Characters: m Square-free

Fix an r and let $m_1, \ldots, m_r$ be distinct odd primes.

$$\begin{aligned} m &= m_1 m_2 \cdots m_r \\ M_1 &= (m_1 - 1)(m_2 - 1)\cdots(m_r - 1) = \phi(m) \\ M_2 &= (m_1 - 2)(m_2 - 2)\cdots(m_r - 2). \end{aligned}$$

M_2 is the number of primitive characters mod m, each of conductor m.

A general primitive character mod m is given by $\chi(u) = \chi_{l_1}(u)\chi_{l_2}(u)\cdots\chi_{l_r}(u)$.

Let $\mathcal{F} = \{\chi : \chi = \chi_{l_1}\chi_{l_2}\cdots\chi_{l_r}\}$.

$$\frac{1}{M_2}\sum_p \frac{\log p}{\log(m/\pi)}\widehat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right) p^{-\frac{1}{2}} \sum_{\chi\in\mathcal{F}}[\chi(p) + \overline{\chi}(p)]$$

$$\frac{1}{M_2}\sum_p \frac{\log p}{\log(m/\pi)}\widehat{\phi}\left(2\frac{\log p}{\log(m/\pi)}\right) p^{-1} \sum_{\chi\in\mathcal{F}}[\chi^2(p) + \overline{\chi}^2(p)]$$

Characters Sums:

$$\sum_{l_i=1}^{m_i-2} \chi_{l_i}(p) = \begin{cases} m_i - 1 - 1 & p \equiv 1(m_i) \\ -1 & \text{otherwise} \end{cases}$$

Define

$$\delta_{m_i}(p,1) = \begin{cases} 1 & p \equiv 1(m_i) \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\begin{aligned} \sum_{\chi \in \mathcal{F}} \chi(p) &= \sum_{l_1=1}^{m_1-2} \cdots \sum_{l_r=1}^{m_r-2} \chi_{l_1}(p) \cdots \chi_{l_r}(p) \\ &= \prod_{i=1}^{r} \sum_{l_i=1}^{m_i-2} \chi_{l_i}(p) \\ &= \prod_{i=1}^{r} \left(-1 + (m_i - 1)\delta_{m_i}(p,1)\right). \end{aligned}$$

Expansion Preliminaries:

$k(s)$ is an s-tuple $(k_1, k_2, \ldots, k_s)$ with $k_1 < k_2 < \cdots < k_s$.

This is just a subset of $(1, 2, \ldots, r)$, 2^r possible choices for $k(s)$.

$$\delta_{k(s)}(p, 1) = \prod_{i=1}^{s} \delta_{m_{k_i}}(p, 1).$$

If $s = 0$ we define $\delta_{k(0)}(p, 1) = 1 \ \forall p$.

Then

$$\begin{aligned} & \prod_{i=1}^{r} (-1 + (m_i - 1)\delta_{m_i}(p, 1)) \\ = \quad & \sum_{s=0}^{r} \sum_{k(s)} (-1)^{r-s} \delta_{k(s)}(p, 1) \prod_{i=1}^{s} (m_{k_i} - 1) \end{aligned}$$

First Sum:

$$\ll \sum_{p}^{m^\sigma} p^{-\frac{1}{2}} \frac{1}{M_2} \left(1 + \sum_{s=1}^{r} \sum_{k(s)} \delta_{k(s)}(p,1) \prod_{i=1}^{s} (m_{k_i} - 1) \right).$$

As $m/M_2 \leq 3^r$, $s = 0$ sum contributes

$$S_{1,0} = \frac{1}{M_2} \sum_{p}^{m^\sigma} p^{-\frac{1}{2}} \ll 3^r m^{\frac{1}{2}\sigma - 1},$$

hence negligible for $\sigma < 2$. Now we study

$$\begin{aligned} S_{1,k(s)} &= \frac{1}{M_2} \prod_{i=1}^{s} (m_{k_i} - 1) \sum_{p}^{m^\sigma} p^{-\frac{1}{2}} \delta_{k(s)}(p,1) \\ &\ll \frac{1}{M_2} \prod_{i=1}^{s} (m_{k_i} - 1) \sum_{n \equiv 1 (m_{k(s)})}^{m^\sigma} n^{-\frac{1}{2}} \\ &\ll \frac{1}{M_2} \prod_{i=1}^{s} (m_{k_i} - 1) \frac{1}{\prod_{i=1}^{s} (m_{k_i})} \sum_{n}^{m^\sigma} n^{-\frac{1}{2}} \\ &\ll 3^r m^{\frac{1}{2}\sigma - 1}. \end{aligned}$$

First Sum (continued):

There are 2^r choices, yielding

$$S_1 \ll 6^r m^{\frac{1}{2}\sigma - 1},$$

which is negligible as m goes to infinity for fixed r if $\sigma < 2$.

Cannot let r go to infinity.

If m is the product of the first r primes,

$$\begin{aligned} \log m &= \sum_{k=1}^{r} \log p_k \\ &= \sum_{p \le r} \log p \approx r \end{aligned}$$

Therefore

$$6^r \approx m^{\log 6} \approx m^{1.79}.$$

Second Sum Expansions:

$$\sum_{l_i=1}^{m_i-2} \chi_{l_i}^2(p) \;=\; \begin{cases} m_i - 1 - 1 & p \equiv \pm 1 (m_i) \\ -1 & \text{otherwise} \end{cases}$$

$$\begin{aligned} & \sum_{\chi \in \mathcal{F}} \chi^2(p) \\ = \;& \sum_{l_1=1}^{m_1-2} \cdots \sum_{l_r=1}^{m_r-2} \chi_{l_1}^2(p) \cdots \chi_{l_r}^2(p) \\ = \;& \prod_{i=1}^{r} \sum_{l_i=1}^{m_i-2} \chi_{l_i}^2(p) \\ = \;& \prod_{i=1}^{r} \left(-1 + (m_i - 1)\delta_{m_i}(p, 1) + (m_i - 1)\delta_{m_i}(p, -1)\right) \end{aligned}$$

Second Sum Bounds:

Handle similarly as before. Say

$$\begin{aligned} p &\equiv 1 \bmod m_{k_1}, \ldots, m_{k_a} \\ p &\equiv -1 \bmod m_{k_a+1}, \ldots, m_{k_b} \end{aligned}$$

How small can p be?

$+1$ congruences imply $p \geq m_{k_1} \cdots m_{k_a} + 1$.

-1 congruences imply $p \geq m_{k_{a+1}} \cdots m_{k_b} - 1$.

Since the product of these two lower bounds is greater than $\prod_{i=1}^{b}(m_{k_i} - 1)$, at least one must be greater than $\left(\prod_{i=1}^{b}(m_{k_i} - 1)\right)^{\frac{1}{2}}$.

There are 3^r pairs, yielding

$$\text{Second Sum} = \sum_{s=0}^{r} \sum_{k(s)} \sum_{j(s)} S_{2,k(s),j(s)} \ll 9^r m^{-\frac{1}{2}}.$$

APPENDIX II:
Random Graphs

Random Graphs (with Peter Sarnak, Yakov Sinai)

- Dustin Steinhauer
- Randy Qian
- Felice Kuan
- Leo Goldmakher

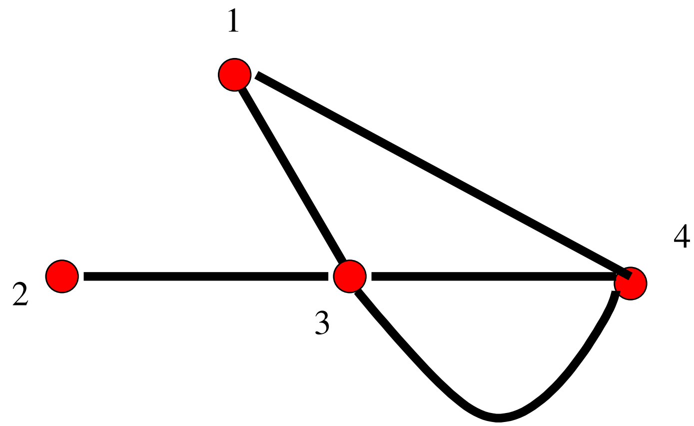
Fat Thin Families

Need a family ***F A T*** enough to do averaging.

Need a family ***THIN*** enough so that everything isn't averaged out.

Real Symmetric Matrices have $\frac{N(N+1)}{2}$ independent entries.

Random Graphs



Degree of a vertex = number of edges leaving the vertex.

Adjacency matrix: a_{ij} = number edges between Vertex i and Vertex j.

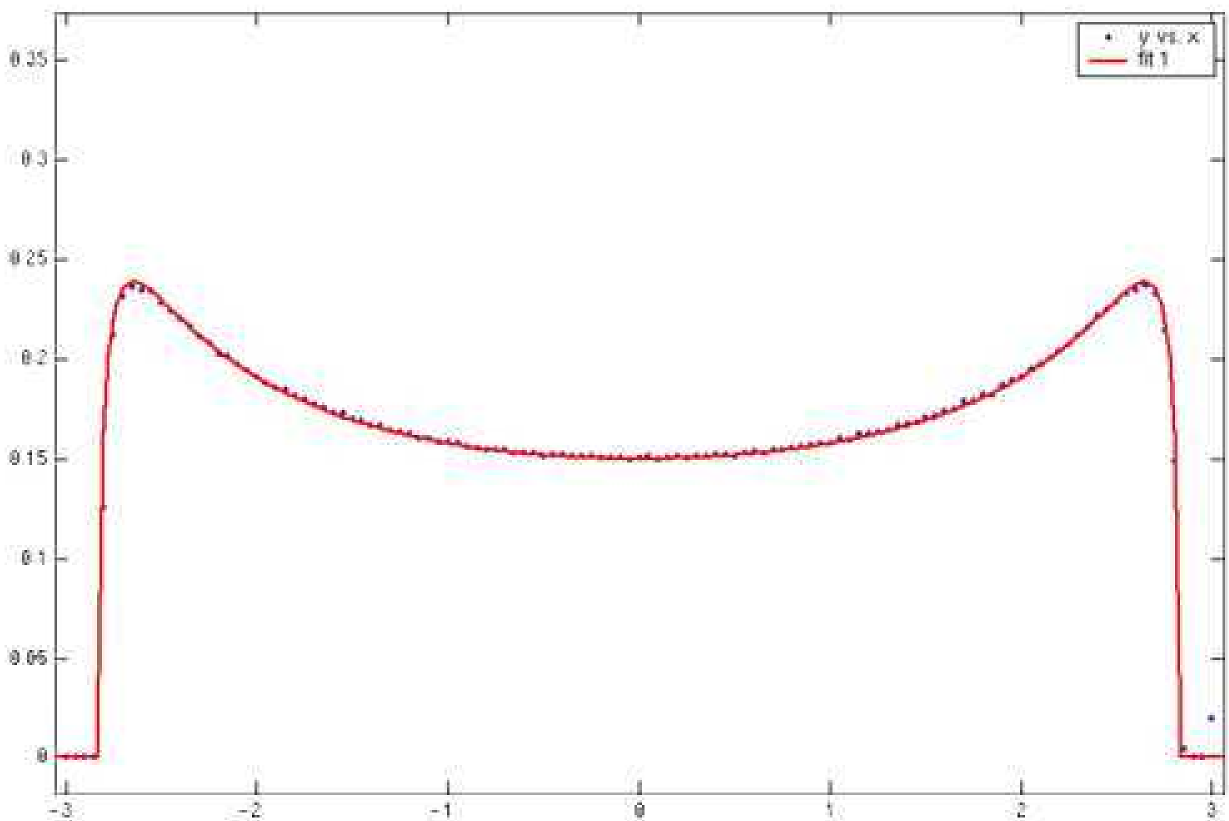
$$A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \end{pmatrix}$$

These are Real Symmetric Matrices.

McKay's Law

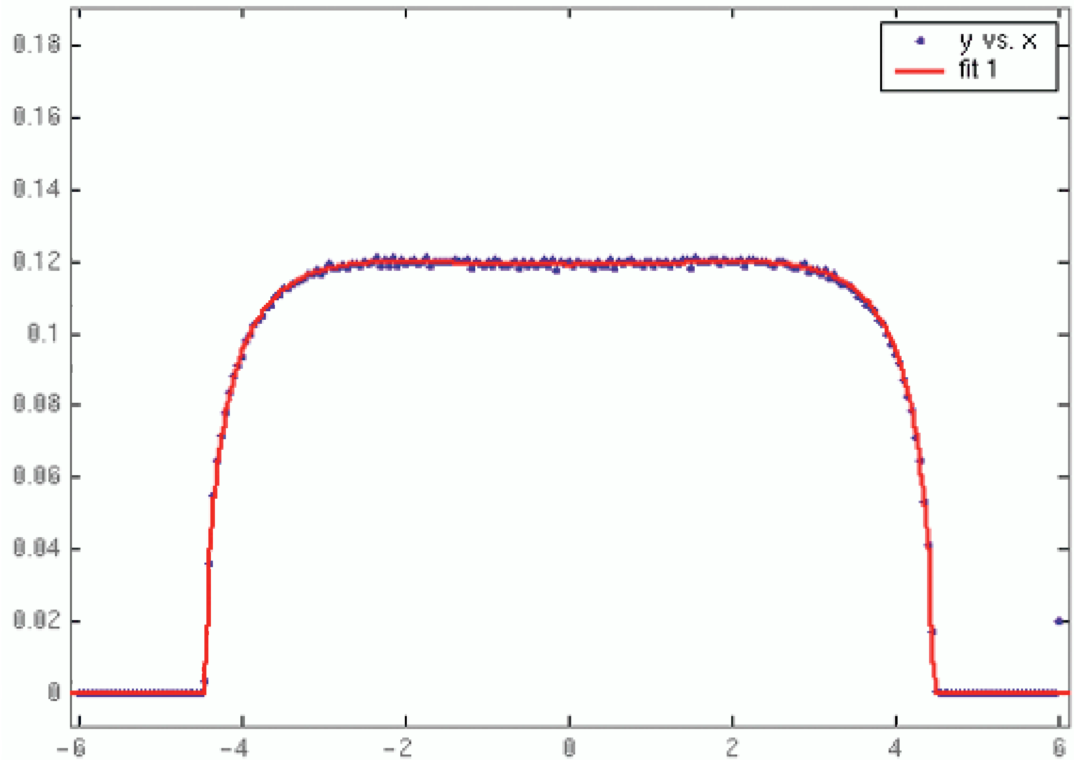
Density of Eigenvalues for d-regular graphs

$$f(x) \;=\; \begin{cases} \frac{d}{2\pi(d^2-x^2)}\sqrt{4(d-1)-x^2} & |x| \le 2\sqrt{d-1} \\ 0 & \text{otherwise.} \end{cases}$$



$$d \;=\; 3.$$

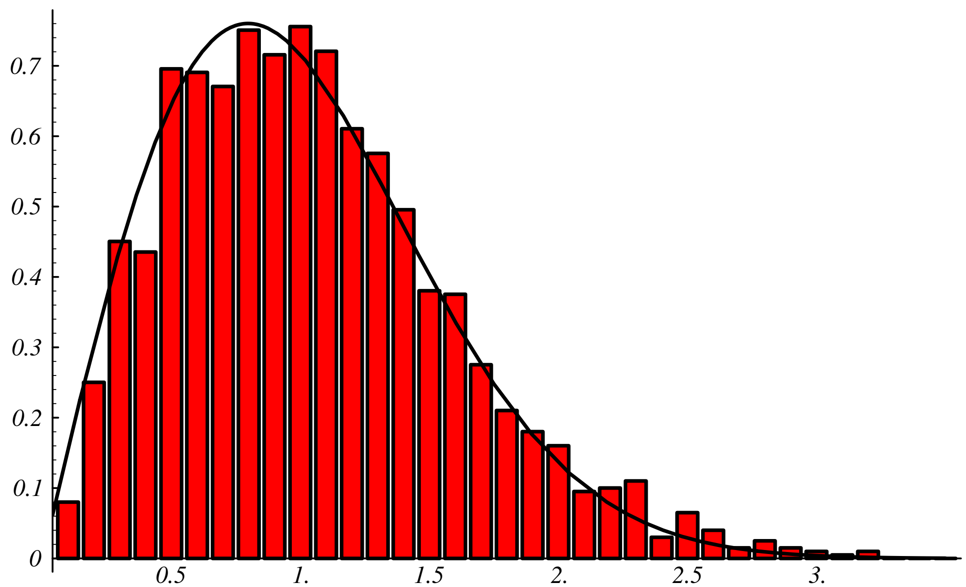
McKay's Law



$$d = 6.$$

Fat Thin: fat enough to average, thin enough to get something different than Semi-circle.

3-Regular, 2000 Vertices and GOE



APPENDIX III:
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