Spectral Statistics of Non-Hermitian Matrix Ensembles

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Introduction

Random Matrix Ensembles

A **random matrix ensemble** is a collection of matrices, with some probability assigned to each matrix.

For us, we will consider ensembles with matrix probabilities given by a product of entry probabilities:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1N} & a_{2N} & a_{3N} & \cdots & a_{NN} \end{pmatrix}$$

Fix p, define

$$\mathrm{Prob}(A) = \prod_{1 \leq i,j \leq N} p(a_{ij}).$$

Random Matrix Ensembles - Examples

Examples:

- Real symmetric matrices
- Hermitian matrices
- Real asymmetric matrices
- Complex symmetric/asymmetric matrices

Real eigenvalues: Real symmetric matrices, Hermitian matrices

Complex eigenvalues: Real asymmetric matrices, Complex symmetric/asymmetric matrices

Classical Random Matrix Theory

The basic question: Given a random matrix ensemble, what does the distribution of eigenvalues look like as we send the matrix dimension $N \to \infty$?

Wigner's Semi-Circle Law (Wigner 1958)

$N \times N$ real symmetric matrices, entries i.i.d.r.v. from a fixed $p(x)$ with mean 0, variance 1, and other moments finite. Then for almost all A, as $N \to \infty$

$$\mu_{A,N}(x) \longrightarrow \begin{cases} \frac{2}{\pi}\sqrt{1-x^2} & \text{if } |x| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

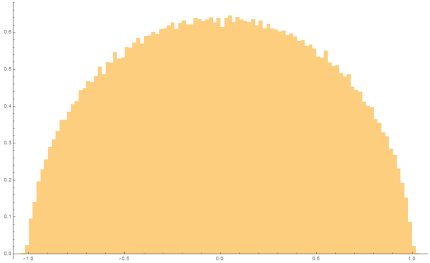
Proof Sketch - Wigner Semicircle

The *kth-moments* M_k of a probability distribution μ are given by $M_k := \int_{-\infty}^{\infty} x^k d\mu$.

Let A be an $N \times N$ matrix with eigenvalues λ_i.

- $M_k(N) := \frac{1}{N} \cdot \mathbb{E}\left[\sum_{i=1}^{N} \left(\frac{\lambda_i}{\sqrt{N}}\right)^k\right]$
- $\sum_{i=1}^{N} \lambda_i^k = \mathrm{Trace}(A^k) = \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} a_{i_1 i_2} \cdots a_{i_k i_1}$

Numerical Support



(Rescaled) 500 Matrices (Gaussian) 250×250

$$p(x) \;=\; \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}$$

The even moments are given by the Catalan numbers.

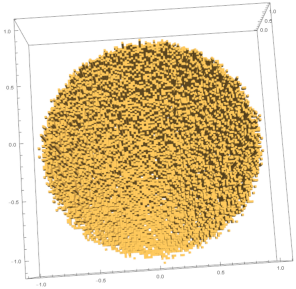
Ensembles with Complex Eigenvalues

Moment analysis has its limitations - for example, in the complex plane, every radially symmetric distribution has all moments zero!

Circular Law - Complex Asymmetric (Tao and Vu, 2008)

Consider the ensemble of $N \times N$ complex asymmetric random matrices, with iidrv complex random variables with mean 0 and variance 1. Then, after normalizing by $\sqrt{N}$, the eigenvalue distribution converges to the uniform distribution on the unit disk.

Numerical Support



(Rescaled) 500 Matrices (Gaussian) 250×250

We will study complex symmetric matrices. It turns out that, for sufficient conditions on the higher moments of the base distribution, we also obtain a circular law:

Circular Law - Complex Symmetric (SMALL 2017)

Consider the ensemble of $N \times N$ complex symmetric random matrices, with iidrv complex random variables with mean 0 and variance 1. Then, after normalizing by $\sqrt{N}$, the eigenvalue distribution converges to the uniform distribution on the unit disk.

Complex Symmetric Matrices - Singular Values

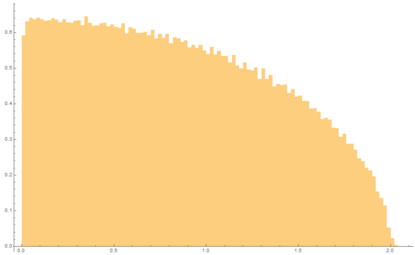
Singular Values

- The *singular values* of a matrix A are defined as the square roots of the eigenvalues of A^*A.
- Since A^*A is Hermitian, these eigenvalues will be real.
- Furthermore, A^*A positive definite implies all the eigenvalues are nonnegative.

Quarter Circular Law - Complex Asymmetric (Marchenko and Pastur, 1967)

Consider the ensemble of $N \times N$ complex asymmetric random matrices, with iidrv complex random variables with mean 0 and variance 1. Then, after normalizing by $\sqrt{N}$, the singular value distribution converges to a quarter circle.

Numerical Support



(Rescaled) 500 Matrices (Gaussian) 250 $\times$ 250

The Complex Symmetric Case

It turns out that, for sufficient conditions on the higher moments of the base distribution, we also obtain a quarter circular law for complex symmetric matrices:

Quarter Circular Law - Complex Symmetric (SMALL 2017)

Consider the ensemble of $N \times N$ complex symmetric random matrices, with iidrv complex random variables with mean 0 and variance 1. Then, after normalizing by $\sqrt{N}$, the singular value distribution converges to a quarter circular law.

Joint Density Functions (Gaussian Distributions)

Joint Density - Real Asymmetric Singular Values (James 1960)

$$\rho_N(z_1, \ldots, z_N) = c_N \prod_{1 \le i < j \le N} |z_j^2 - z_i^2| \prod_{1 \le i \le N} e^{-|z_i|^2}$$

Joint Density - Complex Asymmetric Singular Values (James 1963)

$$\rho_N(z_1, \ldots, z_N) = c_N \prod_{1 \le i < j \le N} |z_j^2 - z_i^2|^2 \prod_{1 \le i \le N} |z_i| \prod_{1 \le i \le N} e^{-|z_i|^2}$$

Joint Density - Complex Symmetric Singular Values (SMALL 2017)

$$\rho_N(z_1, \ldots, z_N) = c_N \prod_{1 \le i < j \le N} |z_j^2 - z_i^2| \prod_{1 \le i \le N} |z_i| \prod_{1 \le i \le N} e^{-|z_i|^2}$$

Joint Densities - 2-tuples of singular values of 2×2 matrices

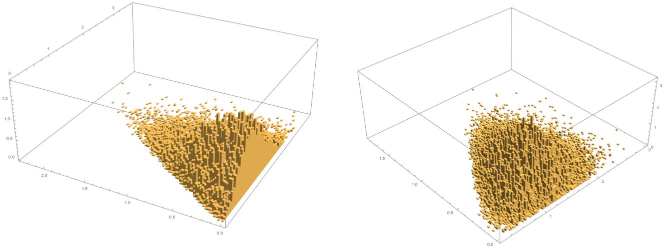


Figure: Real and Complex Asymmetric

Joint Densities - 2-tuples of singular values of 2×2 matrices

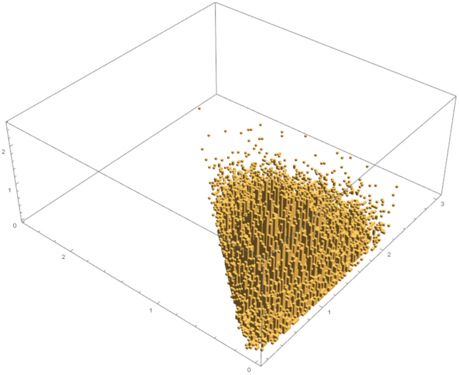


Figure: Complex Symmetric

Structured Families - Complex Checkerboard Matrices

Complex Symmetric Checkerboard Matrices

A complex symmetric (k, w)-checkerboard random matrix is a complex symmetric matrix made up of $k \times k$ blocks of the form

$$B = \begin{pmatrix} w & * & \cdots & * \\ * & w & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & w \end{pmatrix}$$

where w is some fixed constant and each $*$ represents a random variable.

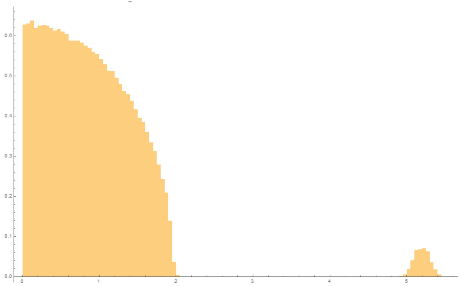
Complex Symmetric Checkerboard - Singular Values

Singular Value Distribution for Complex Symmetric Checkerboard Ensemble (SMALL 2017)

The limiting singular value distribution for the complex symmetric checkerboard ensemble shows split limiting behavior:

1. A "bulk" containing most of the mass following the Quarter Circle Law.
2. A "blip" of density k/N with mean close to Nw/k whose distribution is that of the hollow GOE ensemble (random real symmetric matrices with 0 on the diagonal).

Numerical Support



(Rescaled) 2000 Complex Symmetric (3,1)-Checkerboard Matrices

100×100

Proof Outline - Checkerboard Singular Value Bulk

- Reduce to case of $w = 0$ by using a matrix perturbation argument.
- Apply eigenvalue-trace and method of moments.
- Use combinatorial argument similar to Wigner Semicircle Law proof.

Checkerboard Singular Value Blip

Harder: Define the **empirical blip square singular spectral measure** (EBSSSM) for a matrix A to be

$$\mu_{A,N}^{s^2} := \frac{1}{k}\sum_{\sigma} f_{n(N)}\left(\frac{k^2\sigma}{N^2}\right)\delta\left(x - \frac{1}{N}\left(\sigma - \frac{N^2}{k^2}\right)\right)$$

where σ ranges over singular values of A,

$$f_n(x) = x^{2n}(x-2)^{2n}$$

and $n(N)$ is some slow growing function.

Checkerboard Singular Value Blip

We apply the method of moments to show that this distribution converges to the square of the hollow GOE distribution in the case of the complex symmetric checkerboard ensemble.

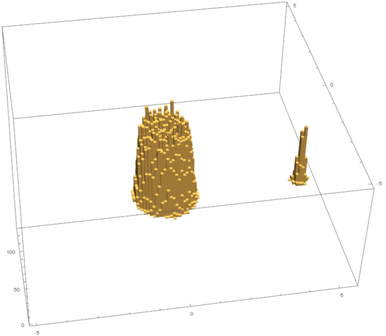
Complex Symmetric Checkerboard - Eigenvalues

Eigenvalue Distribution for Complex Symmetric Checkerboard Ensemble (SMALL 2017)

The limiting singular value distribution for the complex symmetric checkerboard ensemble shows split limiting behavior:

1. A "bulk" containing most of the mass following the Circular Law.
2. A "blip" of density k/N with mean close to Nw/k.

Numerical Support



(Rescaled) 500 Complex Symmetric (3,1)-Checkerboard Matrices

300×300

Checkerboard Eigenvalue Blip

Use knowledge of singular values.

Fact

The largest singular value of a complex symmetric matrix A has magnitude greater than or equal to that of the largest eigenvalue of A.

The absolute values of the eigenvalues must be bounded above by $N/k + \mathcal{O}(1)$. Look at eigenvalues of $\frac{k}{N}A$.

Checkerboard Eigenvalue Blip

Eigenvalues of $\frac{k}{N}A$: Contained within disk of radius $1 + \mathcal{O}\left(\frac{1}{N}\right)$.

- Show that mean of eigenvalue distribution is $\frac{k}{N} + \mathcal{O}\left(N^{-2}\right)$.
- Show that the only eigenvalues contributing to the mean must be on the boundary of the disk.
- Conclude that there must be $\frac{k}{N}$ eigenvalues at 1.

Future Work

- Different placement of w's and different block shapes.
- Complex asymmetric and combinations of checkerboarding with other ensembles.
- Allow the block size to grow as a function of N.

References

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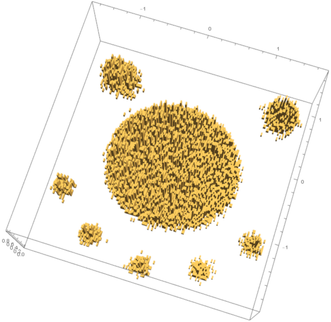

Thank you!