Limiting Distributions in Generalized b-bin Zeckendorf Decompositions

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Introduction

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- What about powers of 3? 5? 10?

- What other sequences can we use besides powers?

The Fibonacci Numbers

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$$g(x) = \sum_{n \geq 1} \boldsymbol{F}_n x^n = \frac{x}{1 - x - x^2} = \frac{1}{\sqrt{5}} \left(\frac{1}{1 - x(\frac{1+\sqrt{5}}{2})} - \frac{1}{1 - x(\frac{1-\sqrt{5}}{2})} \right)$$

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Binet's formula follows from geometric series expansion
$F_n = \frac{(1+\sqrt{5})^n-(1-\sqrt{5})^n}{2^n\sqrt{5}}$.

Previous Results

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Lekkerkerker's Theorem (1952)

The average number of summands in the Zeckendorf decomposition for integers in $[F_n, F_{n+1})$ tends to $\frac{n}{\varphi^2+1} \approx .276n$, where $\varphi = \frac{1+\sqrt{5}}{2}$ is the golden mean.

Previous Results continued

Central Limit Type Theorem (KKMW 2010)

As $n \to \infty$, the distribution of the number of summands in the Zeckendorf decomposition for integers in $[F_n, F_{n+1})$ is Gaussian.

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Remark:
Note that this is equivalent to choose summands from the first n Fibonacci numbers with the largest summand being F_n.

Previous Results continued

- Number of summands
- Gaps between adjacent summands in a decomposition
- Legal decomposition

Main Result

Theorem

Let b_i be the number of terms in the i^{th} bin of a sequence, N the number of bins, and Y_i the number of summands chosen from the i^{th} bin. Then if $\sum_{i=1}^{\infty} \frac{1}{b_i}$ diverges, the distribution of the average number of summands in a decomposition converges to a Gaussian as $N \to \infty$.

The Bin Approach

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- $1, 2, 3, 5, \ldots$

Bins

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Then, the sequence is determined by the rules on its bins.

For example, we can view the Fibonacci sequence as having bins of size 1, and only being allowed to choose numbers from nonadjacent bins.

$$\underbrace{1}\ \underbrace{2}\ \underbrace{3}\ \underbrace{5}\ \cdots$$

Example

Example: Bins of constant length 2, may choose 0 or 1 summand from each bin, adjacent bins allowed

$$\underbrace{1\ 2}\ \underbrace{3\ 6}\ \underbrace{9\ 18}\ \dots$$

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- What happens when the allowed numbers of summands per bin vary?
- What happens when we put various adjacency conditions on the bins?
- In what situations do we retain uniqueness of decomposition?

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We let $A_i \subseteq \{0, 1, \ldots, b_i\}$ be a set of integers so that if $a \in A_i$, we may choose a summands from the i^{th} bin.

Example: $A = \{0, 1, 2\}$, $b_i = 3$ for all i

$$\underbrace{1\ 2\ 4}\ \underbrace{7\ 14\ 28}\ \ldots$$

Theorem

Let $b_n \in \mathbb{N}$ be the size of the n-th bin, and $A_n \subseteq \{0, \dots, b_n\}$ be the set of legal number of summands from the n-th bin. Assume $|A| = |A_n| \geq 2$ is constant. Then the distribution of the number of summands is Gaussian if $\sum \frac{1}{b_n^{m-m'}}$ diverges, where m is the maximal element of A and m' is the second maximal element.

Sketch of the Proof

We will need the following theorem for the proof:

Theorem (Lyapunov CLT)

Let $\{Y_1, Y_2, \dots\}$ be independent random variables, each with finite mean μ_i and variance σ_i^2. Define $s_n^2 = \sum_{i=1}^{n} \sigma_i^2$. Then if there exists a $\delta > 0$ such that $\lim_{n\to\infty} \frac{1}{s_n^{2+\delta}} \sum_{i=1}^{n} \mathbb{E}(|Y_i - \mu_i|^{2+\delta}) = 0$, $\frac{1}{N} \sum_{i=1}^{\infty} Y_i$ converges to a Gaussian as $N \to \infty$.

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The probability of choosing i summands from the n-th bin is

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and the expectation values of Y_n and Y_n^2 are

$$\mathbb{E}[Y_n] = \frac{\sum_{t \in A_n} t\binom{b_n}{t}}{\sum_{t \in A_n} \binom{b_n}{t}}$$

$$\mathbb{E}[Y_n^2] = \frac{\sum_{t \in A_n} t^2\binom{b_n}{t}}{\sum_{t \in A_n} \binom{b_n}{t}}$$

Sketch of Proof continued

We then find that

$$\begin{aligned}\sigma_n^2 &= \mathbb{E}[Y_n^2] - \mathbb{E}[Y_n]^2 \\ &= \frac{\sum_{i,j\in A_n, i\neq j}(i-j)^2\binom{b_n}{i}\binom{b_n}{j}}{2\left(\sum_{t\in A_n}\binom{b_n}{t}\right)^2},\end{aligned}$$

and the absolute centered moment

$$\begin{aligned}\rho_n^{2+\delta} &:= \mathbb{E}\left[\left|Y_n-\mu_n\right|^{2+\delta}\right] \\ &= \frac{\sum_{i\in A_n}\binom{b_n}{i}\left|\sum_{t\in A_n}(i-t)\binom{b_n}{t}\right|^{2+\delta}}{\left(\sum_{t\in A_n}\binom{b_n}{t}\right)^{3+\delta}}\end{aligned}$$

Sketch of Proof continued

We come to the conclusion of the theorem by analyzing σ_n^2 and $\rho_n^{2+\delta}$ asymptotically and applying the Lyapunov Central Limit Theorem.

Sequences with Non-adjacency Rules

Now let's address the question of the behavior of sequences with adjacency conditions.
For instance, if we have constant bin size $b_n = 2$, choose at most 1 summand from each bin, and disallow summands from adjacent bins, we have the following sequence:

$$\underbrace{1, 2}, \ \underbrace{3, 4}, \ \underbrace{5, 8}, \ \underbrace{11, 16}, \cdots$$

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Note that the Fibonacci sequence is an example of such sequences with $b_n = 1$.
There exist previous combinatorial proofs that the distribution of number summands for the Fibonacci sequence is gaussian.
However, to generalize to growing bin sizes, we consider using **dependent Central Limit Theorem type result**.

α-mixing

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Definition

Let $\{X_i\}$ be a sequence of random variables. Then the ith **α-mixing coefficient**, α_i is defined to be

$$\alpha_i = \sup\left\{\left|\mathrm{P}\left(A \cap B\right) - \mathrm{P}\left(A\right)\mathrm{P}\left(B\right)\right| : t \in (-\infty, \infty), A \in X_{-\infty}^{t}, B \in X_{t+i}^{\infty}\right\}$$

where X_a^b is the set of events involving finitely many random variables in the set $\{X_a, \ldots, X_b\}$.

We wish to bound α_i for general bin sizes, and we have the following bound for constant bin sizes.

$$\alpha_i \leq c\left(\left(\frac{\psi_b}{\phi_b}\right)^i\right)$$

where c constant, $\phi_b = \frac{1+\sqrt{4b+1}}{2}$ and $\psi_b = 1 - \phi_b$.

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Future Directions:
If we get bounds for the variance as well as the mixing coefficients, we can use a dependent type of CLT to show that for all bin sizes with growth rate less than $n^{1-\epsilon}$, the distribution of number of summands is Gaussian.

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Thank You