

n -Level Densities of Zeroes of Quadratic Dirichlet L -Functions

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The 4-level Density of Zeroes of Quadratic Dirichlet L -functions

$$\begin{aligned}
& \lim_{X \rightarrow \infty} \frac{\pi^2}{4X} \sum_{d \in D(X)} \left(\prod_{i=1}^4 \left(\int_{\mathbb{R}} f_i(x) dx - \frac{1}{2} \int_{\mathbb{R}} \hat{f}_i(u) du - \frac{2}{\log X} \sum_p \frac{\log p}{\sqrt{p}} \left(\frac{8d}{p} \right) \hat{f}_i \left(\frac{\log p}{\log X} \right) \right) \right. \\
& - 2 \sum_{\{\{a\}, \{b\}, \{c, d\}\}}^6 \prod_{a, b} \left(\int_{\mathbb{R}} f_i(x) dx - \frac{1}{2} \int_{\mathbb{R}} \hat{f}_i(u) du - \frac{2}{\log X} \sum_p \frac{\log p}{\sqrt{p}} \left(\frac{8d}{p} \right) \hat{f}_i \left(\frac{\log p}{\log X} \right) \right) \cdot \\
& \quad \cdot \left(\int_{\mathbb{R}} f_c(x) f_d(x) dx - \frac{1}{2} \int_{\mathbb{R}} \widehat{f_c f_d}(u) du - \frac{2}{\log X} \sum_p \frac{\log p}{\sqrt{p}} \left(\frac{8d}{p} \right) \widehat{f_c f_d} \left(\frac{\log p}{\log X} \right) \right) \\
& + 4 \sum_{\{\{a, b\}, \{c, d\}\}}^3 \prod^2 \left(\int_{\mathbb{R}} f_i(x) f_j(x) dx - \frac{1}{2} \int_{\mathbb{R}} \widehat{f_i f_j}(u) du - \frac{2}{\log X} \sum_p \frac{\log p}{\sqrt{p}} \left(\frac{8d}{p} \right) \widehat{f_i f_j} \left(\frac{\log p}{\log X} \right) \right) \\
& + 8 \sum_{\{\{a\}, \{b, c, d\}\}}^4 \left(\int_{\mathbb{R}} f_a(x) dx - \frac{1}{2} \int_{\mathbb{R}} \hat{f}_a(u) du - \frac{2}{\log X} \sum_p \frac{\log p}{\sqrt{p}} \left(\frac{8d}{p} \right) \hat{f}_a \left(\frac{\log p}{\log X} \right) \right) \cdot \\
& \quad \cdot \left(\int_{\mathbb{R}} f_b(x) f_c(x) f_d(x) dx - \frac{1}{2} \int_{\mathbb{R}} \widehat{f_b f_c f_d}(u) du - \frac{2}{\log X} \sum_p \frac{\log p}{\sqrt{p}} \left(\frac{8d}{p} \right) \widehat{f_b f_c f_d} \left(\frac{\log p}{\log X} \right) \right) \\
& - 48 \left(\int_{\mathbb{R}} f_1 f_2 f_3 f_4(x) dx - \frac{1}{2} \int_{\mathbb{R}} \widehat{f_1 f_2 f_3 f_4}(u) du - \frac{2}{\log X} \sum_p \frac{\log p}{\sqrt{p}} \left(\frac{8d}{p} \right) \widehat{f_1 f_2 f_3 f_4} \left(\frac{\log p}{\log X} \right) \right) \Big)
\end{aligned}$$

$$=$$

The 4-Level Density of Eigenvalues of the Gaussian Symplectic Ensemble

$$\begin{aligned}
& \prod_{i=1}^4 \left(\int_{\mathbb{R}} f_i(x) dx - \frac{1}{2} \int_{\mathbb{R}} \hat{f}_i(u) du + \frac{1}{2} \int_{\mathbb{R}} (1 - \chi(u)) \hat{f}_i(u) du \right) - 2 \sum_{\{\{a\}, \{b\}, \{c, d\}\}}^6 \prod_{a, b} \left(\int_{\mathbb{R}} f_i(x) dx - \frac{1}{2} \int_{\mathbb{R}} \hat{f}_i(u) du + \frac{1}{2} \int_{\mathbb{R}} (1 - \chi(u)) \right. \\
& \cdot \left(\int_{\mathbb{R}} f_c(x) f_d(x) dx - \frac{1}{2} \int_{\mathbb{R}} \widehat{f_c f_d}(u) du + \frac{1}{2} \int_{\mathbb{R}} (1 - \chi(u)) \widehat{f_c f_d}(u) du + \frac{1}{2} \int_{\mathbb{R}^2} (1 - \chi(u_c + u_d)) \chi(u_c - u_d) \right) \hat{f}_c(u_c) \hat{f}_d(u_d) du_c du_d \Big) \\
& + 4 \sum_{\{\{a, b\}, \{c, d\}\}}^3 \prod_{a, b} \left(\int_{\mathbb{R}} f_i(x) f_j(x) dx - \frac{1}{2} \int_{\mathbb{R}} \widehat{f_i f_j}(u) du - \int_{\mathbb{R}} |u| \hat{f}_i(u) \hat{f}_j(u) du - \frac{1}{2} \int_{\mathbb{R}^2} (1 - \chi(u_i + u_j)) \chi(u_i - u_j) \right) \hat{f}_i(u_i) \hat{f}_j(u_j) du_i du_j \\
& + 4 \sum_{\{\{a\}, \{b, c, d\}\}}^4 \left(\int_{\mathbb{R}} f_a(x) dx - \frac{1}{2} \int_{\mathbb{R}} \hat{f}_a(u) du + \frac{1}{2} \int_{\mathbb{R}} (1 - \chi(u)) \hat{f}_a(u) du \right) \cdot \left(2 \int_{\mathbb{R}} f_b(x) f_c(x) f_d(x) dx - \int_{\mathbb{R}} \widehat{f_b f_c f_d}(u) du \right. \\
& - \sum_{\{i, j\} \subset \{b, c, d\}}^3 \int_{\mathbb{R}} |u| \widehat{f_i f_j}(u) \hat{f}_k(u) du + \frac{1}{2} \int_{\mathbb{R}^3} (2 - \chi(u_b + u_c + u_d)) \chi(u_b - u_c + u_d) \chi(u_b - u_c - u_d) \\
& - \chi(u_b + u_c + u_d) \chi(u_b + u_c - u_d) \chi(u_b - u_c - u_d) \Big) \prod_{b, c, d} \hat{f}_i(u_i) du_i - 8 \left(6 \int_{\mathbb{R}} f_1 f_2 f_3 f_4(x) dx - 3 \int_{\mathbb{R}} \widehat{f_1 f_2 f_3 f_4}(u) du \right. \\
& - \sum_{\{a, b\}, \{c, d\}}^3 \int_{\mathbb{R}} |u| \hat{f}_a(u) \hat{f}_b(u) du - 2 \sum_{a=1}^4 \int_{\mathbb{R}} |u| \hat{f}_a(u) \widehat{f_b f_c f_d}(u) du + \frac{1}{2} \int_{\mathbb{R}^4} (6 - \chi_4^*(u_1, u_2, u_3, u_4)) \prod_{i=1}^4 \hat{f}_i(u_i) du_i \Big).
\end{aligned}$$

4-level Density of Zeroes of Quadratic Dirichlet L -functions

$$\begin{aligned}
& \frac{1}{16} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + 2 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1 \hat{f}_2(u) \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + 2 \left(\int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_1 \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_2 \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + 2 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_2 \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& 2 \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_2 \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \frac{1}{8} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_0^{\infty} \hat{f}_1(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \left(\int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_3(u) du \right) \left(\int_0^{\infty} \hat{f}_1(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_2 \hat{f}_3(u) du \right) \left(\int_0^{\infty} \hat{f}_1(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \frac{1}{8} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_0^{\infty} \hat{f}_2(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_3(u) du \right) \left(\int_0^{\infty} \hat{f}_2(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_3(u) du \right) \left(\int_0^{\infty} \hat{f}_2(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_0^{\infty} \hat{f}_1 \hat{f}_2(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \frac{1}{8} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_0^{\infty} \hat{f}_3(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_2(u) du \right) \left(\int_0^{\infty} \hat{f}_3(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_2(u) du \right) \left(\int_0^{\infty} \hat{f}_3(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_0^{\infty} \hat{f}_1 \hat{f}_3(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_0^{\infty} \hat{f}_2 \hat{f}_3(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - 4 \left(\int_0^{\infty} \hat{f}_1 \hat{f}_2 \hat{f}_3(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du \equiv \triangleright \quad \equiv \quad \curvearrowright \quad \curvearrowleft \quad \curvearrowright \quad \curvearrowleft
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{8} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \left(\int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_2 \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_0^{\infty} \hat{f}_2(u_1) \tilde{\chi}(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_0^{\infty} \hat{f}_3(u_1) \tilde{\chi}(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \left(\int_0^{\infty} \hat{f}_2 \hat{f}_3(u_1) \tilde{\chi}(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \frac{1}{8} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} f_2(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} f_2(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} f_2(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_0^{\infty} \hat{f}_1(u_1) \tilde{\chi}(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} f_2(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_0^{\infty} \hat{f}_3(u_1) \tilde{\chi}(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} f_2(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \left(\int_0^{\infty} \hat{f}_1 \hat{f}_3(u_1) \tilde{\chi}(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} f_2(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \left(\int_{-\infty}^{\infty} f_2(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \frac{1}{2} \left(\int_0^{\infty} \hat{f}_3(u_1) \tilde{\chi}(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \left(\int_{-\infty}^{\infty} f_2(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} (f_1 f_2)(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \left(\int_0^{\infty} \hat{f}_3(u_1) \tilde{\chi}(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} (f_1 f_2)(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \frac{1}{8} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_0^{\infty} \hat{f}_1(u_1) \tilde{\chi}(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_0^{\infty} \hat{f}_2(u_1) \tilde{\chi}(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \left(\int_0^{\infty} \hat{f}_1 \hat{f}_2(u_1) \tilde{\chi}(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \frac{1}{2} \left(\int_0^{\infty} \hat{f}_2(u_1) \tilde{\chi}(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(x) dx \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \frac{1}{2} \left(\int_0^{\infty} \hat{f}_1(u_1) \tilde{\chi}(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(x) dx \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(x) dx \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \left(\int_{-\infty}^{\infty} (f_1 f_2)(x) dx \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} (f_1 f_3)(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \left(\int_0^{\infty} \hat{f}_2(u_1) \tilde{\chi}(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} (f_1 f_3)(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \left(\int_{-\infty}^{\infty} \hat{f}_2(x) dx \right) \left(\int_{-\infty}^{\infty} (f_1 f_3)(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} (f_2 f_3)(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \left(\int_0^{\infty} \hat{f}_1(u_1) \tilde{\chi}(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} (f_2 f_3)(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \left(\int_{-\infty}^{\infty} (f_2 f_3)(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - 4 \left(\int_{-\infty}^{\infty} (f_1 f_2 f_3)(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_0^{\infty} \int_0^{\infty} \hat{f}_1(u_1) \hat{f}_2(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \left(\int_0^{\infty} \int_0^{\infty} \hat{f}_1(u_1) \hat{f}_2(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_0^{\infty} \int_0^{\infty} \hat{f}_1(u_1) \hat{f}_3(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \left(\int_{-\infty}^{\infty} f_2(x) dx \right) \left(\int_0^{\infty} \int_0^{\infty} \hat{f}_1(u_1) \hat{f}_3(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_0^{\infty} \int_0^{\infty} \hat{f}_2(u_1) \hat{f}_3(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \left(\int_0^{\infty} \int_0^{\infty} \hat{f}_2(u_1) \hat{f}_3(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& 2 \left(\int_0^{\infty} \int_0^{\infty} \hat{f}_1(u_1) \hat{f}_2(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& 2 \left(\int_0^{\infty} \int_0^{\infty} \hat{f}_2(u_1) \hat{f}_1(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& 2 \left(\int_0^{\infty} \int_0^{\infty} \hat{f}_1(u_1) \hat{f}_2(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du
\end{aligned}$$

$$\begin{aligned}
& -2 \left(\int_0^\infty \int_0^\infty \int_0^\infty \hat{f}_1(u_1) \hat{f}_2(u_2) \hat{f}_3(u_3) (\tilde{\chi}(u_1 - u_2 - u_3) + \tilde{\chi}(-u_1 + u_2 - u_3) - \tilde{\chi}(u_1 + u_2 - u_3) + \tilde{\chi}(-u_1 - u_2 + u_3) - \tilde{\chi}(u_1 - \right. \\
& \left. \frac{1}{2} \left(\int_{-\infty}^\infty \hat{f}_2(u) du \right) \left(\int_{-\infty}^\infty \hat{f}_3(u) du \right) \int_{-\infty}^\infty |u| \hat{f}_1(u) \hat{f}_4(u) du + 4 \left(\int_{-\infty}^\infty |u| \hat{f}_2(u) \hat{f}_3(u) du \right) \int_{-\infty}^\infty |u| \hat{f}_1(u) \hat{f}_4(u) du + \right. \\
& 2 \left(\int_{-\infty}^\infty \hat{f}_2 \hat{f}_3(u) du \right) \int_{-\infty}^\infty |u| \hat{f}_1(u) \hat{f}_4(u) du + \frac{1}{2} \left(\int_{-\infty}^\infty \hat{f}_1(u) du \right) \left(\int_{-\infty}^\infty \hat{f}_3(u) du \right) \int_{-\infty}^\infty |u| \hat{f}_2(u) \hat{f}_4(u) du + \\
& 4 \left(\int_{-\infty}^\infty |u| \hat{f}_1(u) \hat{f}_3(u) du \right) \int_{-\infty}^\infty |u| \hat{f}_2(u) \hat{f}_4(u) du + 2 \left(\int_{-\infty}^\infty \hat{f}_1 \hat{f}_3(u) du \right) \int_{-\infty}^\infty |u| \hat{f}_2(u) \hat{f}_4(u) du + \\
& 2 \left(\int_{-\infty}^\infty \hat{f}_3(u) du \right) \int_{-\infty}^\infty |u| \hat{f}_1 \hat{f}_2(u) \hat{f}_4(u) du + \frac{1}{2} \left(\int_{-\infty}^\infty \hat{f}_1(u) du \right) \left(\int_{-\infty}^\infty \hat{f}_2(u) du \right) \int_{-\infty}^\infty |u| \hat{f}_3(u) \hat{f}_4(u) du + \\
& 4 \left(\int_{-\infty}^\infty |u| \hat{f}_1(u) \hat{f}_2(u) du \right) \int_{-\infty}^\infty |u| \hat{f}_3(u) \hat{f}_4(u) du + 2 \left(\int_{-\infty}^\infty \hat{f}_1 \hat{f}_2(u) du \right) \int_{-\infty}^\infty |u| \hat{f}_3(u) \hat{f}_4(u) du + \\
& 2 \left(\int_{-\infty}^\infty \hat{f}_2(u) du \right) \int_{-\infty}^\infty |u| \hat{f}_1 \hat{f}_3(u) \hat{f}_4(u) du + 2 \left(\int_{-\infty}^\infty \hat{f}_1(u) du \right) \int_{-\infty}^\infty |u| \hat{f}_2 \hat{f}_3(u) \hat{f}_4(u) du + \\
& 16 \int_{-\infty}^\infty |u| \hat{f}_1 \hat{f}_2 \hat{f}_3(u) \hat{f}_4(u) du + \frac{1}{4} \left(\int_{-\infty}^\infty \hat{f}_2(u) du \right) \left(\int_{-\infty}^\infty \hat{f}_3(u) du \right) \int_{-\infty}^\infty \hat{f}_1 \hat{f}_4(u) du + \\
& 2 \left(\int_{-\infty}^\infty |u| \hat{f}_2(u) \hat{f}_3(u) du \right) \int_{-\infty}^\infty \hat{f}_1 \hat{f}_4(u) du + \left(\int_{-\infty}^\infty \hat{f}_2 \hat{f}_3(u) du \right) \int_{-\infty}^\infty \hat{f}_1 \hat{f}_4(u) du + \\
& 2 \left(\int_{-\infty}^\infty \hat{f}_3(u) du \right) \int_{-\infty}^\infty |u| \hat{f}_2(u) \hat{f}_1 \hat{f}_4(u) du + 2 \left(\int_{-\infty}^\infty \hat{f}_2(u) du \right) \int_{-\infty}^\infty |u| \hat{f}_3(u) \hat{f}_1 \hat{f}_4(u) du + 8 \int_{-\infty}^\infty |u| \hat{f}_2 \hat{f}_3(u) \hat{f}_1 \hat{f}_4(u) du + \\
& \frac{1}{4} \left(\int_{-\infty}^\infty \hat{f}_1(u) du \right) \left(\int_{-\infty}^\infty \hat{f}_3(u) du \right) \int_{-\infty}^\infty \hat{f}_2 \hat{f}_4(u) du + 2 \left(\int_{-\infty}^\infty |u| \hat{f}_1(u) \hat{f}_3(u) du \right) \int_{-\infty}^\infty \hat{f}_2 \hat{f}_4(u) du + \\
& \left(\int_{-\infty}^\infty \hat{f}_1 \hat{f}_3(u) du \right) \int_{-\infty}^\infty \hat{f}_2 \hat{f}_4(u) du + 2 \left(\int_{-\infty}^\infty \hat{f}_3(u) du \right) \int_{-\infty}^\infty |u| \hat{f}_1(u) \hat{f}_2 \hat{f}_4(u) du + \\
& 2 \left(\int_{-\infty}^\infty \hat{f}_1(u) du \right) \int_{-\infty}^\infty |u| \hat{f}_3(u) \hat{f}_2 \hat{f}_4(u) du + 8 \int_{-\infty}^\infty |u| \hat{f}_1 \hat{f}_3(u) \hat{f}_2 \hat{f}_4(u) du + 2 \left(\int_{-\infty}^\infty \hat{f}_3(u) du \right) \int_{-\infty}^\infty \hat{f}_1 \hat{f}_2 \hat{f}_4(u) du + \\
& 16 \int_{-\infty}^\infty |u| \hat{f}_3(u) \hat{f}_1 \hat{f}_2 \hat{f}_4(u) du + \frac{1}{4} \left(\int_{-\infty}^\infty \hat{f}_1(u) du \right) \left(\int_{-\infty}^\infty \hat{f}_2(u) du \right) \int_{-\infty}^\infty \hat{f}_3 \hat{f}_4(u) du + \\
& 2 \left(\int_{-\infty}^\infty |u| \hat{f}_1(u) \hat{f}_2(u) du \right) \int_{-\infty}^\infty \hat{f}_3 \hat{f}_4(u) du + \left(\int_{-\infty}^\infty \hat{f}_1 \hat{f}_2(u) du \right) \int_{-\infty}^\infty \hat{f}_3 \hat{f}_4(u) du + \\
& 2 \left(\int_{-\infty}^\infty \hat{f}_2(u) du \right) \int_{-\infty}^\infty |u| \hat{f}_1(u) \hat{f}_3 \hat{f}_4(u) du + 2 \left(\int_{-\infty}^\infty \hat{f}_1(u) du \right) \int_{-\infty}^\infty |u| \hat{f}_2(u) \hat{f}_3 \hat{f}_4(u) du + 8 \int_{-\infty}^\infty |u| \hat{f}_1 \hat{f}_2(u) \hat{f}_3 \hat{f}_4(u) du + \\
& 2 \left(\int_{-\infty}^\infty \hat{f}_2(u) du \right) \int_{-\infty}^\infty \hat{f}_1 \hat{f}_3 \hat{f}_4(u) du + 16 \int_{-\infty}^\infty |u| \hat{f}_2(u) \hat{f}_1 \hat{f}_3 \hat{f}_4(u) du + 2 \left(\int_{-\infty}^\infty \hat{f}_1(u) du \right) \int_{-\infty}^\infty \hat{f}_2 \hat{f}_3 \hat{f}_4(u) du + \\
& 16 \int_{-\infty}^\infty |u| \hat{f}_1(u) \hat{f}_2 \hat{f}_3 \hat{f}_4(u) du + 24 \int_{-\infty}^\infty \hat{f}_1 \hat{f}_2 \hat{f}_3 \hat{f}_4(u) du
\end{aligned}$$

A set of small navigation icons typically found in Beamer presentations, including symbols for back, forward, search, and other slide controls.

$$\begin{aligned}
& - \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_0^{\infty} \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_0^{\infty} \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 - \\
& \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_3(u) du \right) \int_0^{\infty} \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 - \\
& \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_3(u) du \right) \int_0^{\infty} \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 - 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1 \hat{f}_2(u) \hat{f}_3(u) du \right) \int_0^{\infty} \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_3(u) du \right) \int_0^{\infty} \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 - 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_1 \hat{f}_3(u) du \right) \int_0^{\infty} \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_2 \hat{f}_3(u) du \right) \int_0^{\infty} \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 - 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_2 \hat{f}_3(u) du \right) \int_0^{\infty} \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 - \\
& 4 \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_2 \hat{f}_3(u) du \right) \int_0^{\infty} \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 - \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_0^{\infty} \hat{f}_1 \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 - \\
& 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_3(u) du \right) \int_0^{\infty} \hat{f}_1 \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 - 2 \left(\int_{-\infty}^{\infty} \hat{f}_2 \hat{f}_3(u) du \right) \int_0^{\infty} \hat{f}_1 \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_0^{\infty} \hat{f}_2 \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 - 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_3(u) du \right) \int_0^{\infty} \hat{f}_2 \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 - \\
& 2 \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_3(u) du \right) \int_0^{\infty} \hat{f}_2 \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 - 4 \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_0^{\infty} \hat{f}_1 \hat{f}_2 \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \int_0^{\infty} \hat{f}_3 \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 - 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_2(u) du \right) \int_0^{\infty} \hat{f}_3 \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 - \\
& 2 \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_2(u) du \right) \int_0^{\infty} \hat{f}_3 \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 - 4 \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \int_0^{\infty} \hat{f}_1 \hat{f}_3 \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 - \\
& 4 \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \int_0^{\infty} \hat{f}_2 \hat{f}_3 \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 - 48 \int_0^{\infty} \hat{f}_1 \hat{f}_2 \hat{f}_3 \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 - \\
& \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} \hat{f}_1(x) dx - \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} |u| \hat{f}_3(u) \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} \hat{f}_1(x) dx - \\
& 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_2 \hat{f}_3(u) \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} \hat{f}_1(x) dx - \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_2 \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} \hat{f}_1(x) dx - \\
& 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_3(u) \hat{f}_2 \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} \hat{f}_1(x) dx - \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3 \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} \hat{f}_1(x) dx - \\
& 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_3 \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} \hat{f}_1(x) dx - 4 \left(\int_{-\infty}^{\infty} \hat{f}_2 \hat{f}_3 \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} \hat{f}_1(x) dx
\end{aligned}$$

$$\begin{aligned}
& +2 \left(\int_{-\infty}^{\infty} |u| \hat{f}_3(u) \hat{f}_4(u) du \right) \left(\int_0^{\infty} \hat{f}_2(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} f_1(x) dx + \\
& \left(\int_{-\infty}^{\infty} \hat{f}_3 \hat{f}_4(u) du \right) \left(\int_0^{\infty} \hat{f}_2(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} f_1(x) dx + \\
& 2 \left(\int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_4(u) du \right) \left(\int_0^{\infty} \hat{f}_3(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} f_1(x) dx + \\
& \left(\int_{-\infty}^{\infty} \hat{f}_2 \hat{f}_4(u) du \right) \left(\int_0^{\infty} \hat{f}_3(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} f_1(x) dx + \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_0^{\infty} \hat{f}_4(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} f_1(x) dx + \\
& 2 \left(\int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_3(u) du \right) \left(\int_0^{\infty} \hat{f}_4(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} f_1(x) dx + \\
& \left(\int_{-\infty}^{\infty} \hat{f}_2 \hat{f}_3(u) du \right) \left(\int_0^{\infty} \hat{f}_4(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} f_1(x) dx + \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_0^{\infty} \hat{f}_2 \hat{f}_4(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} f_1(x) dx + \\
& \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_0^{\infty} \hat{f}_3 \hat{f}_4(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} f_1(x) dx + 8 \left(\int_0^{\infty} \hat{f}_2 \hat{f}_3 \hat{f}_4(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} f_1(x) dx - \\
& \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_2(x) dx - \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} |u| \hat{f}_3(u) \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_2(x) dx - \\
& 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1 \hat{f}_3(u) \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_2(x) dx - \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_2(x) dx - \\
& 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_3(u) \hat{f}_1 \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_2(x) dx - \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3 \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_2(x) dx - \\
& 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_3 \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_2(x) dx - 4 \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_3 \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_2(x) dx + \\
& 2 \left(\int_{-\infty}^{\infty} |u| \hat{f}_3(u) \hat{f}_4(u) du \right) \left(\int_0^{\infty} \hat{f}_1(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} f_2(x) dx + \\
& \left(\int_{-\infty}^{\infty} \hat{f}_3 \hat{f}_4(u) du \right) \left(\int_0^{\infty} \hat{f}_1(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} f_2(x) dx + \\
& 2 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_4(u) du \right) \left(\int_0^{\infty} \hat{f}_3(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} f_2(x) dx + \\
& \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_4(u) du \right) \left(\int_0^{\infty} \hat{f}_3(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} f_2(x) dx + \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_0^{\infty} \hat{f}_4(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} f_2(x) dx
\end{aligned}$$

$$\begin{aligned}
& +2 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_3(u) du \right) \left(\int_0^{\infty} \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} f_2(x) dx + \\
& \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_3(u) du \right) \left(\int_0^{\infty} \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} f_2(x) dx + \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_0^{\infty} \hat{f}_1 \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} f_2(x) dx + \\
& \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_0^{\infty} \hat{f}_3 \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} f_2(x) dx + 8 \left(\int_0^{\infty} \hat{f}_1 \hat{f}_3 \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} f_2(x) dx + \\
& 2 \left(\int_{-\infty}^{\infty} |u| \hat{f}_3(u) \hat{f}_4(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_1(x) dx \right) \int_{-\infty}^{\infty} f_2(x) dx + \left(\int_{-\infty}^{\infty} \hat{f}_3 \hat{f}_4(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_1(x) dx \right) \int_{-\infty}^{\infty} f_2(x) dx - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_0^{\infty} \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} \hat{f}_1(x) dx \right) \int_{-\infty}^{\infty} f_2(x) dx - \\
& 2 \left(\int_0^{\infty} \hat{f}_3 \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} \hat{f}_1(x) dx \right) \int_{-\infty}^{\infty} f_2(x) dx - 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_3(u) \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} (f_1 f_2)(x) dx - \\
& 2 \left(\int_{-\infty}^{\infty} \hat{f}_3 \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} (f_1 f_2)(x) dx + \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_0^{\infty} \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} (f_1 f_2)(x) dx + \\
& 4 \left(\int_0^{\infty} \hat{f}_3 \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} (f_1 f_2)(x) dx - \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_3(x) dx - \\
& \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_3(x) dx - 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1 \hat{f}_2(u) \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_3(x) dx - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_3(x) dx - 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_1 \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_3(x) dx - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_2 \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_3(x) dx - 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_2 \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_3(x) dx - \\
& 4 \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_2 \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_3(x) dx + 2 \left(\int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_4(u) du \right) \left(\int_0^{\infty} \hat{f}_1(u_1) \bar{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} f_3(x) dx + \\
& \left(\int_{-\infty}^{\infty} \hat{f}_2 \hat{f}_4(u) du \right) \left(\int_0^{\infty} \hat{f}_1(u_1) \bar{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} f_3(x) dx + \\
& 2 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_4(u) du \right) \left(\int_0^{\infty} \hat{f}_2(u_1) \bar{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} f_3(x) dx + \\
& \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_4(u) du \right) \left(\int_0^{\infty} \hat{f}_2(u_1) \bar{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} f_3(x) dx + \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_0^{\infty} \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} f_3(x) dx + \\
& 2 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_2(u) du \right) \left(\int_0^{\infty} \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} f_3(x) dx
\end{aligned}$$

$$\begin{aligned}
& + \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_2(u) du \right) \left(\int_0^{\infty} \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} f_3(x) dx + \\
& \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_0^{\infty} \hat{f}_1 \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} f_3(x) dx + \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_0^{\infty} \hat{f}_2 \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} f_3(x) dx + \\
& 8 \left(\int_0^{\infty} \hat{f}_1 \hat{f}_2 \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} f_3(x) dx + 2 \left(\int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_4(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_1(x) dx \right) \int_{-\infty}^{\infty} f_3(x) dx + \\
& \left(\int_{-\infty}^{\infty} \hat{f}_2 \hat{f}_4(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_1(x) dx \right) \int_{-\infty}^{\infty} f_3(x) dx - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_0^{\infty} \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} \hat{f}_1(x) dx \right) \int_{-\infty}^{\infty} f_3(x) dx - \\
& 2 \left(\int_0^{\infty} \hat{f}_2 \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} \hat{f}_1(x) dx \right) \int_{-\infty}^{\infty} f_3(x) dx + \\
& 2 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_4(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(x) dx \right) \int_{-\infty}^{\infty} f_3(x) dx + \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_4(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(x) dx \right) \int_{-\infty}^{\infty} f_3(x) dx - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_0^{\infty} \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(x) dx \right) \int_{-\infty}^{\infty} f_3(x) dx - \\
& 2 \left(\int_0^{\infty} \hat{f}_1 \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(x) dx \right) \int_{-\infty}^{\infty} f_3(x) dx + \\
& \left(\int_0^{\infty} \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} \hat{f}_1(x) dx \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(x) dx \right) \int_{-\infty}^{\infty} f_3(x) dx - \\
& 2 \left(\int_0^{\infty} \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} (\hat{f}_1 \hat{f}_2)(x) dx \right) \int_{-\infty}^{\infty} f_3(x) dx - 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} (\hat{f}_1 \hat{f}_3)(x) dx - \\
& 2 \left(\int_{-\infty}^{\infty} \hat{f}_2 \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} (\hat{f}_1 \hat{f}_3)(x) dx + \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_0^{\infty} \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} (\hat{f}_1 \hat{f}_3)(x) dx + \\
& 4 \left(\int_0^{\infty} \hat{f}_2 \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} (\hat{f}_1 \hat{f}_3)(x) dx - 2 \left(\int_0^{\infty} \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(x) dx \right) \int_{-\infty}^{\infty} (\hat{f}_1 \hat{f}_3)(x) dx - \\
& 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} (\hat{f}_2 \hat{f}_3)(x) dx - 2 \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} (\hat{f}_2 \hat{f}_3)(x) dx + \\
& \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_0^{\infty} \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} (\hat{f}_2 \hat{f}_3)(x) dx + 4 \left(\int_0^{\infty} \hat{f}_1 \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} (\hat{f}_2 \hat{f}_3)(x) dx - \\
& 2 \left(\int_0^{\infty} \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} \hat{f}_1(x) dx \right) \int_{-\infty}^{\infty} (\hat{f}_2 \hat{f}_3)(x) dx + 8 \left(\int_0^{\infty} \hat{f}_4(u_1) \bar{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} (\hat{f}_1 \hat{f}_2 \hat{f}_3)(x) dx - \\
& \frac{1}{8} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} \hat{f}_4(x) dx
\end{aligned}$$

$$\begin{aligned}
& - \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} f_4(x) dx - \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} f_4(x) dx - \\
& \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} f_4(x) dx - \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} f_4(x) dx - \\
& 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1 \hat{f}_2(u) \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} f_4(x) dx - \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} f_4(x) dx - \\
& 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_1 \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} f_4(x) dx - \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_2 \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} f_4(x) dx - \\
& 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_2 \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} f_4(x) dx - 4 \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_2 \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} f_4(x) dx + \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_0^{\infty} \hat{f}_1(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} f_4(x) dx + \\
& 2 \left(\int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_3(u) du \right) \left(\int_0^{\infty} \hat{f}_1(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} f_4(x) dx + \\
& \left(\int_{-\infty}^{\infty} \hat{f}_2 \hat{f}_3(u) du \right) \left(\int_0^{\infty} \hat{f}_1(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} f_4(x) dx + \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_0^{\infty} \hat{f}_2(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} f_4(x) dx + \\
& 2 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_3(u) du \right) \left(\int_0^{\infty} \hat{f}_2(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} f_4(x) dx + \\
& \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_3(u) du \right) \left(\int_0^{\infty} \hat{f}_2(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} f_4(x) dx + \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_0^{\infty} \hat{f}_1 \hat{f}_2(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} f_4(x) dx + \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_0^{\infty} \hat{f}_3(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} f_4(x) dx + \\
& 2 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_2(u) du \right) \left(\int_0^{\infty} \hat{f}_3(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} f_4(x) dx + \\
& \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_2(u) du \right) \left(\int_0^{\infty} \hat{f}_3(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} f_4(x) dx + \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_0^{\infty} \hat{f}_1 \hat{f}_3(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} f_4(x) dx + \\
& \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_0^{\infty} \hat{f}_2 \hat{f}_3(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} f_4(x) dx + 8 \left(\int_0^{\infty} \hat{f}_1 \hat{f}_2 \hat{f}_3(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} f_4(x) dx + \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_1(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx + \\
& 2 \left(\int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_1(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx + \left(\int_{-\infty}^{\infty} \hat{f}_2 \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_1(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_0^{\infty} \hat{f}_2(u_1) \bar{\chi}(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_0^{\infty} \hat{f}_3(u_1) \bar{\chi}(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx - \\
& 2 \left(\int_0^{\infty} \hat{f}_2 \hat{f}_3(u_1) \bar{\chi}(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx + \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} f_2(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx + \\
& 2 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} f_2(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx + \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} f_2(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_0^{\infty} \hat{f}_1(u_1) \bar{\chi}(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} f_2(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_0^{\infty} \hat{f}_3(u_1) \bar{\chi}(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} f_2(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx - \\
& 2 \left(\int_0^{\infty} \hat{f}_1 \hat{f}_3(u_1) \bar{\chi}(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} f_2(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_1(x) dx \right) \left(\int_{-\infty}^{\infty} f_2(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx + \\
& \left(\int_0^{\infty} \hat{f}_3(u_1) \bar{\chi}(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \left(\int_{-\infty}^{\infty} f_2(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx + \\
& \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} (f_1 f_2)(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx - 2 \left(\int_0^{\infty} \hat{f}_3(u_1) \bar{\chi}(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} (f_1 f_2)(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx + \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx + \\
& 2 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx + \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_0^{\infty} \hat{f}_1(u_1) \bar{\chi}(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_0^{\infty} \hat{f}_2(u_1) \bar{\chi}(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx - \\
& 2 \left(\int_0^{\infty} \hat{f}_1 \hat{f}_2(u_1) \bar{\chi}(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_1(x) dx \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx
\end{aligned}$$

$$\begin{aligned}
& + \left(\int_0^\infty \hat{f}_2(u_1) \tilde{\chi}(u_1) du_1 \right) \left(\int_{-\infty}^\infty f_1(x) dx \right) \left(\int_{-\infty}^\infty f_3(x) dx \right) \int_{-\infty}^\infty f_4(x) dx - \\
& \frac{1}{2} \left(\int_{-\infty}^\infty \hat{f}_1(u) du \right) \left(\int_{-\infty}^\infty f_2(x) dx \right) \left(\int_{-\infty}^\infty f_3(x) dx \right) \int_{-\infty}^\infty f_4(x) dx + \\
& \left(\int_0^\infty \hat{f}_1(u_1) \tilde{\chi}(u_1) du_1 \right) \left(\int_{-\infty}^\infty f_2(x) dx \right) \left(\int_{-\infty}^\infty f_3(x) dx \right) \int_{-\infty}^\infty f_4(x) dx + \\
& \left(\int_{-\infty}^\infty f_1(x) dx \right) \left(\int_{-\infty}^\infty f_2(x) dx \right) \left(\int_{-\infty}^\infty f_3(x) dx \right) \int_{-\infty}^\infty f_4(x) dx - \\
& 2 \left(\int_{-\infty}^\infty (f_1 f_2)(x) dx \right) \left(\int_{-\infty}^\infty f_3(x) dx \right) \int_{-\infty}^\infty f_4(x) dx + \left(\int_{-\infty}^\infty \hat{f}_2(u) du \right) \left(\int_{-\infty}^\infty (f_1 f_3)(x) dx \right) \int_{-\infty}^\infty f_4(x) dx - \\
& 2 \left(\int_0^\infty \hat{f}_2(u_1) \tilde{\chi}(u_1) du_1 \right) \left(\int_{-\infty}^\infty (f_1 f_3)(x) dx \right) \int_{-\infty}^\infty f_4(x) dx - 2 \left(\int_{-\infty}^\infty f_2(x) dx \right) \left(\int_{-\infty}^\infty (f_1 f_3)(x) dx \right) \int_{-\infty}^\infty f_4(x) dx + \\
& \left(\int_{-\infty}^\infty \hat{f}_1(u) du \right) \left(\int_{-\infty}^\infty (f_2 f_3)(x) dx \right) \int_{-\infty}^\infty f_4(x) dx - 2 \left(\int_0^\infty \hat{f}_1(u_1) \tilde{\chi}(u_1) du_1 \right) \left(\int_{-\infty}^\infty (f_2 f_3)(x) dx \right) \int_{-\infty}^\infty f_4(x) dx - \\
& 2 \left(\int_{-\infty}^\infty f_1(x) dx \right) \left(\int_{-\infty}^\infty (f_2 f_3)(x) dx \right) \int_{-\infty}^\infty f_4(x) dx + 8 \left(\int_{-\infty}^\infty (f_1 f_2 f_3)(x) dx \right) \int_{-\infty}^\infty f_4(x) dx - \\
& \frac{1}{2} \left(\int_{-\infty}^\infty \hat{f}_2(u) du \right) \left(\int_{-\infty}^\infty \hat{f}_3(u) du \right) \int_{-\infty}^\infty (f_1 f_4)(x) dx - 4 \left(\int_{-\infty}^\infty |u| \hat{f}_2(u) \hat{f}_3(u) du \right) \int_{-\infty}^\infty (f_1 f_4)(x) dx - \\
& 2 \left(\int_{-\infty}^\infty \hat{f}_2(u) du \right) \int_{-\infty}^\infty (f_1 f_4)(x) dx + \left(\int_{-\infty}^\infty \hat{f}_3(u) du \right) \left(\int_0^\infty \hat{f}_2(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^\infty (f_1 f_4)(x) dx + \\
& \left(\int_{-\infty}^\infty \hat{f}_2(u) du \right) \left(\int_0^\infty \hat{f}_3(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^\infty (f_1 f_4)(x) dx + 4 \left(\int_0^\infty \hat{f}_2 \hat{f}_3(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^\infty (f_1 f_4)(x) dx + \\
& \left(\int_{-\infty}^\infty \hat{f}_3(u) du \right) \left(\int_{-\infty}^\infty f_2(x) dx \right) \int_{-\infty}^\infty (f_1 f_4)(x) dx - 2 \left(\int_0^\infty \hat{f}_3(u_1) \tilde{\chi}(u_1) du_1 \right) \left(\int_{-\infty}^\infty f_2(x) dx \right) \int_{-\infty}^\infty (f_1 f_4)(x) dx + \\
& \left(\int_{-\infty}^\infty \hat{f}_2(u) du \right) \left(\int_{-\infty}^\infty f_3(x) dx \right) \int_{-\infty}^\infty (f_1 f_4)(x) dx - 2 \left(\int_0^\infty \hat{f}_2(u_1) \tilde{\chi}(u_1) du_1 \right) \left(\int_{-\infty}^\infty f_3(x) dx \right) \int_{-\infty}^\infty (f_1 f_4)(x) dx - \\
& 2 \left(\int_{-\infty}^\infty f_2(x) dx \right) \left(\int_{-\infty}^\infty f_3(x) dx \right) \int_{-\infty}^\infty (f_1 f_4)(x) dx + 4 \left(\int_{-\infty}^\infty (f_2 f_3)(x) dx \right) \int_{-\infty}^\infty (f_1 f_4)(x) dx - \\
& \frac{1}{2} \left(\int_{-\infty}^\infty \hat{f}_1(u) du \right) \left(\int_{-\infty}^\infty \hat{f}_3(u) du \right) \int_{-\infty}^\infty (f_2 f_4)(x) dx - 4 \left(\int_{-\infty}^\infty |u| \hat{f}_1(u) \hat{f}_3(u) du \right) \int_{-\infty}^\infty (f_2 f_4)(x) dx - \\
& 2 \left(\int_{-\infty}^\infty \hat{f}_1 \hat{f}_3(u) du \right) \int_{-\infty}^\infty (f_2 f_4)(x) dx + \left(\int_{-\infty}^\infty \hat{f}_3(u) du \right) \left(\int_0^\infty \hat{f}_1(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^\infty (f_2 f_4)(x) dx + \\
& \left(\int_{-\infty}^\infty \hat{f}_1(u) du \right) \left(\int_0^\infty \hat{f}_3(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^\infty (f_2 f_4)(x) dx + 4 \left(\int_0^\infty \hat{f}_1 \hat{f}_3(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^\infty (f_2 f_4)(x) dx
\end{aligned}$$

$$\begin{aligned}
& + \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \int_{-\infty}^{\infty} (f_2 f_4)(x) dx - \\
& 2 \left(\int_0^{\infty} \hat{f}_3(u_1) \tilde{\chi}(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \int_{-\infty}^{\infty} (f_2 f_4)(x) dx + \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} (f_2 f_4)(x) dx - \\
& 2 \left(\int_0^{\infty} \hat{f}_1(u_1) \tilde{\chi}(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} (f_2 f_4)(x) dx - 2 \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} (f_2 f_4)(x) dx + \\
& 4 \left(\int_{-\infty}^{\infty} (f_1 f_3)(x) dx \right) \int_{-\infty}^{\infty} (f_2 f_4)(x) dx - 4 \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} (f_1 f_2 f_4)(x) dx + \\
& 8 \left(\int_0^{\infty} \hat{f}_3(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} (f_1 f_2 f_4)(x) dx + 8 \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} (f_1 f_2 f_4)(x) dx - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \int_{-\infty}^{\infty} (f_3 f_4)(x) dx - 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_2(u) du \right) \int_{-\infty}^{\infty} (f_3 f_4)(x) dx - \\
& 2 \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_2(u) du \right) \int_{-\infty}^{\infty} (f_3 f_4)(x) dx + \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_0^{\infty} \hat{f}_1(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} (f_3 f_4)(x) dx + \\
& \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_0^{\infty} \hat{f}_2(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} (f_3 f_4)(x) dx + 4 \left(\int_0^{\infty} \hat{f}_1 \hat{f}_2(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} (f_3 f_4)(x) dx + \\
& \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \int_{-\infty}^{\infty} (f_3 f_4)(x) dx - 2 \left(\int_0^{\infty} \hat{f}_2(u_1) \tilde{\chi}(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \int_{-\infty}^{\infty} (f_3 f_4)(x) dx + \\
& \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} f_2(x) dx \right) \int_{-\infty}^{\infty} (f_3 f_4)(x) dx - 2 \left(\int_0^{\infty} \hat{f}_1(u_1) \tilde{\chi}(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} f_2(x) dx \right) \int_{-\infty}^{\infty} (f_3 f_4)(x) dx - \\
& 2 \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \left(\int_{-\infty}^{\infty} f_2(x) dx \right) \int_{-\infty}^{\infty} (f_3 f_4)(x) dx + 4 \left(\int_{-\infty}^{\infty} (f_1 f_2)(x) dx \right) \int_{-\infty}^{\infty} (f_3 f_4)(x) dx - \\
& 4 \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \int_{-\infty}^{\infty} (f_1 f_3 f_4)(x) dx + 8 \left(\int_0^{\infty} \hat{f}_2(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} (f_1 f_3 f_4)(x) dx + \\
& 8 \left(\int_{-\infty}^{\infty} f_2(x) dx \right) \int_{-\infty}^{\infty} (f_1 f_3 f_4)(x) dx - 4 \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \int_{-\infty}^{\infty} (f_2 f_3 f_4)(x) dx + \\
& 8 \left(\int_0^{\infty} \hat{f}_1(u_1) \tilde{\chi}(u_1) du_1 \right) \int_{-\infty}^{\infty} (f_2 f_3 f_4)(x) dx + 8 \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \int_{-\infty}^{\infty} (f_2 f_3 f_4)(x) dx - 48 \int_{-\infty}^{\infty} (f_1 f_2 f_3 f_4)(x) dx - \\
& 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_3(u) \hat{f}_4(u) du \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_1(u_1) \hat{f}_2(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 - \\
& 2 \left(\int_{-\infty}^{\infty} \hat{f}_3 \hat{f}_4(u) du \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_1(u_1) \hat{f}_2(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 + \\
& \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_4(x) dx \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_1(u_1) \hat{f}_2(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2
\end{aligned}$$

$$\begin{aligned}
& -2 \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \left(\int_{-\infty}^{\infty} f_4(x) dx \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_1(u_1) \hat{f}_2(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 + \\
& 4 \left(\int_{-\infty}^{\infty} (f_3 f_4)(x) dx \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_1(u_1) \hat{f}_2(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 - \\
& 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_4(u) du \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_1(u_1) \hat{f}_3(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 - \\
& 2 \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_1(u_1) \hat{f}_3(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 + \\
& \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} f_4(x) dx \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_1(u_1) \hat{f}_3(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 - \\
& 2 \left(\int_{-\infty}^{\infty} \hat{f}_2(x) dx \right) \left(\int_{-\infty}^{\infty} f_4(x) dx \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_1(u_1) \hat{f}_3(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 + \\
& 4 \left(\int_{-\infty}^{\infty} (f_2 f_4)(x) dx \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_1(u_1) \hat{f}_3(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 - \\
& 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_4(u) du \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_2(u_1) \hat{f}_3(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 - \\
& 2 \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_2(u_1) \hat{f}_3(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 + \\
& \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} f_4(x) dx \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_2(u_1) \hat{f}_3(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 - \\
& 2 \left(\int_{-\infty}^{\infty} \hat{f}_1(x) dx \right) \left(\int_{-\infty}^{\infty} f_4(x) dx \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_2(u_1) \hat{f}_3(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 + \\
& 4 \left(\int_{-\infty}^{\infty} (f_1 f_4)(x) dx \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_2(u_1) \hat{f}_3(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 + \\
& 4 \left(\int_{-\infty}^{\infty} f_4(x) dx \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_1 \hat{f}_2(u_1) \hat{f}_3(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 + \\
& 4 \left(\int_{-\infty}^{\infty} f_4(x) dx \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_2(u_1) \hat{f}_1 \hat{f}_3(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 + \\
& 4 \left(\int_{-\infty}^{\infty} f_4(x) dx \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_1(u_1) \hat{f}_2 \hat{f}_3(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_1(u_1) \hat{f}_4(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 - \\
& 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_3(u) du \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_1(u_1) \hat{f}_4(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2
\end{aligned}$$

$$\begin{aligned}
& -2 \left(\int_{-\infty}^{\infty} \hat{f}_2 \hat{f}_3(u) du \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_1(u_1) \hat{f}_4(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 + \\
& \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(x) dx \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_1(u_1) \hat{f}_4(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 + \\
& \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(x) dx \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_1(u_1) \hat{f}_4(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 - \\
& 2 \left(\int_{-\infty}^{\infty} \hat{f}_2(x) dx \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(x) dx \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_1(u_1) \hat{f}_4(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 + \\
& 4 \left(\int_{-\infty}^{\infty} (\hat{f}_2 \hat{f}_3)(x) dx \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_1(u_1) \hat{f}_4(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_2(u_1) \hat{f}_4(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 - \\
& 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_3(u) du \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_2(u_1) \hat{f}_4(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 - \\
& 2 \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_3(u) du \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_2(u_1) \hat{f}_4(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 + \\
& \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_1(x) dx \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_2(u_1) \hat{f}_4(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 + \\
& \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(x) dx \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_2(u_1) \hat{f}_4(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 - \\
& 2 \left(\int_{-\infty}^{\infty} \hat{f}_1(x) dx \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(x) dx \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_2(u_1) \hat{f}_4(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 + \\
& 4 \left(\int_{-\infty}^{\infty} (\hat{f}_1 \hat{f}_3)(x) dx \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_2(u_1) \hat{f}_4(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 - \\
& 2 \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_1 \hat{f}_2(u_1) \hat{f}_4(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 + \\
& 4 \left(\int_{-\infty}^{\infty} \hat{f}_3(x) dx \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_1 \hat{f}_2(u_1) \hat{f}_4(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_3(u_1) \hat{f}_4(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 - \\
& 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_2(u) du \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_3(u_1) \hat{f}_4(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 - \\
& 2 \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_2(u) du \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_3(u_1) \hat{f}_4(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2
\end{aligned}$$

$$\begin{aligned}
& + \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_3(u_1) \hat{f}_4(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 + \\
& \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} f_2(x) dx \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_3(u_1) \hat{f}_4(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 - \\
& 2 \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \left(\int_{-\infty}^{\infty} f_2(x) dx \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_3(u_1) \hat{f}_4(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 + \\
& 4 \left(\int_{-\infty}^{\infty} (f_1 f_2)(x) dx \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_3(u_1) \hat{f}_4(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 - \\
& 2 \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_1 \hat{f}_3(u_1) \hat{f}_4(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 + \\
& 4 \left(\int_{-\infty}^{\infty} f_2(x) dx \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_1 \hat{f}_3(u_1) \hat{f}_4(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 - \\
& 2 \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_2 \hat{f}_3(u_1) \hat{f}_4(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 + \\
& 4 \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_2 \hat{f}_3(u_1) \hat{f}_4(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 - \\
& 16 \int_0^{\infty} \int_0^{\infty} \hat{f}_1 \hat{f}_2 \hat{f}_3(u_1) \hat{f}_4(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 - \\
& 2 \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_2(u_1) \hat{f}_1 \hat{f}_4(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 + \\
& 4 \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_2(u_1) \hat{f}_1 \hat{f}_4(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 - \\
& 2 \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_3(u_1) \hat{f}_1 \hat{f}_4(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 + \\
& 4 \left(\int_{-\infty}^{\infty} f_2(x) dx \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_3(u_1) \hat{f}_1 \hat{f}_4(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 - \\
& 8 \int_0^{\infty} \int_0^{\infty} \hat{f}_2 \hat{f}_3(u_1) \hat{f}_1 \hat{f}_4(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 - \\
& 2 \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_1(u_1) \hat{f}_2 \hat{f}_4(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 + \\
& 4 \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_1(u_1) \hat{f}_2 \hat{f}_4(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 - \\
& 2 \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_3(u_1) \hat{f}_2 \hat{f}_4(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2
\end{aligned}$$

$$\begin{aligned}
& +4 \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_3(u_1) \hat{f}_2 \hat{f}_4(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 - \\
& 8 \int_0^{\infty} \int_0^{\infty} \hat{f}_1 \hat{f}_3(u_1) \hat{f}_2 \hat{f}_4(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 - \\
& 16 \int_0^{\infty} \int_0^{\infty} \hat{f}_3(u_1) \hat{f}_1 \hat{f}_2 \hat{f}_4(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 - \\
& 2 \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_1(u_1) \hat{f}_3 \hat{f}_4(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 + \\
& 4 \left(\int_{-\infty}^{\infty} f_2(x) dx \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_1(u_1) \hat{f}_3 \hat{f}_4(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 - \\
& 2 \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_2(u_1) \hat{f}_3 \hat{f}_4(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 + \\
& 4 \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \int_0^{\infty} \int_0^{\infty} \hat{f}_2(u_1) \hat{f}_3 \hat{f}_4(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 - \\
& 8 \int_0^{\infty} \int_0^{\infty} \hat{f}_1 \hat{f}_2(u_1) \hat{f}_3 \hat{f}_4(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 - \\
& 16 \int_0^{\infty} \int_0^{\infty} \hat{f}_2(u_1) \hat{f}_1 \hat{f}_3 \hat{f}_4(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 - \\
& 16 \int_0^{\infty} \int_0^{\infty} \hat{f}_1(u_1) \hat{f}_2 \hat{f}_3 \hat{f}_4(u_2) (-\tilde{\chi}(u_1 - u_2) - \tilde{\chi}(-u_1 + u_2) + \tilde{\chi}(u_1 + u_2)) du_1 du_2 + \\
& 4 \left(\int_{-\infty}^{\infty} f_4(x) dx \right) \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \hat{f}_1(u_1) \hat{f}_2(u_2) \hat{f}_3(u_3) \left(\tilde{\chi}(u_1 - u_2 - u_3) + \tilde{\chi}(-u_1 + u_2 - u_3) - \tilde{\chi}(u_1 + u_2 - u_3) + \tilde{\chi}(-u_1 - u_2 + u_3) - \tilde{\chi}(u_1 - u_2 + u_3) - \tilde{\chi}(-u_1 + u_2 + u_3) + \tilde{\chi}(u_1 + u_2 + u_3) \right) du_1 du_2 du_3 - \\
& 2 \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \hat{f}_1(u_1) \hat{f}_2(u_2) \hat{f}_4(u_3) \left(\tilde{\chi}(u_1 - u_2 - u_3) + \tilde{\chi}(-u_1 + u_2 - u_3) - \tilde{\chi}(u_1 + u_2 - u_3) + \tilde{\chi}(-u_1 - u_2 + u_3) - \tilde{\chi}(u_1 - u_2 + u_3) - \tilde{\chi}(-u_1 + u_2 + u_3) + \tilde{\chi}(u_1 + u_2 + u_3) \right) du_1 du_2 du_3 + \\
& 4 \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \hat{f}_1(u_1) \hat{f}_2(u_2) \hat{f}_4(u_3) \left(\tilde{\chi}(u_1 - u_2 - u_3) + \tilde{\chi}(-u_1 + u_2 - u_3) - \tilde{\chi}(u_1 + u_2 - u_3) + \tilde{\chi}(-u_1 - u_2 + u_3) - \tilde{\chi}(u_1 - u_2 + u_3) - \tilde{\chi}(-u_1 + u_2 + u_3) + \tilde{\chi}(u_1 + u_2 + u_3) \right) du_1 du_2 du_3 - \\
& 2 \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \hat{f}_1(u_1) \hat{f}_3(u_2) \hat{f}_4(u_3) \left(\tilde{\chi}(u_1 - u_2 - u_3) + \tilde{\chi}(-u_1 + u_2 - u_3) - \tilde{\chi}(u_1 + u_2 - u_3) + \tilde{\chi}(-u_1 - u_2 + u_3) - \tilde{\chi}(u_1 - u_2 + u_3) - \tilde{\chi}(-u_1 + u_2 + u_3) + \tilde{\chi}(u_1 + u_2 + u_3) \right) du_1 du_2 du_3
\end{aligned}$$

$$\begin{aligned}
& +4\left(\int_{-\infty}^{\infty} f_2(x) dx\right) \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \hat{f}_1(u_1) \hat{f}_3(u_2) \hat{f}_4(u_3) \left(\tilde{\chi}(u_1 - u_2 - u_3) + \tilde{\chi}(-u_1 + u_2 - u_3) - \tilde{\chi}(u_1 + u_2 - u_3) + \tilde{\chi}(-u_1 - u_2 + u_3) - \tilde{\chi}(u_1 - u_2 + u_3) - \tilde{\chi}(-u_1 + u_2 + u_3) + \tilde{\chi}(u_1 + u_2 + u_3)\right) du_1 du_2 du_3 - \\
& 2\left(\int_{-\infty}^{\infty} \hat{f}_1(u) du\right) \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \hat{f}_2(u_1) \hat{f}_3(u_2) \hat{f}_4(u_3) \left(\tilde{\chi}(u_1 - u_2 - u_3) + \tilde{\chi}(-u_1 + u_2 - u_3) - \tilde{\chi}(u_1 + u_2 - u_3) + \tilde{\chi}(-u_1 - u_2 + u_3) - \tilde{\chi}(u_1 - u_2 + u_3) - \tilde{\chi}(-u_1 + u_2 + u_3) + \tilde{\chi}(u_1 + u_2 + u_3)\right) du_1 du_2 du_3 + \\
& 4\left(\int_{-\infty}^{\infty} f_1(x) dx\right) \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \hat{f}_2(u_1) \hat{f}_3(u_2) \hat{f}_4(u_3) \left(\tilde{\chi}(u_1 - u_2 - u_3) + \tilde{\chi}(-u_1 + u_2 - u_3) - \tilde{\chi}(u_1 + u_2 - u_3) + \tilde{\chi}(-u_1 - u_2 + u_3) - \tilde{\chi}(u_1 - u_2 + u_3) - \tilde{\chi}(-u_1 + u_2 + u_3) + \tilde{\chi}(u_1 + u_2 + u_3)\right) du_1 du_2 du_3 - \\
& 8 \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \hat{f}_1(u_1) \hat{f}_2(u_2) \hat{f}_3(u_3) \left(\tilde{\chi}(u_1 - u_2 - u_3) + \tilde{\chi}(-u_1 + u_2 - u_3) - \tilde{\chi}(u_1 + u_2 - u_3) + \tilde{\chi}(-u_1 - u_2 + u_3) - \tilde{\chi}(u_1 - u_2 + u_3) - \tilde{\chi}(-u_1 + u_2 + u_3) + \tilde{\chi}(u_1 + u_2 + u_3)\right) du_1 du_2 du_3 - 8 \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \hat{f}_2(u_1) \hat{f}_1 \hat{f}_3(u_2) \hat{f}_4(u_3) \left(\tilde{\chi}(u_1 - u_2 - u_3) + \tilde{\chi}(-u_1 + u_2 - u_3) - \tilde{\chi}(u_1 + u_2 - u_3) + \tilde{\chi}(-u_1 - u_2 + u_3) - \tilde{\chi}(u_1 - u_2 + u_3) - \tilde{\chi}(-u_1 + u_2 + u_3) + \tilde{\chi}(u_1 + u_2 + u_3)\right) du_1 du_2 du_3 - 8 \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \hat{f}_1(u_1) \hat{f}_2 \hat{f}_3(u_2) \hat{f}_4(u_3) \left(\tilde{\chi}(u_1 - u_2 - u_3) + \tilde{\chi}(-u_1 + u_2 - u_3) - \tilde{\chi}(u_1 + u_2 - u_3) + \tilde{\chi}(-u_1 - u_2 + u_3) - \tilde{\chi}(u_1 - u_2 + u_3) - \tilde{\chi}(-u_1 + u_2 + u_3) + \tilde{\chi}(u_1 + u_2 + u_3)\right) du_1 du_2 du_3 - \\
& 8 \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \hat{f}_2(u_1) \hat{f}_3(u_2) \hat{f}_1 \hat{f}_4(u_3) \left(\tilde{\chi}(u_1 - u_2 - u_3) + \tilde{\chi}(-u_1 + u_2 - u_3) - \tilde{\chi}(u_1 + u_2 - u_3) + \tilde{\chi}(-u_1 - u_2 + u_3) - \tilde{\chi}(u_1 - u_2 + u_3) - \tilde{\chi}(-u_1 + u_2 + u_3) + \tilde{\chi}(u_1 + u_2 + u_3)\right) du_1 du_2 du_3 - 8 \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \hat{f}_1(u_1) \hat{f}_3(u_2) \hat{f}_2 \hat{f}_4(u_3) \left(\tilde{\chi}(u_1 - u_2 - u_3) + \tilde{\chi}(-u_1 + u_2 - u_3) - \tilde{\chi}(u_1 + u_2 - u_3) + \tilde{\chi}(-u_1 - u_2 + u_3) - \tilde{\chi}(u_1 - u_2 + u_3) - \tilde{\chi}(-u_1 + u_2 + u_3) + \tilde{\chi}(u_1 + u_2 + u_3)\right) du_1 du_2 du_3 - 8 \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \hat{f}_1(u_1) \hat{f}_2(u_2) \hat{f}_3 \hat{f}_4(u_3) \left(\tilde{\chi}(u_1 - u_2 - u_3) + \tilde{\chi}(-u_1 + u_2 - u_3) - \tilde{\chi}(u_1 + u_2 - u_3) + \tilde{\chi}(-u_1 - u_2 + u_3) - \tilde{\chi}(u_1 - u_2 + u_3) - \tilde{\chi}(-u_1 + u_2 + u_3) + \tilde{\chi}(u_1 + u_2 + u_3)\right) du_1 du_2 du_3 - \\
& 8 \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \hat{f}_1(u_1) \hat{f}_2(u_2) \hat{f}_3(u_3) \hat{f}_4(u_4) \left(-\tilde{\chi}(u_1 - u_2 - u_3 - u_4) - \tilde{\chi}(-u_1 + u_2 - u_3 - u_4) + \tilde{\chi}(u_1 + u_2 - u_3 - u_4) - \tilde{\chi}(-u_1 - u_2 + u_3 - u_4) + \tilde{\chi}(u_1 - u_2 + u_3 - u_4) + \tilde{\chi}(-u_1 + u_2 + u_3 - u_4) - \tilde{\chi}(u_1 + u_2 + u_3 - u_4) - \tilde{\chi}(-u_1 - u_2 + u_3 + u_4) + \tilde{\chi}(u_1 - u_2 - u_3 + u_4) + \tilde{\chi}(-u_1 + u_2 - u_3 + u_4) - \tilde{\chi}(u_1 + u_2 - u_3 + u_4) + \tilde{\chi}(-u_1 - u_2 + u_3 + u_4) - \tilde{\chi}(u_1 - u_2 + u_3 + u_4) - \tilde{\chi}(-u_1 + u_2 + u_3 + u_4) + \tilde{\chi}(u_1 + u_2 + u_3 + u_4)\right) du_1 du_2 du_3 du_4 \text{ help it's alive!}
\end{aligned}$$

$$=$$

The 4-Level Density of Eigenvalues of the Gaussian Symplectic Ensemble

$$\begin{aligned}
& \frac{1}{16} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \frac{1}{16} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \frac{1}{16} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \frac{1}{16} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& 2 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1 \hat{f}_2(u) \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \frac{1}{16} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \frac{1}{16} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \frac{1}{16} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \frac{1}{16} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + 2 \left(\int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_1 \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_2 \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} \hat{f}_2 \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + 2 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_2 \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& 2 \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_2 \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \frac{1}{8} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_1(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \frac{1}{8} \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_1(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \left(\int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_1(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \frac{1}{8} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \left(\int_{-\infty}^{\infty} \hat{f}_1(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \frac{1}{8} \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \left(\int_{-\infty}^{\infty} \hat{f}_1(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_2 \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_1(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \frac{1}{8} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \frac{1}{8} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \frac{1}{8} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \frac{1}{8} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_1(x) dx \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \left(\int_{-\infty}^{\infty} \hat{f}_1(x) dx \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} (f_1 f_2)(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \left(\int_{-\infty}^{\infty} (f_1 f_2)(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \frac{1}{8} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \frac{1}{8} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \frac{1}{8} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \frac{1}{8} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_1(x) dx \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} \hat{f}_1(x) dx \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(x) dx \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(x) dx \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_1(x) dx \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(x) dx \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \left(\int_{-\infty}^{\infty} (f_1 f_2)(x) dx \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} (f_1 f_3)(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} (f_1 f_3)(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \left(\int_{-\infty}^{\infty} \hat{f}_2(x) dx \right) \left(\int_{-\infty}^{\infty} (f_1 f_3)(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} (f_2 f_3)(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} (f_2 f_3)(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \left(\int_{-\infty}^{\infty} \hat{f}_1(x) dx \right) \left(\int_{-\infty}^{\infty} (f_2 f_3)(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - 4 \left(\int_{-\infty}^{\infty} (f_1 f_2 f_3)(x) dx \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_1 - u_2)) \chi(u_1 + u_2)) \hat{f}_1(u_1) \hat{f}_2(u_2) du_1 du_2 \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_1 - u_2)) \chi(u_1 + u_2)) \hat{f}_1(u_1) \hat{f}_2(u_2) du_1 du_2 \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_3(x) dx \right) \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_1 - u_2)) \chi(u_1 + u_2)) \hat{f}_1(u_1) \hat{f}_2(u_2) du_1 du_2 \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_1 - u_3)) \chi(u_1 + u_3)) \hat{f}_1(u_1) \hat{f}_3(u_3) du_1 du_3 \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_1 - u_3)) \chi(u_1 + u_3)) \hat{f}_1(u_1) \hat{f}_3(u_3) du_1 du_3 \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_2(x) dx \right) \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_1 - u_3)) \chi(u_1 + u_3)) \hat{f}_1(u_1) \hat{f}_3(u_3) du_1 du_3 \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_2 - u_3)) \chi(u_2 + u_3)) \hat{f}_2(u_2) \hat{f}_3(u_3) du_2 du_3 \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_2 - u_3)) \chi(u_2 + u_3)) \hat{f}_2(u_2) \hat{f}_3(u_3) du_2 du_3 \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du + \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_1(x) dx \right) \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_2 - u_3)) \chi(u_2 + u_3)) \hat{f}_2(u_2) \hat{f}_3(u_3) du_2 du_3 \right) \int_{-\infty}^{\infty} \hat{f}_4(u) du - \\
& \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (2 - \chi(u_1 - u_2 - u_3)) \chi(u_1 + u_2 - u_3) \chi(u_1 + u_2 + u_3) - \chi(u_1 - u_2 - u_3) \chi(u_1 - u_2 + u_3) \chi(u_1 + u_2 + u_3) \right) \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_4(u) du - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_4(u) du + \\
& 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_4(u) du - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_4(u) du + \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_4(u) du + \\
& 2 \left(\int_{-\infty}^{\infty} \hat{f}_2 \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_4(u) du + \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_4(u) du - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_4(u) du + \\
& 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_4(u) du
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_4(u) du + \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_4(u) du + \\
& 2 \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_4(u) du + 2 \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} |u| \hat{f}_1 \hat{f}_2(u) \hat{f}_4(u) du - \\
& 2 \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \int_{-\infty}^{\infty} |u| \hat{f}_1 \hat{f}_2(u) \hat{f}_4(u) du + \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \int_{-\infty}^{\infty} |u| \hat{f}_3(u) \hat{f}_4(u) du - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \int_{-\infty}^{\infty} |u| \hat{f}_3(u) \hat{f}_4(u) du + \\
& 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_2(u) du \right) \int_{-\infty}^{\infty} |u| \hat{f}_3(u) \hat{f}_4(u) du - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \int_{-\infty}^{\infty} |u| \hat{f}_3(u) \hat{f}_4(u) du + \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \int_{-\infty}^{\infty} |u| \hat{f}_3(u) \hat{f}_4(u) du + \\
& 2 \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_2(u) du \right) \int_{-\infty}^{\infty} |u| \hat{f}_3(u) \hat{f}_4(u) du + 2 \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \int_{-\infty}^{\infty} |u| \hat{f}_1 \hat{f}_3(u) \hat{f}_4(u) du - \\
& 2 \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \int_{-\infty}^{\infty} |u| \hat{f}_1 \hat{f}_3(u) \hat{f}_4(u) du + 2 \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \int_{-\infty}^{\infty} |u| \hat{f}_2 \hat{f}_3(u) \hat{f}_4(u) du - \\
& 2 \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \int_{-\infty}^{\infty} |u| \hat{f}_2 \hat{f}_3(u) \hat{f}_4(u) du + 16 \int_{-\infty}^{\infty} |u| \hat{f}_1 \hat{f}_2 \hat{f}_3(u) \hat{f}_4(u) du - \\
& \frac{1}{16} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 + \\
& \frac{1}{16} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 + \\
& \frac{1}{16} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 - \\
& \frac{1}{16} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 - \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 + \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 + \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 - \\
& 2 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1 \hat{f}_2(u) \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 + \\
& \frac{1}{16} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 - \\
& \frac{1}{16} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 + \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 - \\
& \frac{1}{16} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 + \\
& \frac{1}{16} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 - \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 + \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 - \\
& 2 \left(\int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_1 \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 - \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_2 \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 + \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} \hat{f}_2 \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 - \\
& 2 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_2 \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 - 2 \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_2 \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 + \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_4(u) du
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{4} \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_4(u) du + 2 \left(\int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_4(u) du - \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_4(u) du + \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_4(u) du + \\
& \left(\int_{-\infty}^{\infty} \hat{f}_2 \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_4(u) du + 2 \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_1 \hat{f}_4(u) du - \\
& 2 \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_1 \hat{f}_4(u) du + 2 \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \int_{-\infty}^{\infty} |u| \hat{f}_3(u) \hat{f}_1 \hat{f}_4(u) du - \\
& 2 \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \int_{-\infty}^{\infty} |u| \hat{f}_3(u) \hat{f}_1 \hat{f}_4(u) du + 8 \int_{-\infty}^{\infty} |u| \hat{f}_2 \hat{f}_3(u) \hat{f}_1 \hat{f}_4(u) du + \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} \hat{f}_2 \hat{f}_4(u) du - \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} \hat{f}_2 \hat{f}_4(u) du + 2 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} \hat{f}_2 \hat{f}_4(u) du - \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \int_{-\infty}^{\infty} \hat{f}_2 \hat{f}_4(u) du + \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \int_{-\infty}^{\infty} \hat{f}_2 \hat{f}_4(u) du + \\
& \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} \hat{f}_2 \hat{f}_4(u) du + 2 \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_2 \hat{f}_4(u) du - \\
& 2 \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_2 \hat{f}_4(u) du + 2 \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \int_{-\infty}^{\infty} |u| \hat{f}_3(u) \hat{f}_2 \hat{f}_4(u) du - \\
& 2 \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \int_{-\infty}^{\infty} |u| \hat{f}_3(u) \hat{f}_2 \hat{f}_4(u) du + 8 \int_{-\infty}^{\infty} |u| \hat{f}_1 \hat{f}_3(u) \hat{f}_2 \hat{f}_4(u) du + \\
& 2 \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_2 \hat{f}_4(u) du - 2 \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_2 \hat{f}_4(u) du + \\
& 16 \int_{-\infty}^{\infty} |u| \hat{f}_3(u) \hat{f}_1 \hat{f}_2 \hat{f}_4(u) du + \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \int_{-\infty}^{\infty} \hat{f}_3 \hat{f}_4(u) du - \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \int_{-\infty}^{\infty} \hat{f}_3 \hat{f}_4(u) du + 2 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_2(u) du \right) \int_{-\infty}^{\infty} \hat{f}_3 \hat{f}_4(u) du - \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \int_{-\infty}^{\infty} \hat{f}_3 \hat{f}_4(u) du + \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \int_{-\infty}^{\infty} \hat{f}_3 \hat{f}_4(u) du
\end{aligned}$$

$$\begin{aligned}
& + \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_2(u) du \right) \int_{-\infty}^{\infty} \hat{f}_3 \hat{f}_4(u) du + 2 \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_3 \hat{f}_4(u) du - \\
& 2 \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_3 \hat{f}_4(u) du + 2 \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_3 \hat{f}_4(u) du - \\
& 2 \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_3 \hat{f}_4(u) du + 8 \int_{-\infty}^{\infty} |u| \hat{f}_1 \hat{f}_2(u) \hat{f}_3 \hat{f}_4(u) du + \\
& 2 \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_3 \hat{f}_4(u) du - 2 \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_3 \hat{f}_4(u) du + \\
& 16 \int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_1 \hat{f}_3 \hat{f}_4(u) du + 2 \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \int_{-\infty}^{\infty} \hat{f}_2 \hat{f}_3 \hat{f}_4(u) du - \\
& 2 \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \int_{-\infty}^{\infty} \hat{f}_2 \hat{f}_3 \hat{f}_4(u) du + 16 \int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_2 \hat{f}_3 \hat{f}_4(u) du + 24 \int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_2 \hat{f}_3 \hat{f}_4(u) du - \\
& \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_1(x) dx + \\
& \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \left(\int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_1(x) dx - \\
& \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} |u| \hat{f}_3(u) \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_1(x) dx + \\
& \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} |u| \hat{f}_3(u) \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_1(x) dx - 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_2 \hat{f}_3(u) \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_1(x) dx + \\
& \frac{1}{8} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 \right) \int_{-\infty}^{\infty} f_1(x) dx - \\
& \frac{1}{8} \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 \right) \int_{-\infty}^{\infty} f_1(x) dx + \\
& \left(\int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 \right) \int_{-\infty}^{\infty} f_1(x) dx - \\
& \frac{1}{8} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 \right) \int_{-\infty}^{\infty} f_1(x) dx + \\
& \frac{1}{8} \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 \right) \int_{-\infty}^{\infty} f_1(x) dx + \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_2 \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 \right) \int_{-\infty}^{\infty} f_1(x) dx - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_2 \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_1(x) dx + \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \left(\int_{-\infty}^{\infty} \hat{f}_2 \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_1(x) dx - 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_3(u) \hat{f}_2 \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_1(x) dx
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3 \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_1(x) dx + \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} \hat{f}_3 \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_1(x) dx - 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_3 \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_1(x) dx - \\
& 4 \left(\int_{-\infty}^{\infty} \hat{f}_2 \hat{f}_3 \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_1(x) dx - \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_2(x) dx + \\
& \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_2(x) dx - \\
& \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} |u| \hat{f}_3(u) \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_2(x) dx + \\
& \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} |u| \hat{f}_3(u) \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_2(x) dx - 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1 \hat{f}_3(u) \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_2(x) dx + \\
& \frac{1}{8} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 \right) \int_{-\infty}^{\infty} f_2(x) dx - \\
& \frac{1}{8} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 \right) \int_{-\infty}^{\infty} f_2(x) dx + \\
& \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 \right) \int_{-\infty}^{\infty} f_2(x) dx - \\
& \frac{1}{8} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 \right) \int_{-\infty}^{\infty} f_2(x) dx + \\
& \frac{1}{8} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 \right) \int_{-\infty}^{\infty} f_2(x) dx + \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 \right) \int_{-\infty}^{\infty} f_2(x) dx - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_2(x) dx + \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_2(x) dx - 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_3(u) \hat{f}_1 \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_2(x) dx - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3 \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_2(x) dx + \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} \hat{f}_3 \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_2(x) dx - 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_3 \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_2(x) dx - \\
& 4 \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_3 \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_2(x) dx + 2 \left(\int_{-\infty}^{\infty} |u| \hat{f}_3(u) \hat{f}_4(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_1(x) dx \right) \int_{-\infty}^{\infty} f_2(x) dx - \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 \right) \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \int_{-\infty}^{\infty} f_2(x) dx
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{4} \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 \right) \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \int_{-\infty}^{\infty} f_2(x) dx + \\
& \left(\int_{-\infty}^{\infty} \hat{f}_3 \hat{f}_4(u) du \right) \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \int_{-\infty}^{\infty} f_2(x) dx - 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_3(u) \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} (f_1 f_2)(x) dx + \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 \right) \int_{-\infty}^{\infty} (f_1 f_2)(x) dx - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 \right) \int_{-\infty}^{\infty} (f_1 f_2)(x) dx - \\
& 2 \left(\int_{-\infty}^{\infty} \hat{f}_3 \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} (f_1 f_2)(x) dx - \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_3(x) dx + \\
& \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_3(x) dx - \\
& \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_3(x) dx + \\
& \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_3(x) dx - 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1 \hat{f}_2(u) \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_3(x) dx + \\
& \frac{1}{8} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 \right) \int_{-\infty}^{\infty} f_3(x) dx - \\
& \frac{1}{8} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 \right) \int_{-\infty}^{\infty} f_3(x) dx + \\
& \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 \right) \int_{-\infty}^{\infty} f_3(x) dx - \\
& \frac{1}{8} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 \right) \int_{-\infty}^{\infty} f_3(x) dx + \\
& \frac{1}{8} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 \right) \int_{-\infty}^{\infty} f_3(x) dx + \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 \right) \int_{-\infty}^{\infty} f_3(x) dx - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_3(x) dx + \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_3(x) dx - 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_1 \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_3(x) dx - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_2 \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_3(x) dx + \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} \hat{f}_2 \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_3(x) dx - 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_2 \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} f_3(x) dx
\end{aligned}$$

$$\begin{aligned}
& -4 \left(\int_{-\infty}^{\infty} f_1 \hat{f}_2 f_4(u) du \right) \int_{-\infty}^{\infty} f_3(x) dx + 2 \left(\int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_4(u) du \right) \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \int_{-\infty}^{\infty} f_3(x) dx - \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 \right) \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \int_{-\infty}^{\infty} f_3(x) dx + \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 \right) \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \int_{-\infty}^{\infty} f_3(x) dx + \\
& \left(\int_{-\infty}^{\infty} \hat{f}_2 \hat{f}_4(u) du \right) \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \int_{-\infty}^{\infty} f_3(x) dx + 2 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_4(u) du \right) \left(\int_{-\infty}^{\infty} f_2(x) dx \right) \int_{-\infty}^{\infty} f_3(x) dx - \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 \right) \left(\int_{-\infty}^{\infty} f_2(x) dx \right) \int_{-\infty}^{\infty} f_3(x) dx + \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 \right) \left(\int_{-\infty}^{\infty} f_2(x) dx \right) \int_{-\infty}^{\infty} f_3(x) dx + \\
& \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_4(u) du \right) \left(\int_{-\infty}^{\infty} f_2(x) dx \right) \int_{-\infty}^{\infty} f_3(x) dx + \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 \right) \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \left(\int_{-\infty}^{\infty} f_2(x) dx \right) \int_{-\infty}^{\infty} f_3(x) dx - \\
& \left(\int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 \right) \left(\int_{-\infty}^{\infty} (f_1 f_2)(x) dx \right) \int_{-\infty}^{\infty} f_3(x) dx - 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} (f_1 f_3)(x) dx + \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 \right) \int_{-\infty}^{\infty} (f_1 f_3)(x) dx - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 \right) \int_{-\infty}^{\infty} (f_1 f_3)(x) dx - \\
& 2 \left(\int_{-\infty}^{\infty} \hat{f}_2 \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} (f_1 f_3)(x) dx - \left(\int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 \right) \left(\int_{-\infty}^{\infty} f_2(x) dx \right) \int_{-\infty}^{\infty} (f_1 f_3)(x) dx - \\
& 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} (f_2 f_3)(x) dx + \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 \right) \int_{-\infty}^{\infty} (f_2 f_3)(x) dx - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 \right) \int_{-\infty}^{\infty} (f_2 f_3)(x) dx - \\
& 2 \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} (f_2 f_3)(x) dx - \left(\int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 \right) \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \int_{-\infty}^{\infty} (f_2 f_3)(x) dx + \\
& 4 \left(\int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 \right) \int_{-\infty}^{\infty} (f_1 f_2 f_3)(x) dx - \\
& \frac{1}{8} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} f_4(x) dx + \\
& \frac{1}{8} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} f_4(x) dx
\end{aligned}$$

$$\begin{aligned}
& - \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} f_4(x) dx + \\
& \frac{1}{8} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} f_4(x) dx - \\
& \frac{1}{8} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} f_4(x) dx - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} f_4(x) dx - \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} f_4(x) dx + \\
& \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} f_4(x) dx - \\
& \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} f_4(x) dx + \\
& \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} f_4(x) dx - 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1 \hat{f}_2(u) \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} f_4(x) dx + \\
& \frac{1}{8} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \int_{-\infty}^{\infty} f_4(x) dx - \\
& \frac{1}{8} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \int_{-\infty}^{\infty} f_4(x) dx + \\
& \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \int_{-\infty}^{\infty} f_4(x) dx - \\
& \frac{1}{8} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \int_{-\infty}^{\infty} f_4(x) dx + \\
& \frac{1}{8} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \int_{-\infty}^{\infty} f_4(x) dx + \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \int_{-\infty}^{\infty} f_4(x) dx - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} f_4(x) dx + \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} f_4(x) dx - 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_1 \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} f_4(x) dx - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_2 \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} f_4(x) dx + \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} \hat{f}_2 \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} f_4(x) dx - 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_2 \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} f_4(x) dx - \\
& 4 \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_2 \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} f_4(x) dx + \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_1(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{4} \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx + \\
& 2 \left(\int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx - \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx + \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx + \\
& \left(\int_{-\infty}^{\infty} \hat{f}_2 \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx + \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} f_2(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx - \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} f_2(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx + \\
& 2 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} f_2(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx - \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \left(\int_{-\infty}^{\infty} f_2(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx + \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \left(\int_{-\infty}^{\infty} f_2(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx + \\
& \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} f_2(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_1(x) dx \right) \left(\int_{-\infty}^{\infty} f_2(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx + \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \left(\int_{-\infty}^{\infty} \hat{f}_1(x) dx \right) \left(\int_{-\infty}^{\infty} f_2(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx + \\
& \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} (f_1 f_2)(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx - \\
& \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \left(\int_{-\infty}^{\infty} (f_1 f_2)(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx + \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx - \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx + \\
& 2 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx + \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx + \\
& \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_1(x) dx \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx + \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} \hat{f}_1(x) dx \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(x) dx \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx + \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(x) dx \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx + \\
& \left(\int_{-\infty}^{\infty} \hat{f}_1(x) dx \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(x) dx \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx - \\
& 2 \left(\int_{-\infty}^{\infty} (\hat{f}_1 \hat{f}_2)(x) dx \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx + \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} (\hat{f}_1 f_3)(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx - \\
& \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} (\hat{f}_1 f_3)(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx - \\
& 2 \left(\int_{-\infty}^{\infty} \hat{f}_2(x) dx \right) \left(\int_{-\infty}^{\infty} (\hat{f}_1 f_3)(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx + \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} (\hat{f}_2 f_3)(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx - \\
& \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} (\hat{f}_2 f_3)(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx - \\
& 2 \left(\int_{-\infty}^{\infty} \hat{f}_1(x) dx \right) \left(\int_{-\infty}^{\infty} (\hat{f}_2 f_3)(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx + 8 \left(\int_{-\infty}^{\infty} (\hat{f}_1 \hat{f}_2 f_3)(x) dx \right) \int_{-\infty}^{\infty} f_4(x) dx - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} (\hat{f}_1 \hat{f}_4)(x) dx + \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} (\hat{f}_1 \hat{f}_4)(x) dx - 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} (\hat{f}_1 \hat{f}_4)(x) dx + \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \int_{-\infty}^{\infty} (\hat{f}_1 \hat{f}_4)(x) dx - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \int_{-\infty}^{\infty} (\hat{f}_1 \hat{f}_4)(x) dx - \\
& 2 \left(\int_{-\infty}^{\infty} \hat{f}_2 \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} (\hat{f}_1 \hat{f}_4)(x) dx + \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(x) dx \right) \int_{-\infty}^{\infty} (\hat{f}_1 \hat{f}_4)(x) dx
\end{aligned}$$

$$\begin{aligned}
& - \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \left(\int_{-\infty}^{\infty} f_2(x) dx \right) \int_{-\infty}^{\infty} (f_1 f_4)(x) dx + \\
& \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} (f_1 f_4)(x) dx - \\
& \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} (f_1 f_4)(x) dx - \\
& 2 \left(\int_{-\infty}^{\infty} f_2(x) dx \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} (f_1 f_4)(x) dx + 4 \left(\int_{-\infty}^{\infty} (f_2 f_3)(x) dx \right) \int_{-\infty}^{\infty} (f_1 f_4)(x) dx - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} (f_2 f_4)(x) dx + \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} (f_2 f_4)(x) dx - 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} (f_2 f_4)(x) dx + \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \int_{-\infty}^{\infty} (f_2 f_4)(x) dx - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \int_{-\infty}^{\infty} (f_2 f_4)(x) dx - \\
& 2 \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} (f_2 f_4)(x) dx + \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \int_{-\infty}^{\infty} (f_2 f_4)(x) dx - \\
& \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \int_{-\infty}^{\infty} (f_2 f_4)(x) dx + \\
& \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} (f_2 f_4)(x) dx - \\
& \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} (f_2 f_4)(x) dx - \\
& 2 \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} (f_2 f_4)(x) dx + 4 \left(\int_{-\infty}^{\infty} (f_1 f_3)(x) dx \right) \int_{-\infty}^{\infty} (f_2 f_4)(x) dx - \\
& 4 \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} (f_1 f_2 f_4)(x) dx + 4 \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \int_{-\infty}^{\infty} (f_1 f_2 f_4)(x) dx + \\
& 8 \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} (f_1 f_2 f_4)(x) dx - \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \int_{-\infty}^{\infty} (f_3 f_4)(x) dx + \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \int_{-\infty}^{\infty} (f_3 f_4)(x) dx - 4 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_2(u) du \right) \int_{-\infty}^{\infty} (f_3 f_4)(x) dx + \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \int_{-\infty}^{\infty} (f_3 f_4)(x) dx - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \int_{-\infty}^{\infty} (f_3 f_4)(x) dx
\end{aligned}$$

$$\begin{aligned}
& -2 \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_2(u) du \right) \int_{-\infty}^{\infty} (f_3 f_4)(x) dx + \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \int_{-\infty}^{\infty} (f_3 f_4)(x) dx - \\
& \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \int_{-\infty}^{\infty} (f_3 f_4)(x) dx + \\
& \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} f_2(x) dx \right) \int_{-\infty}^{\infty} (f_3 f_4)(x) dx - \\
& \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} f_2(x) dx \right) \int_{-\infty}^{\infty} (f_3 f_4)(x) dx - \\
& 2 \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \left(\int_{-\infty}^{\infty} f_2(x) dx \right) \int_{-\infty}^{\infty} (f_3 f_4)(x) dx + 4 \left(\int_{-\infty}^{\infty} (f_1 f_2)(x) dx \right) \int_{-\infty}^{\infty} (f_3 f_4)(x) dx - \\
& 4 \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \int_{-\infty}^{\infty} (f_1 f_3 f_4)(x) dx + 4 \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \int_{-\infty}^{\infty} (f_1 f_3 f_4)(x) dx + \\
& 8 \left(\int_{-\infty}^{\infty} f_2(x) dx \right) \int_{-\infty}^{\infty} (f_1 f_3 f_4)(x) dx - 4 \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \int_{-\infty}^{\infty} (f_2 f_3 f_4)(x) dx + \\
& 4 \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \int_{-\infty}^{\infty} (f_2 f_3 f_4)(x) dx + 8 \left(\int_{-\infty}^{\infty} f_1(x) dx \right) \int_{-\infty}^{\infty} (f_2 f_3 f_4)(x) dx - \\
& 48 \int_{-\infty}^{\infty} (f_1 f_2 f_3 f_4)(x) dx - 2 \left(\int_{-\infty}^{\infty} |u| \hat{f}_3(u) \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_1 - u_2)) \chi(u_1 + u_2)) \hat{f}_1(u_1) \hat{f}_2(u_2) du_1 du_2 + \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_1 - u_2)) \chi(u_1 + u_2)) \hat{f}_1(u_1) \hat{f}_2(u_2) du_1 du_2 - \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_1 - u_2)) \chi(u_1 + u_2)) \hat{f}_1(u_1) \hat{f}_2(u_2) du_1 du_2 \\
& \left(\int_{-\infty}^{\infty} \hat{f}_3 \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_1 - u_2)) \chi(u_1 + u_2)) \hat{f}_1(u_1) \hat{f}_2(u_2) du_1 du_2 - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 \right) \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_1 - u_2)) \chi(u_1 + u_2)) \hat{f}_1(u_1) \hat{f}_2(u_2) du_1 du_2 + \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} f_4(x) dx \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_1 - u_2)) \chi(u_1 + u_2)) \hat{f}_1(u_1) \hat{f}_2(u_2) du_1 du_2 - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \left(\int_{-\infty}^{\infty} f_4(x) dx \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_1 - u_2)) \chi(u_1 + u_2)) \hat{f}_1(u_1) \hat{f}_2(u_2) du_1 du_2 - \\
& \left(\int_{-\infty}^{\infty} f_3(x) dx \right) \left(\int_{-\infty}^{\infty} f_4(x) dx \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_1 - u_2)) \chi(u_1 + u_2)) \hat{f}_1(u_1) \hat{f}_2(u_2) du_1 du_2 + \\
& 2 \left(\int_{-\infty}^{\infty} (f_3 f_4)(x) dx \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_1 - u_2)) \chi(u_1 + u_2)) \hat{f}_1(u_1) \hat{f}_2(u_2) du_1 du_2 - \\
& 2 \left(\int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_1 - u_3)) \chi(u_1 + u_3)) \hat{f}_1(u_1) \hat{f}_3(u_3) du_1 du_3
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_1 - u_3)) \chi(u_1 + u_3)) \hat{f}_1(u_1) \hat{f}_3(u_3) du_1 du_3 - \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_1 - u_3)) \chi(u_1 + u_3)) \hat{f}_1(u_1) \hat{f}_3(u_3) du_1 du_3 \\
& \left(\int_{-\infty}^{\infty} \hat{f}_2 \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_1 - u_3)) \chi(u_1 + u_3)) \hat{f}_1(u_1) \hat{f}_3(u_3) du_1 du_3 - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(x) dx \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_1 - u_3)) \chi(u_1 + u_3)) \hat{f}_1(u_1) \hat{f}_3(u_3) du_1 du_3 + \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_4(x) dx \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_1 - u_3)) \chi(u_1 + u_3)) \hat{f}_1(u_1) \hat{f}_3(u_3) du_1 du_3 - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} \hat{f}_4(x) dx \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_1 - u_3)) \chi(u_1 + u_3)) \hat{f}_1(u_1) \hat{f}_3(u_3) du_1 du_3 - \\
& \left(\int_{-\infty}^{\infty} \hat{f}_2(x) dx \right) \left(\int_{-\infty}^{\infty} \hat{f}_4(x) dx \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_1 - u_3)) \chi(u_1 + u_3)) \hat{f}_1(u_1) \hat{f}_3(u_3) du_1 du_3 + \\
& 2 \left(\int_{-\infty}^{\infty} (\hat{f}_2 \hat{f}_4)(x) dx \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_1 - u_3)) \chi(u_1 + u_3)) \hat{f}_1(u_1) \hat{f}_3(u_3) du_1 du_3 - \\
& 2 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_2 - u_3)) \chi(u_2 + u_3)) \hat{f}_2(u_2) \hat{f}_3(u_3) du_2 du_3 + \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_2 - u_3)) \chi(u_2 + u_3)) \hat{f}_2(u_2) \hat{f}_3(u_3) du_2 du_3 - \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_2 - u_3)) \chi(u_2 + u_3)) \hat{f}_2(u_2) \hat{f}_3(u_3) du_2 du_3 \\
& \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_4(u) du \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_2 - u_3)) \chi(u_2 + u_3)) \hat{f}_2(u_2) \hat{f}_3(u_3) du_2 du_3 - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} (1 - \chi(u_4)) \hat{f}_4(u_4) du_4 \right) \left(\int_{-\infty}^{\infty} \hat{f}_1(x) dx \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_2 - u_3)) \chi(u_2 + u_3)) \hat{f}_2(u_2) \hat{f}_3(u_3) du_2 du_3 + \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_4(x) dx \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_2 - u_3)) \chi(u_2 + u_3)) \hat{f}_2(u_2) \hat{f}_3(u_3) du_2 du_3 - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} \hat{f}_4(x) dx \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_2 - u_3)) \chi(u_2 + u_3)) \hat{f}_2(u_2) \hat{f}_3(u_3) du_2 du_3 - \\
& \left(\int_{-\infty}^{\infty} \hat{f}_1(x) dx \right) \left(\int_{-\infty}^{\infty} \hat{f}_4(x) dx \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_2 - u_3)) \chi(u_2 + u_3)) \hat{f}_2(u_2) \hat{f}_3(u_3) du_2 du_3 + \\
& 2 \left(\int_{-\infty}^{\infty} (\hat{f}_1 \hat{f}_4)(x) dx \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_2 - u_3)) \chi(u_2 + u_3)) \hat{f}_2(u_2) \hat{f}_3(u_3) du_2 du_3 - \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_1 - u_4)) \chi(u_1 + u_4)) \hat{f}_1(u_1) \hat{f}_4(u_4) du_1 du_4
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{4} \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_1 - u_4) \chi(u_1 + u_4)) \hat{f}_1(u_1) \hat{f}_4(u_4) du_1 du_4 - \\
& 2 \left(\int_{-\infty}^{\infty} |u| \hat{f}_2(u) \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_1 - u_4) \chi(u_1 + u_4)) \hat{f}_1(u_1) \hat{f}_4(u_4) du_1 du_4 + \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_1 - u_4) \chi(u_1 + u_4)) \hat{f}_1(u_1) \hat{f}_4(u_4) du_1 du_4 - \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_1 - u_4) \chi(u_1 + u_4)) \hat{f}_1(u_1) \hat{f}_4(u_4) du_1 du_4 \\
& \left(\int_{-\infty}^{\infty} \hat{f}_2 \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_1 - u_4) \chi(u_1 + u_4)) \hat{f}_1(u_1) \hat{f}_4(u_4) du_1 du_4 + \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(x) dx \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_1 - u_4) \chi(u_1 + u_4)) \hat{f}_1(u_1) \hat{f}_4(u_4) du_1 du_4 - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(x) dx \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_1 - u_4) \chi(u_1 + u_4)) \hat{f}_1(u_1) \hat{f}_4(u_4) du_1 du_4 + \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(x) dx \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_1 - u_4) \chi(u_1 + u_4)) \hat{f}_1(u_1) \hat{f}_4(u_4) du_1 du_4 - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(x) dx \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_1 - u_4) \chi(u_1 + u_4)) \hat{f}_1(u_1) \hat{f}_4(u_4) du_1 du_4 - \\
& \left(\int_{-\infty}^{\infty} \hat{f}_2(x) dx \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(x) dx \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_1 - u_4) \chi(u_1 + u_4)) \hat{f}_1(u_1) \hat{f}_4(u_4) du_1 du_4 + \\
& 2 \left(\int_{-\infty}^{\infty} (\hat{f}_2 \hat{f}_3)(x) dx \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_1 - u_4) \chi(u_1 + u_4)) \hat{f}_1(u_1) \hat{f}_4(u_4) du_1 du_4 + \\
& \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_2 - u_3) \chi(u_2 + u_3)) \hat{f}_2(u_2) \hat{f}_3(u_3) du_2 du_3 \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_1 - u_4) \chi(u_1 + u_4)) \hat{f}_1(u_1) \hat{f}_4(u_4) du_1 du_4 - \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_2 - u_4) \chi(u_2 + u_4)) \hat{f}_2(u_2) \hat{f}_4(u_4) du_2 du_4 + \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_2 - u_4) \chi(u_2 + u_4)) \hat{f}_2(u_2) \hat{f}_4(u_4) du_2 du_4 - \\
& 2 \left(\int_{-\infty}^{\infty} |u| \hat{f}_1(u) \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_2 - u_4) \chi(u_2 + u_4)) \hat{f}_2(u_2) \hat{f}_4(u_4) du_2 du_4 + \\
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& \frac{1}{4} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_2 - u_4) \chi(u_2 + u_4)) \hat{f}_2(u_2) \hat{f}_4(u_4) du_2 du_4 \\
& \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_3(u) du \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_2 - u_4) \chi(u_2 + u_4)) \hat{f}_2(u_2) \hat{f}_4(u_4) du_2 du_4
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_1(x) dx \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_2 - u_4) \chi(u_2 + u_4)) \hat{f}_2(u_2) \hat{f}_4(u_4) du_2 du_4 - \\
& \frac{1}{2} \left(\int_{-\infty}^{\infty} (1 - \chi(u_3)) \hat{f}_3(u_3) du_3 \right) \left(\int_{-\infty}^{\infty} \hat{f}_1(x) dx \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_2 - u_4) \chi(u_2 + u_4)) \hat{f}_2(u_2) \hat{f}_4(u_4) du_2 du_4 + \\
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& \left(\int_{-\infty}^{\infty} \hat{f}_1(x) dx \right) \left(\int_{-\infty}^{\infty} \hat{f}_3(x) dx \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_2 - u_4) \chi(u_2 + u_4)) \hat{f}_2(u_2) \hat{f}_4(u_4) du_2 du_4 + \\
& 2 \left(\int_{-\infty}^{\infty} (\hat{f}_1 \hat{f}_3)(x) dx \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_2 - u_4) \chi(u_2 + u_4)) \hat{f}_2(u_2) \hat{f}_4(u_4) du_2 du_4 + \\
& \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_1 - u_3) \chi(u_1 + u_3)) \hat{f}_1(u_1) \hat{f}_3(u_3) du_1 du_3 \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_2 - u_4) \chi(u_2 + u_4)) \hat{f}_2(u_2) \hat{f}_4(u_4) du_2 du_4 - \\
& \frac{1}{4} \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_3 - u_4) \chi(u_3 + u_4)) \hat{f}_3(u_3) \hat{f}_4(u_4) du_3 du_4 + \\
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& \frac{1}{4} \left(\int_{-\infty}^{\infty} (1 - \chi(u_1)) \hat{f}_1(u_1) du_1 \right) \left(\int_{-\infty}^{\infty} (1 - \chi(u_2)) \hat{f}_2(u_2) du_2 \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_3 - u_4) \chi(u_3 + u_4)) \hat{f}_3(u_3) \hat{f}_4(u_4) du_3 du_4 \\
& \left(\int_{-\infty}^{\infty} \hat{f}_1 \hat{f}_2(u) du \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_3 - u_4) \chi(u_3 + u_4)) \hat{f}_3(u_3) \hat{f}_4(u_4) du_3 du_4 + \\
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& \left(\int_{-\infty}^{\infty} \hat{f}_1(x) dx \right) \left(\int_{-\infty}^{\infty} \hat{f}_2(x) dx \right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (1 - \chi(u_3 - u_4) \chi(u_3 + u_4)) \hat{f}_3(u_3) \hat{f}_4(u_4) du_3 du_4
\end{aligned}$$

$$\begin{aligned}
& +2\left(\int_{-\infty}^{\infty} (f_1 f_2)(x) dx\right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(1 - \chi(u_3 - u_4)\chi(u_3 + u_4)\right) \hat{f}_3(u_3) \hat{f}_4(u_4) du_3 du_4 + \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(1 - \chi(u_1 - u_2)\chi(u_1 + u_2)\right) \hat{f}_1(u_1) \hat{f}_2(u_2) du_1 du_2\right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(1 - \chi(u_3 - u_4)\chi(u_3 + u_4)\right) \hat{f}_3(u_3) \hat{f}_4(u_4) du_3 du_4 + \left(\int_{-\infty}^{\infty} \left(1 - \chi(u_4)\right) \hat{f}_4(u_4) du_4\right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(2 - \chi(u_1 - u_2 - u_3)\chi(u_1 + u_2 - u_3)\chi(u_1 + u_2 + u_3) - \chi(u_1 - u_2 - u_3)\chi(u_1 - u_2 + u_3)\chi(u_1 + u_2 + u_3)\right) \hat{f}_1(u_1) \hat{f}_2(u_2) \hat{f}_3(u_3) du_1 du_2 du_3 + 2\left(\int_{-\infty}^{\infty} f_4(x) dx\right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(2 - \chi(u_1 - u_2 - u_3)\chi(u_1 + u_2 - u_3)\chi(u_1 + u_2 + u_3) - \chi(u_1 - u_2 - u_3)\chi(u_1 - u_2 + u_3)\chi(u_1 + u_2 + u_3)\right) \hat{f}_1(u_1) \hat{f}_2(u_2) \hat{f}_3(u_3) du_1 du_2 du_3 - \left(\int_{-\infty}^{\infty} \hat{f}_3(u) du\right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(2 - \chi(u_1 - u_2 - u_4)\chi(u_1 + u_2 - u_4)\chi(u_1 + u_2 + u_4) - \chi(u_1 - u_2 - u_4)\chi(u_1 - u_2 + u_4)\chi(u_1 + u_2 + u_4)\right) \hat{f}_1(u_1) \hat{f}_2(u_2) \hat{f}_4(u_4) du_1 du_2 du_4 + \left(\int_{-\infty}^{\infty} \left(1 - \chi(u_3)\right) \hat{f}_3(u_3) du_3\right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(2 - \chi(u_1 - u_2 - u_4)\chi(u_1 + u_2 - u_4)\chi(u_1 + u_2 + u_4) - \chi(u_1 - u_2 - u_4)\chi(u_1 - u_2 + u_4)\chi(u_1 + u_2 + u_4)\right) \hat{f}_1(u_1) \hat{f}_2(u_2) \hat{f}_4(u_4) du_1 du_2 du_4 + 2\left(\int_{-\infty}^{\infty} f_3(x) dx\right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(2 - \chi(u_1 - u_2 - u_4)\chi(u_1 + u_2 - u_4)\chi(u_1 + u_2 + u_4) - \chi(u_1 - u_2 - u_4)\chi(u_1 - u_2 + u_4)\chi(u_1 + u_2 + u_4)\right) \hat{f}_1(u_1) \hat{f}_2(u_2) \hat{f}_4(u_4) du_1 du_2 du_4 - \left(\int_{-\infty}^{\infty} \hat{f}_2(u) du\right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(2 - \chi(u_1 - u_3 - u_4)\chi(u_1 + u_3 - u_4)\chi(u_1 + u_3 + u_4) - \chi(u_1 - u_3 - u_4)\chi(u_1 - u_3 + u_4)\chi(u_1 + u_3 + u_4)\right) \hat{f}_1(u_1) \hat{f}_3(u_3) \hat{f}_4(u_4) du_1 du_3 du_4 + \left(\int_{-\infty}^{\infty} \left(1 - \chi(u_2)\right) \hat{f}_2(u_2) du_2\right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(2 - \chi(u_1 - u_3 - u_4)\chi(u_1 + u_3 - u_4)\chi(u_1 + u_3 + u_4) - \chi(u_1 - u_3 - u_4)\chi(u_1 - u_3 + u_4)\chi(u_1 + u_3 + u_4)\right) \hat{f}_1(u_1) \hat{f}_3(u_3) \hat{f}_4(u_4) du_1 du_3 du_4 + 2\left(\int_{-\infty}^{\infty} f_2(x) dx\right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(2 - \chi(u_1 - u_3 - u_4)\chi(u_1 + u_3 - u_4)\chi(u_1 + u_3 + u_4) - \chi(u_1 - u_3 - u_4)\chi(u_1 - u_3 + u_4)\chi(u_1 + u_3 + u_4)\right) \hat{f}_1(u_1) \hat{f}_3(u_3) \hat{f}_4(u_4) du_1 du_3 du_4 - \left(\int_{-\infty}^{\infty} \hat{f}_1(u) du\right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(2 - \chi(u_2 - u_3 - u_4)\chi(u_2 + u_3 - u_4)\chi(u_2 + u_3 + u_4) - \chi(u_2 - u_3 - u_4)\chi(u_2 - u_3 + u_4)\chi(u_2 + u_3 + u_4)\right) \hat{f}_2(u_2) \hat{f}_3(u_3) \hat{f}_4(u_4) du_2 du_3 du_4 + \left(\int_{-\infty}^{\infty} \left(1 - \chi(u_1)\right) \hat{f}_1(u_1) du_1\right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(2 - \chi(u_2 - u_3 - u_4)\chi(u_2 + u_3 - u_4)\chi(u_2 + u_3 + u_4) - \chi(u_2 - u_3 - u_4)\chi(u_2 - u_3 + u_4)\chi(u_2 + u_3 + u_4)\right) \hat{f}_2(u_2) \hat{f}_3(u_3) \hat{f}_4(u_4) du_2 du_3 du_4
\end{aligned}$$

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& +2\left(\int_{-\infty}^{\infty} f_1(x) dx\right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(2 - \chi(u_2 - u_3 - u_4)\chi(u_2 + u_3 - u_4)\chi(u_2 + u_3 + u_4) - \chi(u_2 - u_3 - u_4)\chi(u_2 - \right. \\
& \left. u_3 + u_4)\chi(u_2 + u_3 + u_4)\right) \hat{f}_2(u_2)\hat{f}_3(u_3)\hat{f}_4(u_4) du_2 du_3 du_4 - 4 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(6 - \chi(u_1 - u_2 - u_3 - u_4)\chi(u_1 + \right. \\
& \left. u_2 - u_3 - u_4)\chi(u_1 + u_2 + u_3 - u_4)\chi(u_1 + u_2 + u_3 + u_4) - \chi(u_1 - u_2 - u_3 - u_4)\chi(u_1 - u_2 + u_3 - u_4)\chi(u_1 + u_2 + \right. \\
& \left. u_3 - u_4)\chi(u_1 + u_2 + u_3 + u_4) - \chi(u_1 - u_2 - u_3 - u_4)\chi(u_1 + u_2 - u_3 - u_4)\chi(u_1 + u_2 - u_3 + u_4)\chi(u_1 + u_2 + u_3 + \right. \\
& \left. u_4) - \chi(u_1 - u_2 - u_3 - u_4)\chi(u_1 - u_2 - u_3 + u_4)\chi(u_1 + u_2 - u_3 + u_4)\chi(u_1 + u_2 + u_3 + u_4) - \chi(u_1 - u_2 - u_3 - \right. \\
& \left. u_4)\chi(u_1 - u_2 + u_3 - u_4)\chi(u_1 - u_2 + u_3 + u_4)\chi(u_1 + u_2 + u_3 + u_4) - \chi(u_1 - u_2 - u_3 - u_4)\chi(u_1 - u_2 - u_3 + \right. \\
& \left. u_4)\chi(u_1 - u_2 + u_3 + u_4)\chi(u_1 + u_2 + u_3 + u_4)\right) \hat{f}_1(u_1)\hat{f}_2(u_2)\hat{f}_3(u_3)\hat{f}_4(u_4) du_1 du_2 du_3 du_4. \text{ if you can read this, you} \\
& \text{don't need glasses}
\end{aligned}$$

Random Matrix Theory

Example question: What is the expected largest eigenvalue?

Consider a real symmetric 2×2 matrix

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

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Characteristic polynomial is

$$\begin{aligned} c_A(\lambda) &= \det(\lambda I - A) \\ &= \lambda^2 - (a + c)\lambda + (ac - b^2), \end{aligned}$$

with roots

$$\lambda_{\pm} = \frac{(a + c) \pm \sqrt{4b^2 + (a - c)^2}}{2}.$$

Suppose a, b, c are drawn independently from a probability distribution P on the interval $[0, 1]$.

The largest eigenvalue is

$$\lambda_+ = \lambda_+(a, b, c) = \frac{(a + c) + \sqrt{4b^2 + (a - c)^2}}{2}.$$

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The largest eigenvalue is

$$\lambda_+ = \lambda_+(a, b, c) = \frac{(a + c) + \sqrt{4b^2 + (a - c)^2}}{2}.$$

So, the expected largest eigenvalue is

$$\mathbb{E}(\lambda_+) = \int_0^1 \int_0^1 \int_0^1 \lambda_+(a, b, c) \underbrace{P(a)P(b)P(c) da \, db \, dc}_{dA}.$$

If $P(x)$ is the uniform distribution, $\mathbb{E}(\lambda_+) \approx 1.05$.

Classical Compact Groups of Matrices

$G(N)$	Matrix Ensemble
$U(N)$	Unitary $N \times N$ matrices
$SO(\text{even})$	Unitary orthogonal $N \times N$ matrices (N even)
$SO(\text{odd})$	Unitary orthogonal $N \times N$ matrices (N odd)
$USp(N)$	Unitary symplectic $N \times N$ matrices

- ▶ Unitary matrix A : $AA^* = A^*A = I$.
- ▶ Eigenvalues are $e^{i\theta}$, $\theta \in [-\pi, \pi)$.

eigenvalues $e^{i\theta}$ on unit circle $\leftrightarrow$ eigenangles θ on straight line

- ▶ $G(N)$ is a probability space
 - ▶ can integrate (Haar measure)
 - ▶ can study eigenvalue statistics

Eigenvalue Statistics

- ▶ Physics: eigenvalues $\leftrightarrow$ energy levels of complex systems
- ▶ Spacings, distributions, clusters, . . .

Eigenvalue Statistics

- ▶ Physics: eigenvalues $\leftrightarrow$ energy levels of complex systems
- ▶ Spacings, distributions, clusters, ...

Important statistics:

- ▶ n -level correlation: distribution of clusters of n eigenvalues
 - ▶ (Katz and Sarnak) Same for all classical compact groups!
- ▶ n -level density: distribution of eigenvalues near $\theta = 0$.
 - ▶ (Katz and Sarnak) Different for each group!

Analytic Number Theory

Studies questions like:

- ▶ How many prime numbers in $\{1, \dots, n\}$?
- ▶ How large are the spacings between prime numbers?
- ▶ How common are clusters of primes?

Techniques used for these questions can be used to study primes in arithmetic progressions, elliptic curves, number fields, ...

The Riemann-Zeta Function

The Riemann-Zeta function $\zeta(s)$ ($s \in \mathbb{C}$, $\Re(s) > 1$) is defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

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$$\Lambda(s) = g(s)\zeta(s) = \Lambda(1-s).$$

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Riemann Hypothesis: All the zeroes with $0 < \Re(s) < 1$ are of the form $\frac{1}{2} + i\gamma$, $\gamma \in \mathbb{R}$.

Why care about $\zeta(s)$?

- ▶ **Euler product** for ζ relates the prime numbers to the integers
- ▶ Locations of prime numbers connected to locations of zeroes of $\zeta(s)$:
 - ▶ **Explicit formula** relating # of prime numbers in $[0, x]$ to a sum over the nontrivial zeroes of $\zeta(s)$
 - ▶ Error term is smallest if all zeroes have real part $\frac{1}{2}$

Why care about $\zeta(s)$?

- ▶ **Euler product** for ζ relates the prime numbers to the integers
- ▶ Locations of prime numbers connected to locations of zeroes of $\zeta(s)$:
 - ▶ **Explicit formula** relating # of prime numbers in $[0, x]$ to a sum over the nontrivial zeroes of $\zeta(s)$
 - ▶ Error term is smallest if all zeroes have real part $\frac{1}{2}$
- ▶ **Spacing statistics** appear to be given by RMT ensembles

L -functions generalize the ζ function

An L -function $L(s, f)$ is defined by a series

$$L(s, f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s}.$$

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Quadratic Dirichlet L -Functions

Let $\chi_d : \mathbb{Z} \rightarrow \mathbb{Z}$ be a quadratic Dirichlet character with conductor d . The L -function for χ_d is defined by

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Family : $\mathcal{F} = \{L(s, \chi_d) \mid d = 8d', d' \in \mathbb{N} \text{ odd, squarefree}\}.$

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GRH: zeroes of $L(s, f)$ are $\frac{1}{2} + i\gamma_L^{(i)},$

$$\dots \leq \gamma_L^{(-2)} \leq \gamma_L^{(-1)} < 0 \leq \gamma_L^{(1)} \leq \gamma_L^{(2)} \leq \dots$$

n -level Densities - Number Theory

Let $f_1, \dots, f_n$ be even Schwartz functions. For a fixed $L(s, \chi_d)$ we let

$$D(L; f_1, \dots, f_n) = \sum_{\substack{j_1, \dots, j_n \\ j_i \neq \pm j_k}} f_1(\tilde{\gamma}_L^{(j_1)}) \cdots f_n(\tilde{\gamma}_L^{(j_n)}).$$

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n -level Densities - Random Matrix Theory

Let $f_1, \dots, f_n$ be even Schwartz functions. For a fixed $A \in G(N)$ we let

$$D(A; f_1, \dots, f_n) = \sum_{\substack{j_1, \dots, j_n \\ j_i \neq \pm j_k}} f_1(\tilde{\theta}_L^{(j_1)}) \cdots f_n(\tilde{\theta}_L^{(j_n)}).$$

We **average over** $G(N)$:

$$D(G(N); f) = \int_{G(N)} D(A; f_1, \dots, f_n) dA.$$

We study the limit

$$\lim_{N \rightarrow \infty} D(G(N); f) = \int_{\mathbb{R}_{\geq 0}^n} f(\mathbf{x}) W_G^{(n)}(\mathbf{x}) d\mathbf{x}.$$

The Katz-Sarnak Density Conjecture

- ▶ We associate one of the matrix groups G to our family of L -functions
(Quadratic Dirichlet L -functions: $G = \mathbf{USp}$ (symplectic).)
- ▶ Katz-Sarnak Density Conjecture:

$$\underbrace{\lim_{X \rightarrow \infty} D(\mathcal{F}, X; f)}_{\text{density of zeros}} = \underbrace{\int_{\mathbb{R}^n} f(\mathbf{x}) W_G^{(n)}(\mathbf{x}) d\mathbf{x}}_{\text{eigenvalue density}}.$$

Thesis Results

Previous results:

Let $f_1, \dots, f_n$ be even Schwartz functions, and consider the density conjecture

$$\lim_{X \rightarrow \infty} D(\mathcal{F}(X); f_1, \dots, f_n) = \int_{R^n_{\geq 0}} W_{USp}^{(n)}(\mathbf{x}) \prod_{i=1}^n f_i(x_i) dx_i. \quad (*)$$

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Theorem (Levinson, 2011)

If $\hat{f}_1, \dots, \hat{f}_n$ are supported in $\sum_{i=1}^n |u_i| < 2$, then equality holds in $(*)$ for $n = 4, 5, 6$.

In Need of Support

- ▶ If $\hat{f}_1$ is supported in $(-1, 1)$, then

$$\int_{|u|>1} \hat{f}_1(u) du = 0.$$

- ▶ If $\hat{f}_1, \hat{f}_2$ are supported in $|u_1| + |u_2| < 1$, then

$$\int_{|u_1|+|u_2|>1} \hat{f}_1(u_1) \hat{f}_2(u_2) du_1 du_2 = 0.$$

- ▶ More support $\leftrightarrow$ less cancellation

Key Step

- ▶ RMT and NT already computed; equality for $n \leq 3$
- ▶ Combinatorial / Fourier obstructions:

$$\int_{|u|>1} \hat{f}(u) du = 2 \int_0^\infty (1 - \mathbb{I}_{[1,\infty)}(u)) \hat{f}(u) du$$
$$\int_{|u|>1} \widehat{fg}(u) du = \int \int_{|u|>1} \hat{f}(v) \hat{g}(v-u) dv du$$

- ▶ Key step: find **canonical** forms for integrands (Fourier transform identities) and indicator functions.

Canonical Forms

$$\int_{\mathbb{R}^2} (1 - \chi(u+v)\chi(u-v)) \widehat{f_1 f_2}(u) \widehat{f_3 f_4}(v) du dv$$

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$$\int_{\mathbb{R}^4} (1 - \chi(u_1 + u_2 + u_3 + u_4)\chi(u_1 + u_2 - u_3 - u_4)) \prod_{i=1}^4 \widehat{f_i}(u_i) du_i,$$

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$\Downarrow$ replace $\chi = \mathbb{I}_{[-1,1]}$ with $\tilde{\chi} = \mathbb{I}_{[1,\infty)}$

$\Downarrow$ split over hyperoctants of $\mathbb{R}^4$

$$\sum \text{ terms of the form } \int_{\mathbb{R}_{\geq 0}^4} \tilde{\chi}(\sum \pm u_i) \tilde{\chi}(\sum \pm u_i) \prod_{i=1}^4 \hat{f}_i(u_i) du_i.$$

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4a*. Special case: all $\hat{f}_i$'s have same support	0
4b*. General case: Rewrite terms in canonical form #2	350
5. Simplify terms using support argument	50
6. Simplify terms using combinatorial argument	0

Current efforts

Two goals:

- ▶ **Counting** pieces that appear identically in both expressions
- ▶ **Establishing** equality between remaining pieces

Acknowledgments

- ▶ Thanks to my advisor, Steven J. Miller
- ▶ NSF Grant DMS0970067
- ▶ Questions?