

Generalized More-Sum-Than Difference Sets

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Motivation

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Natural question: What are the sizes of the sum/difference sets?

Definitions

A finite set of integers, $|A|$ its size. Form

- Sumset: $A + A = \{a_i + a_j : a_i, a_j \in A\}$.
- Difference set: $A - A = \{a_i - a_j : a_i, a_j \in A\}$.

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Definition

Difference dominated: $|A - A| > |A + A|$

Balanced: $|A - A| = |A + A|$

Sum dominated (or MSTD): $|A + A| > |A - A|$.

History

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- $x + y = y + x$, but $x - y \neq y - x$.

Nathanson, *Problems in Additive Number Theory*. "With the right way of counting the vast majority of sets satisfy $|A - A| > |A + A|$."

History

Martin-O'Bryant: If each set $A \subseteq [0, n - 1]$ is equally likely, then a positive percentage of sets are sum-dominant in the limit. More precisely:

$$\lim_{n \rightarrow \infty} \frac{\#\{A \subseteq [0, n - 1]; A \text{ is sum-dominant}\}}{2^n} \approx 0.00045.$$

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Martin-O'Bryant: We have the expected values

- $|A + A| \sim 2n - 11,$
- $|A - A| \sim 2n - 7.$

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If x is near n there are many possibilities for a_1, a_2 .

With high probability, the middle will be full.

The trick is to control the fringes.

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We are concerned with adding sets, not multiplying them, therefore we define

$$kA = \underbrace{A + \cdots + A}_{k \text{ times}}.$$

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Furthermore, we are only working with integers, therefore

$$[a, b] = \{a, a + 1, \dots, b\}.$$

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- Can we find a set A such that $|kA + kA| > |kA - kA|$ for all k ?

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- Can we find a set A such that $|kA + kA| > |kA - kA|$?
YES!
- Can we find a set A such that $|A + A| > |A - A|$ and $|2A + 2A| > |2A - 2A|$? **YES!**
- Can we find a set A such that $|kA + kA| > |kA - kA|$ for all k ? **NO!**

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YES!
- Can we find a set A such that $|A + A| > |A - A|$ and $|2A + 2A| > |2A - 2A|$? **YES!**
- Can we find a set A such that $|kA + kA| > |kA - kA|$ for all k ? **NO! (No such set exists.)**

Initial Observations

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If A is symmetric ($A = c - A$ for some c) then

$$|A + A| = |A + (c - A)| = |A - A|.$$

$$|2A + 2A| > |2A - 2A|$$

Example: $|2A + 2A| > |2A - 2A|$

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$$A = \{0, 1, 3, 4, 5, 9\} \cup [33, 56] \cup \{79, 83, 84, 85, 87, 88, 89\}$$

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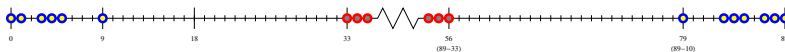
$$|2A + 2A| > |2A - 2A|$$

$$A + A = [0, 9] \cup \{10, 12, 13, 14, 18\} \cup [33, 145] \\ \cup \{158, 162, 163, 164, 166, 167\} \cup [168, 178]$$

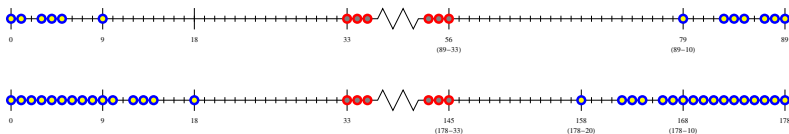


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A

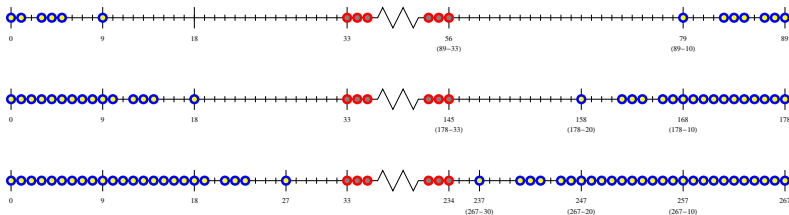


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 $A + A$ 

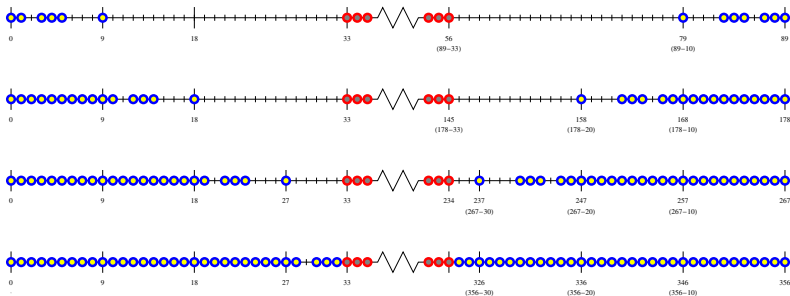
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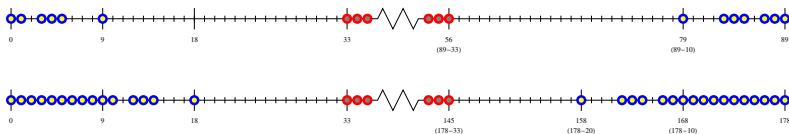


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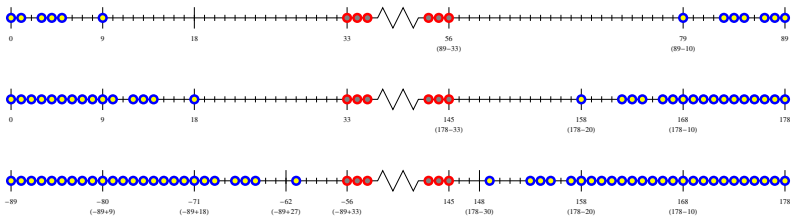


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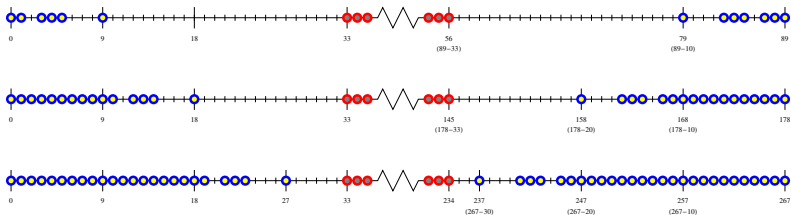
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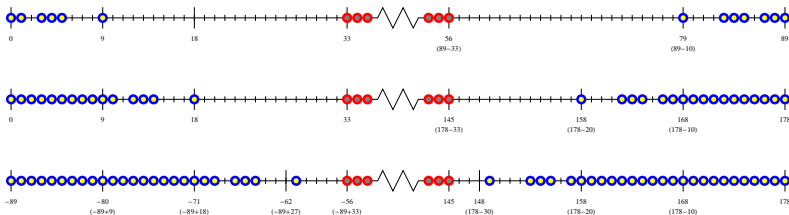
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The left fringe of $A + A$ is $L + L$, the right fringe is $R + R$.

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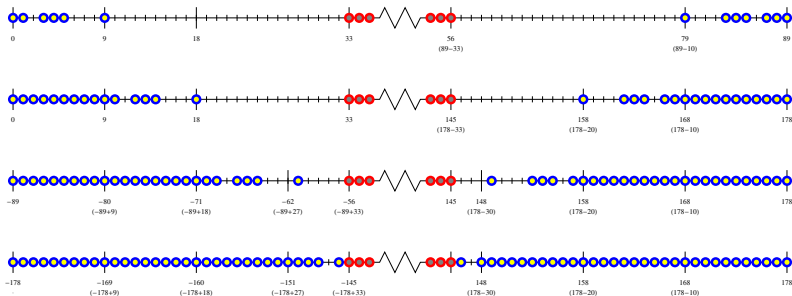
Say that L is the left fringe of A , R the right fringe.

The left fringe of $A + A$ is $L + L$, the right fringe is $R + R$.

The left fringe of $A - A$ is $L - R$, the right fringe is $R - L$.

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$$A + A - A - A$$



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For all nontrivial choices of s_1, d_1, s_2, d_2 , $\exists A \subseteq \mathbb{Z}$ such that $|s_1 A - d_1 A| > |s_2 A - d_2 A|$.

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Example: We can have $|A + A + A + A| > |A + A + A - A|$:

$$A = \{0, 1, 3, 4, 5, 9, 33, 34, 35, 50, 54, 55, 56, 58, 59, 60\}$$

k -Generational Sets

Question: Does a set A exist such that $|A + A| > |A - A|$ and $|A + A + A + A| > |A + A - A - A|$? If yes call it 2-generational.

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Equivalently, $A, 2A$ are sum-dominant.

More generally, A is k -generational if $|cA + cA| > |cA - cA|$ for all $1 \leq c \leq k$.

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Yes!

$$A = \{0, 1, 3, 4, 7, 26, 27, 29, 30, 33, 37, 38, 40, 41, 42, 43, 46, 49, 50, 52, 53, 54, 72, 75, 76, 78, 79, 80\}$$

In fact, we can find a k -generational set for all k .

k -Generational Sets

Idea of proof: We can find A_j such that
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Combine the A_j using the method of base expansion.

Base Expansion

Example: $A_1 = \{0, 1\}$, $A_2 = \{0, 2, 3, 4, 7, 11, 12, 14\}$

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Figure: A_2 and $40 \cdot A_1 + A_2$

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$$A_1 = \{0, 1\}, A_2 = \{0, 2, 3, 4, 7, 11, 12, 14\}$$



Figure: $A_2 + A_2$ and $(40 \cdot A_1 + A_2) + (40 \cdot A_1 + A_2)$

Base Expansion

$$A_1 = \{0, 1\}, A_2 = \{0, 2, 3, 4, 7, 11, 12, 14\}$$



Figure: $A_2 - A_2$ and $(40 \cdot A_1 + A_2) - (40 \cdot A_1 + A_2)$

Base Expansion

Base Expansion: For sets A_1, A_2 and $m \in \mathbb{N}$ sufficiently large (relative to A_1, A_2) the set

$$A = m \cdot A_1 + A_2$$

behaves like the direct product $A_1 \times A_2 \subseteq \mathbb{Z} \times \mathbb{Z}$.

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In particular:

$$|xA - yA| = |xA_1 - yA_1| \cdot |xA_2 - yA_2|$$

whenever $x + y$ is small relative to m .

Generalization

For nontrivial x_j, y_j, w_j, z_j ($2 \leq j \leq k$), we can find an A such that $|x_j A - y_j A| > |w_j A - z_j A|$ for all j .

Generalization

For nontrivial x_j, y_j, w_j, z_j ($2 \leq j \leq k$), we can find an A such that $|x_j A - y_j A| > |w_j A - z_j A|$ for all j .

Example: We can find an A such that

$$|A + A| > |A - A|$$

$$|A + A - A| > |A + A + A|$$

$$|5A - 2A| > |A - 6A|$$

$$\vdots$$

$$|1870A - 141A| > |1817A - 194A|$$

Limiting behavior of kA

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It turns out that all sets have a sort of limiting behavior.

Stabilizing Fringes

Example: $A = \{0, 3, 5, 6, 8, 9, 10, 11, 12, 15, 16, 20\}$

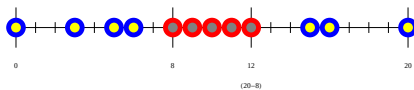


Figure: A

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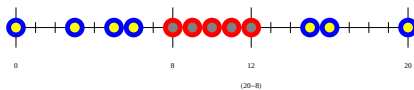


Figure: A



Figure: $A + A$

$|kA - kA|$ vs. $|kA + kA|$

Nathanson: For any set A , as k goes to infinity kA eventually becomes stabilized before k reaches $\max(A)^2 \cdot |A|$.

$|kA - kA|$ vs. $|kA + kA|$

Theorem

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$|kA - kA|$ vs. $|kA + kA|$

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For any set A , as k goes to infinity kA eventually becomes stabilized before k reaches $\max(A)$. Furthermore kA will become difference-dominated or balanced k reaches $2 \cdot \max(A)$.

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Proof Idea:

- The middle will quickly become full, and the remaining fringes are finite.

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- The middle will quickly become full, and the remaining fringes are finite.
- $kA \subseteq kA - kA$. Any sum can eventually be written as a difference.

$$|kA - kA| \text{ vs. } |kA + kA|$$

Theorem

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Proof Idea:

- The middle will quickly become full, and the remaining fringes are finite.
- $kA \subseteq kA - kA$. Any sum can eventually be written as a difference.

Because the form stabilizes, this means $kA - kA \supseteq kA + kA$ when k large.

Other Results

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More generally, any order and number of (nontrivial) comparisons.

Thanks

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