Distribution of Missing Sums in Sumsets

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SMALL Research Program 2011, Williams College

2011 Young Mathematicians Conference
The Ohio State University, August 21, 2011

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Key Question: What is the structure of $A + A$?

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Note: Both theorem can be more naturally stated in terms of missing sums (independent of n).

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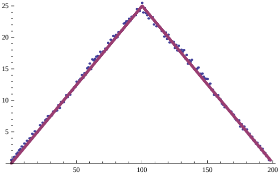

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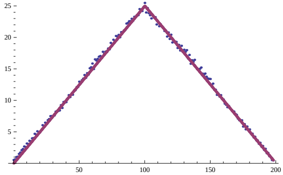

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- Key fact: if $k < n$, then $\mathrm{P}(k \notin A+A) \sim \left(\frac{3}{4}\right)^{k/2}$

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and Log(P($A + A$ has k missing sums)) is eventually linear:

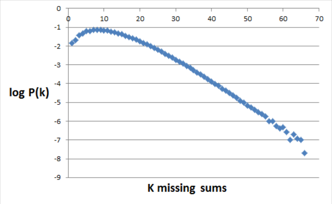

Figure: Log P(k missing sums) for $5 \cdot 10^8$ trial A of size 120

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$$\left(\frac{1}{2}\right)^{(k+i)/2} \ll P(A+A \text{ is missing } [k, k+i]) \ll \left(\frac{1}{2}\right)^{(k+i)/2} (1+\epsilon_i)^k$$

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Bounds on the Distribution

Bound on Distribution: Upper Bound

Weaker Upper bound: $P(A + A \text{ has } k \text{ missing sums}) < 0.93^k$

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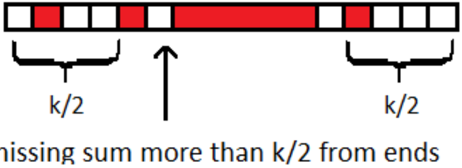

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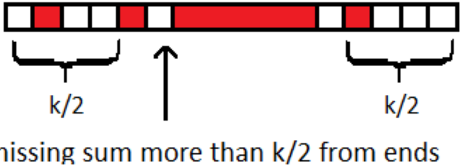

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Note: Since the expectation is around 10 and expansion decay, all the higher moments are bounded as $n \to \infty$.

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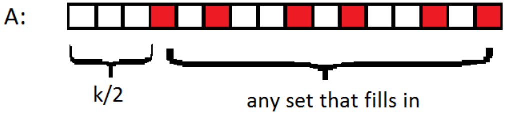

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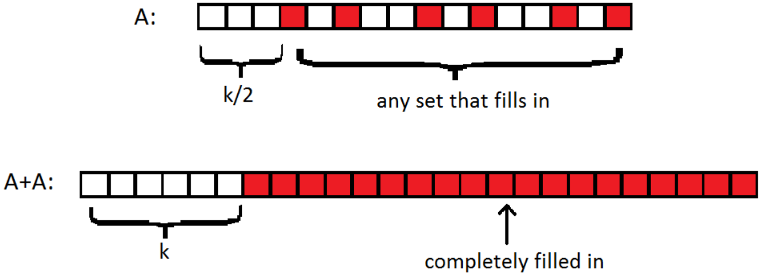

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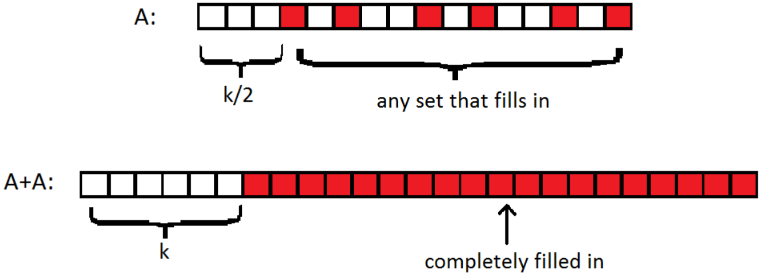

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Therefore just need $\mathrm{E}(|A+A|^2)$:

$$\begin{aligned} \mathrm{E}(|A+A|^2) &= \frac{1}{2^n} \sum_{A \subseteq [0,n-1]} |A+A|^2 \\ &= \frac{1}{2^n} \sum_{A \subseteq [0,n-1]} \sum_{i,j \in A+A} 1 \\ &= \frac{1}{2^n} \sum_{0 \leq i,j \leq 2n-2} \sum_{A : i,j \in A+A} 1 \\ &= \sum_{0 \leq i,j \leq 2n-2} \mathrm{P}(A : i \text{ and } j \in A+A) \end{aligned}$$

Problem: Dependent Random Variables

It is sufficient to study $\mathrm{P}(A : i \text{ and } j \notin A + A)$:

$$\begin{aligned}\mathrm{P}(A : i \text{ and } j \in A + A) &= 1 - \mathrm{P}(A : i \notin A + A) - \mathrm{P}(A : j \notin A + A) \\ &\quad + \mathrm{P}(A : i \text{ and } j \notin A + A).\end{aligned}$$

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- Conditions:

$$i = 3: \quad 0 \text{ or } 3 \notin A \text{ and } 1 \text{ or } 2 \notin A \qquad j = 7: \quad 0 \text{ or } 7 \notin A \text{ and } 1 \text{ or } 6 \notin A \text{ and } 2 \text{ or } 5 \notin A \text{ and } 3 \text{ or } 4 \notin A.$$

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$i = 3$: 0 or $3 \notin A$ and 1 or $2 \notin A$

$j = 7$: 0 or $7 \notin A$ and 1 or $6 \notin A$ and 2 or $5 \notin A$ and 3 or $4 \notin A$.

- Since there are common integers in both lists, the events $3 \notin A+A$ and $7 \notin A+A$ are dependent.

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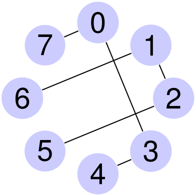

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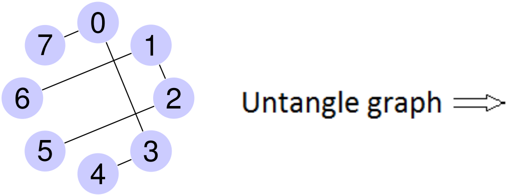

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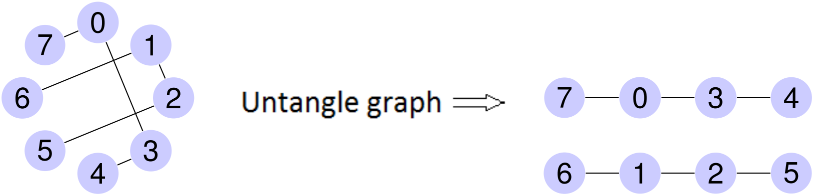

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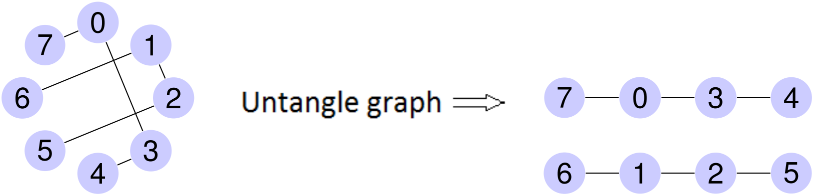

- One-to-one correspondence between conditions/edges (and integers/vertices).

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- Each integer is equally likely so can ignore vertex labels!

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If $7, 0, 4, 6, 2 \notin A$, then $3, 7 \notin A + A$

Lemma(Lazarev, 2011)

$$P(i, j \notin A + A) = P(\text{pick a vertex cover for graph})$$

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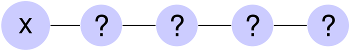

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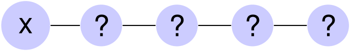

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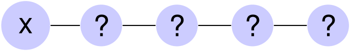

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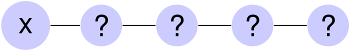

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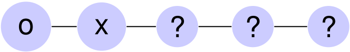

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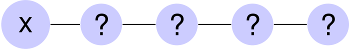

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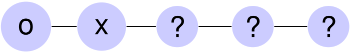

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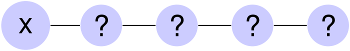

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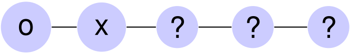

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In particular

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- And to get variance, need to sum up over all $i < j < 2n$....

Variance Formula

$$Var|A+A| = -40 + 4\sum_{i<j<n} P(i,j \notin A+A)$$

$$= -40 + O(c^n)$$

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So clearly

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- General idea: The better we know $P(a_1, \cdots, a_j \notin A + A)$, the better we know the $P(A + A \text{ has } k \text{ missing sums })$

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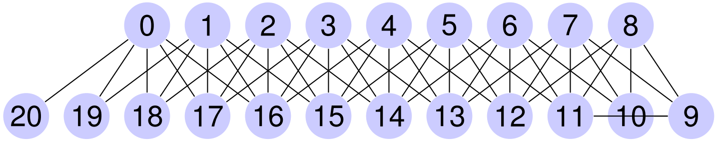

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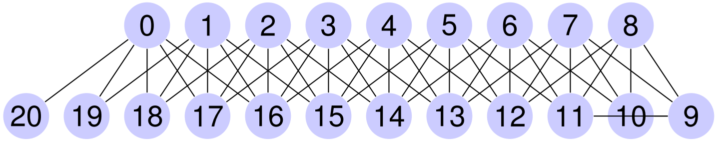

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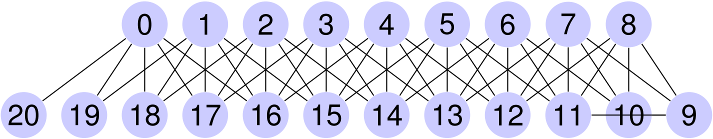

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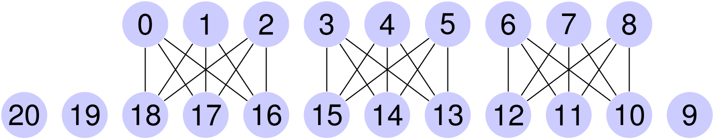

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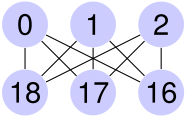

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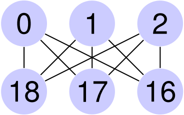

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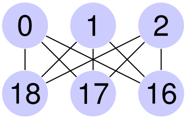

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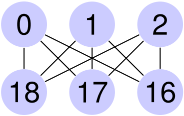

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- In general $P(k, k+1, k+2, k+3, k+4) \leq \left(\frac{1}{4}\right)^{(k+4)/6} \sim 0.79^{k+4}$ which is a slight improvement!

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- Essentially the only way to miss a block of i consecutive sums is to miss all elements before the block as well.

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- Distribution of missing differences in $A - A = \{x - y : x, y \in A\}$

Thank you!