Identifying Symmetry Groups of Low-Lying Zeros of Families of L-Functions

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Measures of Spacings: n-Level Correlations

$\{\alpha_j\}$ an increasing sequence of numbers, $B \subset \mathbb{R}^{n-1}$ a compact box. Define the n-level correlation by

$$\lim_{N\to\infty} \frac{\#\left\{ \left(\alpha_{j_1} - \alpha_{j_2}, \dots, \alpha_{j_{n-1}} - \alpha_{j_n}\right) \in B, j_i \neq j_k \leq N \right\}}{N}$$

Results on Zeros (assuming GRH):

- Normalized spacings of $\zeta(s)$ starting at 10^{20} (Odlyzko)
- Pair and triple correlations of $\zeta(s)$ (Montgomery, Hejhal)
- n-level correlations for all automorphic cupsidal L-fns (Rudnick-Sarnak)
- n-level correlations for the classical compact groups (Katz-Sarnak)
- **insensitive to any finite set of zeros**
- **universality of answer**

Measures of Spacings:
n-Level Density and Families

Let g_i be even Schwartz functions whose Fourier Transform is compactly supported. Let $L(s, f)$ be an L-function with zeros $\frac{1}{2} + i\gamma_f$ ($\gamma_f \in \mathbb{R}$) and conductor C_f. Define the n-level density by

$$D_{n,f}(g) \ = \ \sum_{\substack{j_1,\dots,j_n \\ j_i \neq \pm j_k}} g_1\left(\gamma_{f,j_1}\frac{\log C_f}{2\pi}\right)\cdots g_n\left(\gamma_{f,j_n}\frac{\log C_f}{2\pi}\right)$$

- Individual zeros contribute in limit
- Most of contribution is from low zeros
- Average over similar L-functions (family)

To any geometric family, Katz-Sarnak predict the n-level density depends only on a symmetry group (a classical compact group) attached to the family.

n-Level Density

As $N \to \infty$:

$$\frac{1}{|\mathcal{F}_N|}\sum_{f\in\mathcal{F}_N} D_{n,f}(g) \;=\; \frac{1}{|\mathcal{F}_N|}\sum_{f\in\mathcal{F}_N}\sum_{\substack{j_1,\dots,j_n\\ j_i\neq\pm j_k}}\prod_i g_i\left(\gamma_{f,j_i}\frac{\log C_t}{2\pi}\right)$$

$$\longrightarrow \int\cdots\int g(x)W_{n,\mathcal{G}(\mathcal{F})}(x)dx$$

$$\longrightarrow \int\cdots\int \widehat{g}(u)\widehat{W}_{n,\mathcal{G}(\mathcal{F})}(u)du.$$

Conjecture: Distribution of Zeros near Central Point agrees with Distribution of Eigenvalues near 1 of a Classical Compact Group.

1-Level Densities

Let $\mathcal{G}$ be one of the classical compact groups: Unitary, Symplectic, Orthogonal (or $\mathrm{SO(even)}, \mathrm{SO(odd)}$).

For test functions g with $\mathrm{supp}(\widehat{g}) \subset (-1, 1)$, 1-level density of $\mathcal{G}$ is

$$\widehat{g}(0) \ - \ c_{\mathcal{G}}\,\frac{g(0)}{2},$$

where

$$c_{\mathcal{G}} \ = \ \begin{cases} 0 & \mathcal{G} \text{ is Unitary} \\ 1 & \mathcal{G} \text{ is Symplectic} \\ -1 & \mathcal{G} \text{ is Orthogonal.} \end{cases}$$

Some Results: Simple Families

- **Orthogonal:**

  Iwaniec-Luo-Sarnak: 1-level density for $H_k^{\pm}(N)$, N square-free.

  Miller, Young: families of elliptic curves.

  Güloğlu: 1-level for $\{\mathrm{Sym}^r f : f \in H_k(1)\}$, r odd.

- **Symplectic:**

  Rubinstein: n-level densities for $L(s, \chi_d)$.

  Güloğlu: 1-level for $\{\mathrm{Sym}^r f : f \in H_k(1)\}$, r even.

- **Unitary:**

  Hughes-Rudnick, Miller: Families of Primitive Dirichlet Characters.

Identifying the Symmetry Groups

Often an analysis of the monodromy group in the function field case suggests the answer.

All simple families studied to date are built from GL_1 or GL_2 L-functions.

Tools: Explicit Formula, Orthogonality of Characters / Petersson Formula.

How to identify symmetry group in general? One possibility is by the signs of the functional equation:

Folklore Conjecture: If all signs are even and no corresponding family with odd signs, Symplectic symmetry; otherwise $\mathrm{SO(even)}$.

Some Results: Rankin-Selberg Convolution of Families

Notation:

- π a fixed cuspidal automorphic representation of GL_n over $\mathbb{Q}$;
- ϕ a fixed Hecke-Maass cusp form of level 1;
- $\mathcal{F}_{\chi_d}$: family of primitive quadratic characters;
- $\mathcal{F}_{H_k}$: family of weight k cuspidal newforms of level 1;
- $\mathcal{F}_{\mathrm{sym}^2 H_k}$: family of symmetric squares of above family.
- **Orthogonal:**

  Rubinstein: $\pi \times \mathcal{F}_{\chi_d}$ if $L(s, \mathrm{sym}^2\pi)$ is entire.

  Dueñez-Miller: $\phi \times \mathcal{F}_{\mathrm{sym}^2 H_k}$.
- **Symplectic:**

  Rubinstein: $\pi \times \mathcal{F}_{\chi_d}$ if $L(s, \mathrm{sym}^2\pi)$ has a pole.

  Dueñez-Miller: $\phi \times \mathcal{F}_{H_k}$.

Explicit Formula

- π: cuspidal automorphic representation on GL_n.
- $R_\pi > 0$: analytic conductor of $L(s,\pi)$.
- By GRH the non-trivial zeros are $\frac{1}{2} + \mathrm{i}\gamma_{\pi,j}$.
- Satake parameters $\{\alpha_{\pi,i}(p)\}_{i=1}^n$; $\lambda_\pi(p^\nu) = \sum_{i=1}^n \alpha_{\pi,i}(p)^\nu$.
- $L(s,\pi) = \sum_n \frac{\lambda_\pi(n)}{n^s} = \prod_p \prod_{i=1}^n (1 - \alpha_{\pi,i}(p)p^{-s})^{-1}$.

$$\sum_j g\left(\gamma_{\pi,j}\frac{\log R_\pi}{2\pi}\right) = \widehat{g}(0) - 2\sum_p \sum_{\nu=1}^{\infty} \widehat{g}\left(\nu\frac{\log p}{\log R_\pi}\right)\frac{\lambda_\pi(p^\nu)\log p}{p^{\nu/2}\log R_\pi}$$

g even Schwartz function, $\widehat{g}$ compactly supported.

Usually no contribution from $\nu \geq 3$ (Ramanujan Conjectures).

1-Level Density for a Family

Assuming conductors constant in family $\mathcal{F}$, have to study

$$\lambda_f(p^\nu) \ = \ \alpha_{f,1}(p)^\nu + \cdots + \alpha_{f,n}(p)^\nu$$

$$S_1(\mathcal{F}) \ = \ -2\sum_p \hat{g}\left(\frac{\log p}{\log R}\right)\frac{\log p}{\sqrt{p}\log R}\left[\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\lambda_f(p)\right]$$

$$S_2(\mathcal{F}) \ = \ -2\sum_p \hat{g}\left(2\frac{\log p}{\log R}\right)\frac{\log p}{p\log R}\left[\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\lambda_f(p^2)\right]$$

- Except for families of elliptic curves with rank, first sum zero in all known cases (rank zero families).

- The corresponding classical compact group is determined by

$$\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\lambda_f(p^2) \ = \ c_{\mathcal{F}} \ = \ \begin{cases} 0 & \text{Unitary} \\ 1 & \text{Symplectic} \\ -1 & \text{Orthogonal.} \end{cases}$$

1-Level Density for Rankin-Selberg Convolution of Families

Families $\mathcal{F}$ and $\mathcal{G}$.

Satake parameters $\{\alpha_{f,i}(p)\}_{i=1}^{n}$ and $\{\beta_{g,j}(p)\}_{j=1}^{m}$.

Family $\mathcal{F}\times\mathcal{G}$, $L(s, f\times g)$ has parameters $\{\alpha_{f,i}(p)\beta_{g,j}(p)\}_{i=1\dots n, j=1\dots m}$.

$$\begin{aligned} a_{f\times g}(p^\nu) &= \sum_{i=1}^{n}\sum_{j=1}^{m}\alpha_{f,i}(p)^\nu\beta_{g,j}(p)^\nu \\ &= \sum_{i=1}^{n}\alpha_{f,i}(p)^\nu\sum_{j=1}^{m}\beta_{g,j}(p)^\nu \\ &= \lambda_f(p^\nu)\cdot\lambda_g(p^\nu). \end{aligned}$$

Technical restriction: need f and g unrelated (i.e., g is not the contragredient of f) for our applications.

1-Level Density for Rankin-Selberg Convolution of Families (cont)

To analyze $S_\nu(\mathcal{F} \times \mathcal{G})$ we must study

$$\frac{1}{|\mathcal{F} \times \mathcal{G}|} \sum_{f \times g \in \mathcal{F} \times \mathcal{G}} \lambda_f(p^\nu) \cdot \lambda_g(p^\nu) = \left[\frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} \lambda_f(p^\nu)\right] \cdot \left[\frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} \lambda_g(p^\nu)\right]$$

- $\nu = 1$: If one of the families is rank zero, so is $\mathcal{F} \times \mathcal{G}$; $S_1(\mathcal{F} \times \mathcal{G})$ will not contribute.

- $\nu = 2$: $c_{\mathcal{F} \times \mathcal{G}} = c_{\mathcal{F}} \cdot c_{\mathcal{G}}$.

If each family is of rank 0, the symmetry type of the convolution is the product of the symmetry types.

One-Parameter Families of Elliptic Curves over $\mathbb{Q}(T)$

One-parameter family $\mathcal{E} : y^2 \ = \ x^3 + A(T)x + B(T)$ of rank r over $\mathbb{Q}(T)$.

$a_t(p) = \lambda_t(p)\sqrt{p}$ is of size $\sqrt{p}$, $a_{t+mp}(p) = a_t(p)$.

Rational Elliptic Surfaces (Rosen and Silverman): If rank r over $\mathbb{Q}(T)$:

$$\lim_{X\to\infty} \frac{1}{X} \sum_{p\le X} \left(\frac{1}{p} \sum_{t \bmod p} a_t(p) \right) \log p \ = \ -\mathbf{r}$$

Surfaces with $j(T)$ non-constant (Michel):

$$\frac{1}{p} \sum_{t \bmod p} a_t(p^2) \ = \ -p + O\left(\sqrt{p}\right).$$

1-Level Density of One-Parameter Families of Elliptic Curves

Rescale zeros by average log-conductor (for convenience).
First sum contributes

$$\begin{aligned}
S_1(\mathcal{E}) &= -2\sum_p \hat{g}\left(\frac{\log p}{\log R}\right)\frac{\log p}{\sqrt{p}\log R}\frac{1}{N}\sum_{t\in[N,2N]}\lambda_t(p) \\
&\sim -2\sum_p \hat{g}\left(\frac{\log p}{\log R}\right)\frac{\log p}{\sqrt{p}\log R}\frac{1}{N}\frac{N}{p}\sum_{t \bmod p}\frac{a_t(p)}{\sqrt{p}} \\
&\sim -2\sum_p \hat{g}\left(\frac{\log p}{\log R}\right)\frac{\log p}{\sqrt{p}\log R}\frac{\mathbf{-r}}{\sqrt{\mathbf{p}}} \\
&\sim 2r\sum_p \hat{g}\left(\frac{\log p}{\log R}\right)\frac{\log p}{p\log R} \\
&\sim rg(0).
\end{aligned}$$

1-Level Density of One-Parameter Families of Elliptic Curves

Second sum handled similarly:

$$\begin{aligned}
S_2(\mathcal{E}) &= -2\sum_p \hat{g}\left(2\frac{\log p}{\log R}\right)\frac{\log p}{p\log R}\frac{1}{N}\sum_{t\in[N,2N]}\lambda_t(p^2)\\
&\sim -2\sum_p \hat{g}\left(2\frac{\log p}{\log R}\right)\frac{\log p}{p\log R}\frac{1}{N}\frac{N}{p}\sum_{t \bmod p}\frac{a_t(p^2)}{p}\\
&\sim -2\sum_p \hat{g}\left(2\frac{\log p}{\log R}\right)\frac{\log p}{p\log R}\frac{\mathbf{-1}\cdot p^2}{p^2}\\
&\sim 1\cdot 2\sum_p \hat{g}\left(2\frac{\log p}{\log R}\right)\frac{\log p}{p\log R}\\
&\sim 1\cdot\frac{g(0)}{2}.
\end{aligned}$$

1-Level Density of One-Parameter Families of Elliptic Curves

Combining we find that the 1-level density of a one-parameter family of Elliptic Curves of rank r over $\mathbb{Q}(T)$ is

$$\widehat{g}(0) + \frac{g(0)}{2} + rg(0).$$

Agrees with scaling limits of

$$\begin{pmatrix} I_{r\times r} & \\ & \mathrm{SO(even)} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} I_{r\times r} & \\ & \mathrm{SO(odd)} \end{pmatrix}.$$

What is the symmetry type of $\mathcal{E}_{r_1} \times \mathcal{E}_{r_2}$, $\mathcal{E}_{r_i}$ one-parameter family of ECs of rank r_i?

1-Level Density for $\mathcal{E}_{r_1} \times \mathcal{E}_{r_2}$

To evaluate $S_1(\mathcal{E}_{r_1} \times \mathcal{E}_{r_2})$ must study

$$\sum_p \hat{g}\left(\frac{\log p}{\log R}\right) \frac{\log p}{\sqrt{p}\log R} \frac{1}{NM} \sum_{\substack{t_1 \in [N,2N] \\ t_2 \in [M,2M]}} \lambda_{t_1,t_2}(p)$$

$$\sim \sum_p \hat{g}\left(\frac{\log p}{\log R}\right) \frac{\log p}{\sqrt{p}\log R} \left[\frac{1}{N}\frac{N}{p}\sum_{t_1(p)} \frac{a_{t_1}(p)}{\sqrt{p}}\right] \left[\frac{1}{M}\frac{M}{p}\sum_{t_2(p)} \frac{\widetilde{a}_{t_2}(p)}{\sqrt{p}}\right]$$

$$\sim \sum_p \hat{g}\left(\frac{\log p}{\log R}\right) \frac{\log p}{\sqrt{p}\log R} \frac{-\mathbf{r_1}}{\sqrt{\mathbf{p}}} \frac{-\mathbf{r_2}}{\sqrt{\mathbf{p}}}$$

$$\sim O\left(\frac{1}{\log R}\right).$$

Ranks of families play no role in new family.

Similar to Goldfeld's results on quadratic twists of a fixed elliptic curve.

Twisting by a Fixed Form

Consider a GL_2 form $L(s,f)$ with Satake parameters $\alpha, \overline{\alpha}$ of size 1.

$$\lambda_f(p) \;=\; \alpha + \overline{\alpha}, \quad \alpha\overline{\alpha} \;= 1$$

$$\begin{aligned}\lambda_f(p^2) &= \alpha^2 + \overline{\alpha}^2 \\ &= \alpha^2 + 1 + \overline{\alpha}^2 - 1 \\ &= \lambda_{\mathrm{sym}^2 f}(p) \; - \; 1 \\ &= \left[\lambda_f(p)\lambda_f(p) - 1\right] \; - \; 1\end{aligned}$$

Thus

$$\begin{aligned}\lambda_{f\times g}(p^2) &= \lambda_f(p^2)\cdot\lambda_g(p^2) \\ &= \lambda_{\mathrm{sym}^2 f}(p)\lambda_g(p^2) \; - \; \lambda_g(p^2).\end{aligned}$$

The minus sign changes the sign of the contribution from p^2, flips symmetry: Symplectic $\longleftrightarrow$ Orthogonal.

Twisting by a Fixed Form

$\pi = \widetilde{\pi}$ cuspidal automorphic representation on GL_n.

$$\frac{1}{X} \sum_{p \leq X} \lambda_\pi(p^2) \log p \sim \begin{cases} 1 & \text{if } L(s, \mathrm{sym}^2 \pi) \text{ has a pole} \\ -1 & \text{if } L(s, \mathrm{sym}^2 \pi) \text{ is entire.} \end{cases}$$

If χ_d is a quadratic character then for almost all primes, $\chi_d(p^2) = 1$.

Summary

- Theory of Low-Lying Zeros is more than a theory of signs of functional equations.

- Similar to Rudnick-Sarnak, second moment of Satake parameters seems to determine behavior.

- Ranks of families $\mathcal{F}$ and $\mathcal{G}$ do not play a role in the main term of the low-lying zeros of $\mathcal{F} \times \mathcal{G}$ (though potential lower order correction).

- Danger in generalizing from just GL_1 and GL_2 examples; attempts to use Trace Formulas to handle families on GL_n.

Appendix I: Numerical Data on the Independence of the Family Zeros of Elliptic Curves wrt the Remaining Zeros

By the Birch and Swinnerton-Dyer conjecture and the Silverman specialization theorem, in a one-parameter family of elliptic curves of rank r over $\mathbb{Q}(T)$, eventually all curves have rank at least r. The theoretical evidence suggests these family zeros are independent of the remaining zeros. While the numerical evidence is contrary (the family zeros seem to repel the nearby zeros), this is likely an effect of small log-conductors. The convergence is probably no better than $\log N_E$, and in the regions investigated this is at most 25. Nevertheless, unlike the excess rank data (except for Watkins' example), as the log-conductor increases we are able to noticeably see the repulsion decrease.

DEFINITIONS

$$D_{n,\mathcal{F}_N}(\phi) = \frac{1}{|\mathcal{F}_N|} \sum_{E_t \in \mathcal{F}_N} \sum_{\substack{j_1,\dots,j_n \\ j_i \neq \pm j_k}} \prod_i \phi_i \left(\gamma_{t,j_i} \frac{\log C_t}{2\pi} \right)$$

$D^{(r)}_{n,\mathcal{F}_N}(\phi)$: n-level density with contribution of r zeros at central point removed.

$\mathcal{F}_N$: Rational one-parameter family of elliptic curves of rank r over $\mathbb{Q}(T)$, $t \in [N, 2N]$, conductors monotone.

ASSUMPTIONS

1-parameter family of Ell Curves, rank r over $\mathbb{Q}(T)$, rational surface. Assume

- GRH;
- $j(T)$ non-constant;
- Sq-Free Sieve if $\Delta(T)$ has irred. poly. factor of degree ≥ 4.

Pass to positive percent sub-seq where conductors polynomial of degree m.
ϕ_i even Schwartz, support σ_i:

- $\sigma_1 < \min\left(\frac{1}{2}, \frac{2}{3m}\right)$ for 1-level.
- $\sigma_1 + \sigma_2 < \frac{1}{3m}$ for 2-level.

MAIN RESULT

Theorem (M–): Under previous conditions, as $N \to \infty$, $n = 1, 2$:

$$D^{(r)}_{n,\mathcal{F}_N}(\phi) \longrightarrow \int \phi(x) W_{\mathcal{G}}(x) dx,$$

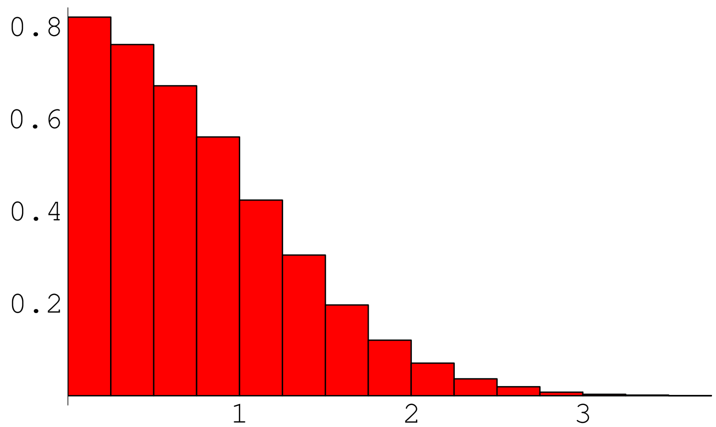
where

$$\mathcal{G} = \begin{cases} \text{SO} & \text{if half odd} \\ \text{SO(even)} & \text{if all even} \\ \text{SO(odd)} & \text{if all odd} \end{cases}$$

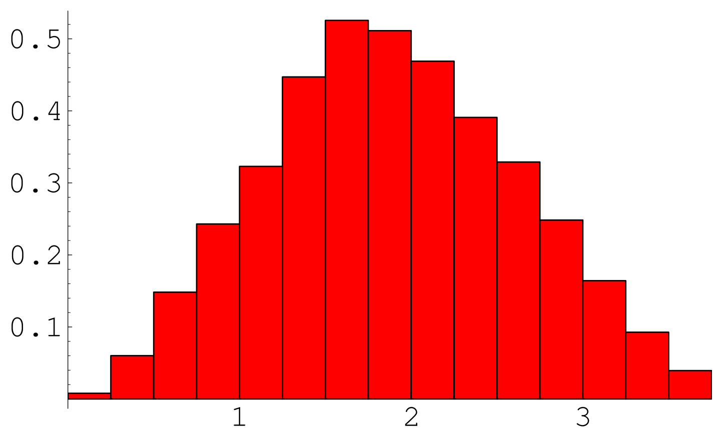
1 and 2-level densities confirm Katz-Sarnak, Birch and Swinnderton-Dyer predictions for small support.

- Agree with Independent Model, note universality;
- Dependence on $\mathcal{F}$ through lower order correction terms.

Theoretical Distribution of First Normalized Zero



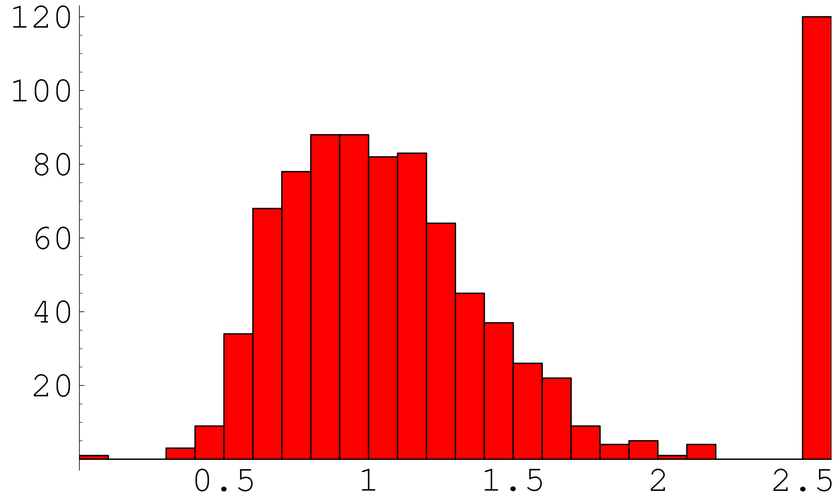
First normalized eigenvalue: 230,400 from $SO(6)$ with Haar Measure



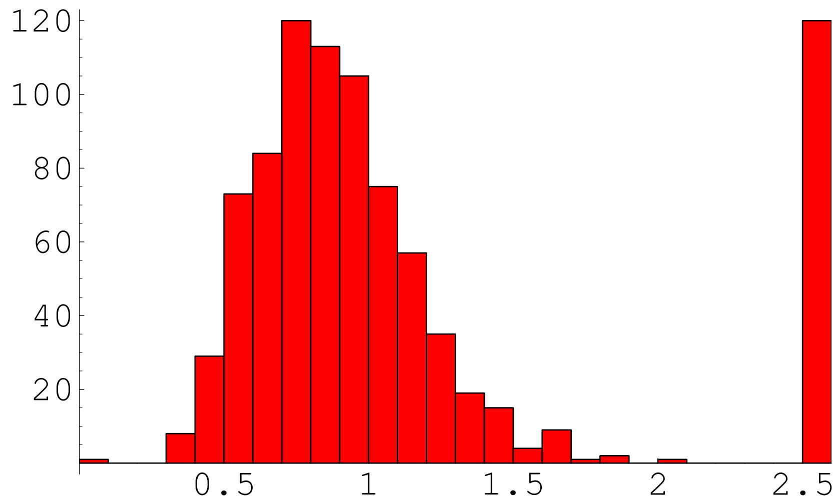
First normalized eigenvalue: 322,560 from $SO(7)$ with Haar Measure

Rank 0 Curves: 1st Normalized Zero

(Far left and right bins just for formatting)

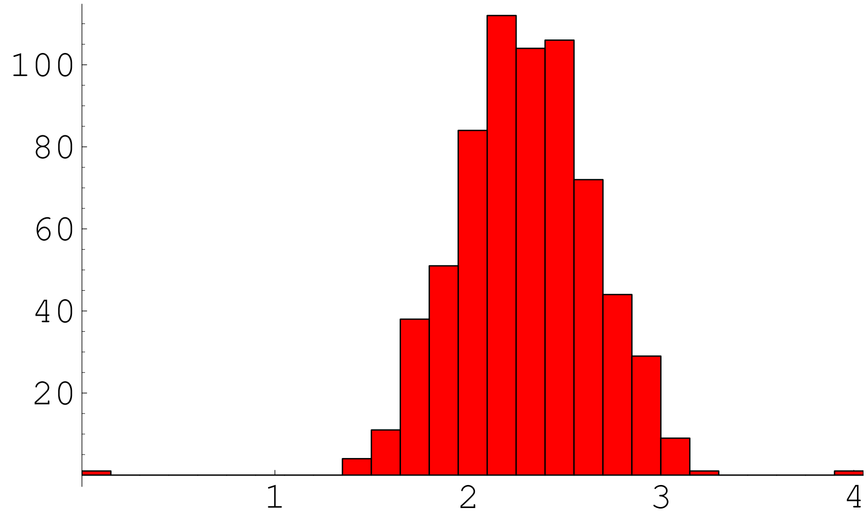


750 **curves**, log(**cond**) $\in$ [3.2, 12.6]; **mean** = 1.04

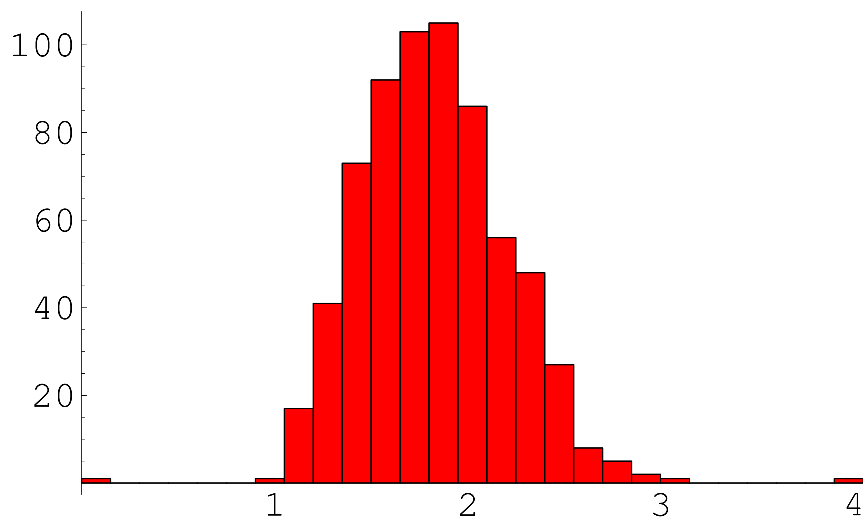


750 **curves**, log(**cond**) $\in$ [12.6, 14.9]; **mean** = .88

Rank 2 Curves: 1st Normalized Zero

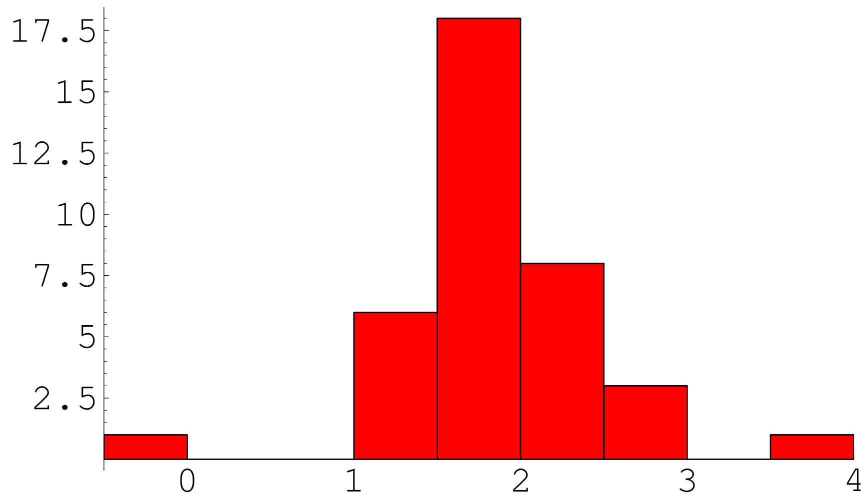


665 **curves**, $\log(\textbf{cond}) \in [10, 10.3125]$; **mean** $= 2.30$

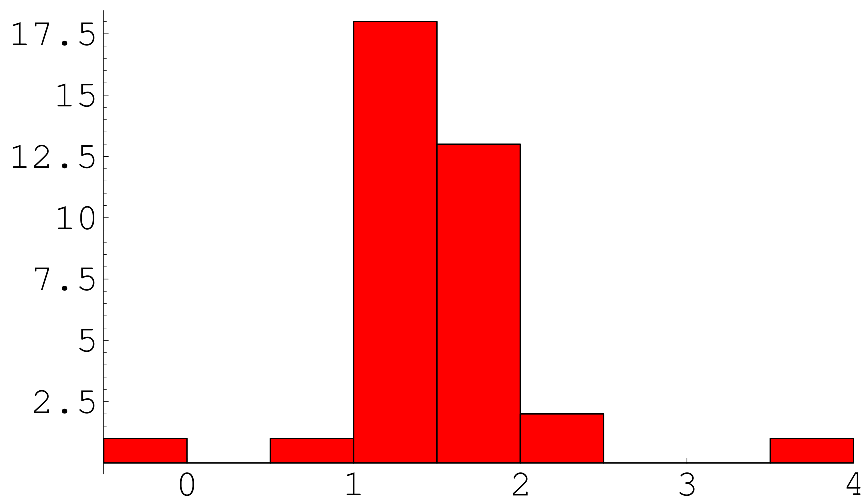


665 **curves**, $\log(\textbf{cond}) \in [16, 16.5]$; **mean** $= 1.82$

Rank 2 Curves: $y^2 = x^3 - T^2x + T^2$: 1st Normalized Zero



35 **curves**, $\log(\textbf{cond}) \in [7.8, 16.1]$; **mean** $= 2.24$



34 **curves**, $\log(\textbf{cond}) \in [16.2, 23.3]$; **mean** $= 2.00$

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