Distribution of Gaps between Summands in Zeckendorf Decompositions

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Introduction

A few questions

- How can we write a number as a sum of powers of 2?

Example: $2011 = 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^4 + 2^3 + 2^1 + 2^0$

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- What can we say about the distribution of the summands?
- What about powers of 3? 5? 10?
- What other sequences can we use besides powers?

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$F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, \ldots$.

- Generating function: $g(x) = \sum_{n>0} \boldsymbol{F}_n x^n$.

$$
\begin{aligned}
(1) \quad &\Rightarrow \sum_{n\geq 2} \boldsymbol{F}_{n+1} x^{n+1} = \sum_{n\geq 2} \boldsymbol{F}_n x^{n+1} + \sum_{n\geq 2} \boldsymbol{F}_{n-1} x^{n+1} \\
&\Rightarrow \sum_{n\geq 3} \boldsymbol{F}_n x^n = \sum_{n\geq 2} \boldsymbol{F}_n x^{n+1} + \sum_{n\geq 1} \boldsymbol{F}_n x^{n+2} \\
&\Rightarrow \sum_{n\geq 3} \boldsymbol{F}_n x^n = x \sum_{n\geq 2} \boldsymbol{F}_n x^n + x^2 \sum_{n\geq 1} \boldsymbol{F}_n x^n \\
&\Rightarrow g(x) - \boldsymbol{F}_1 x - \boldsymbol{F}_2 x^2 = x(g(x) - \boldsymbol{F}_1 x) + x^2 g(x) \\
&\Rightarrow g(x) = x/(1 - x - x^2).
\end{aligned}
$$

The Fibonacci Numbers, Continued

Partial Fraction Expansion:

$$g(x) = \frac{x}{1 - x - x^2} = \frac{1}{\sqrt{5}} \left(\frac{1}{1 - x(\frac{1+\sqrt{5}}{2})} - \frac{1}{1 - x(\frac{1-\sqrt{5}}{2})} \right).$$

- Binet's formula follows from geometric series expansion $\frac{1}{1-r} = \sum_{k=0}^{\infty} r^k$.

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Lekkerkerker's Theorem (1952)

The average number of summands in the Zeckendorf decomposition for integers in $[F_n, F_{n+1})$ tends to $\frac{n}{\varphi^2+1} \approx .276n$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden mean.

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Central Limit Type Theorem (KKMW 2010)

As $n \to \infty$, the distribution of the number of summands in the Zeckendorf decomposition for integers in $[F_n, F_{n+1})$ is Gaussian.

Preliminaries: The Cookie Problem

The Cookie Problem

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Example:10 cookies and 5 people ($C=10$, $P=5$):

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For $N \in [F_n, F_{n+1})$, the largest summand is F_n.

$$N = F_{i_1} + F_{i_2} + \cdots + F_{i_{k-1}} + F_n,$$
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$$d_1 := i_1 - 1,\ d_j := i_j - i_{j-1} - 2\ (j > 1).$$

$$d_1 + d_2 + \cdots + d_k = n - 2k + 1,\ d_j \geq 0.$$

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Cookie counting $\Rightarrow p_{n,k} = \binom{n-2k+1-k-1}{k-1} = \binom{n-k}{k-1}$.

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By cookie problem, the number of decompositions with largest summand F_n is $\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-1-k}{k} = F_{n-1}$.

Distribution of Gaps

For $F_{i_1} + F_{i_2} + \cdots + F_{i_n}$, the gaps are the differences $i_n - i_{n-1}, i_{n-1} - i_{n-2}, \ldots, i_2 - i_1$.

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Let $P_n(k)$ be the probability that a gap for a decomposition in $[F_n, F_{n+1})$ is of length k.

What is $P(k) = \lim_{n\to\infty} P_n(k)$?

Main Results

Theorem (Base B Gap Distribution)

For base B decompositions, $P(0) = \frac{(B-1)(B-2)}{B^2}$, *and for* $k \geq 1$, $P(k) = c_B B^{-k}$, *with* $c_B = \frac{(B-1)(3B-2)}{B^2}$.

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Theorem (Zeckendorf Gap Distribution)

For Zeckendorf decompositions, $P(k) = \frac{\phi(\phi-1)}{\phi^k}$ *for* $k \geq 2$*, with* $\phi = \frac{1+\sqrt{5}}{2}$ *the golden mean.*

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$$P(k) = \lim_{n\to\infty} \frac{\sum_{i=1}^{n-k} x_{i,i+k}}{F_{n-1}\frac{n}{\phi^2+1}}.$$

Calculating $x_{i,i+k}$

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For the indices greater than $i+k$: $F_{n-k-2-i}$ choices.

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$i = n-k-1$: 0.

$i = n-k$: F_{n-k-1}.

$$\sum_{i=1,n-k} x_{i,i+k} = F_{n-k-1} + \sum_{i=1}^{n-k-2} F_{i-1}F_{n-k-i-2}$$
$$= F_{n-k-1} + \sum_{i=0}^{n-k-3} F_i F_{n-k-i-3}$$

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$(g(x))^2 = \frac{A}{(1-\phi x)^2} + \frac{B}{(1-x(1-\phi))^2} + \frac{C}{1-\phi x} + \frac{D}{1-(1-\phi)x}$

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Using geometric series, the x^m coefficient is
$A(m+1)\phi^m + B(m+1)(1-\phi)^m + C\phi^+ D(1-\phi)^m$.

Consider the ratio:

$$\begin{aligned}
\frac{P(k+1)}{P(k)} &= \lim_{n\to\infty} \frac{P_n(k+1)}{P_n(k)} \\
&= \lim_{n\to\infty} \frac{F_{n-k-2} + \sum_{i=0}^{n-k-4} F_i F_{n-k-i-4}}{F_{n-k-1} + \sum_{i=0}^{n-k-3} F_i F_{n-k-i-4}} \\
&= \frac{F_{n-k-2} + A(n-k-3)\phi^{n-k-4} + B(n-k-3)(1-\phi)^{n-k-4} + C\phi^{n-k-4} + D(1-\phi)^{n-k-4}}{F_{n-k-1} + A(n-k-2)\phi^{n-k-3} + B(n-k-2)(1-\phi)^{n-k-3} + C\phi^{n-k-3} + D(1-\phi)^{n-k-3}} \\
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- For general b, the percent of gaps of length 0 is $\frac{(b-1)(b-2)}{2}$, geometric series for longer gaps.

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- Length zero gaps are allowed, for example, $20 = 2 \times 10^1$ has one length zero gap.
- Delta spike for $k = 0$, geometric series otherwise.
- For general b, the percent of gaps of length 0 is $\frac{(b-1)(b-2)}{2}$, geometric series for longer gaps.