On Summand Minimality of Generalized Zeckendorf Decompositions

Katherine Cordwell[1], Magda Hlavacek[2]
SMALL REU, Williams College

ktcordwell@gmail.com
mhlavacek@hmc.edu

[1]University of Maryland, College Park
[2]Harvey Mudd College

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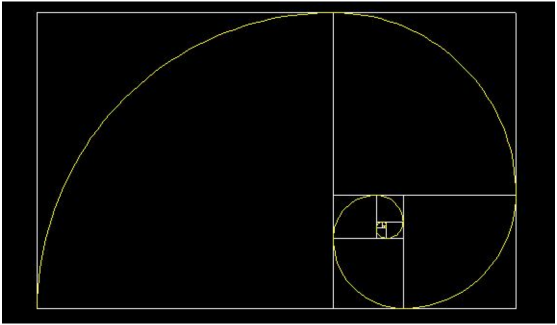
Fibonacci Sequence

$$F_{n+1} = F_n + F_{n-1}$$
$$1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ \ldots$$

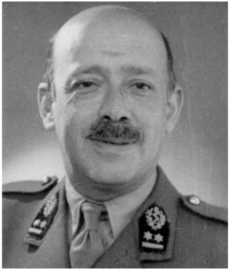


Fibonacci Spiral

Zeckendorf Decompositions

Theorem (Zeckendorf)

Every positive integer has a unique representation as a sum of nonadjacent Fibonacci numbers.



Edouard Zeckendorf

Summand Minimality

Example

- $18 = 13 + 5 = F_6 + F_4$, legal decomposition, two summands
- $18 = 13 + 3 + 2 = F_6 + F_3 + F_2$, nonlegal decomposition, three summands

Question

Definition

The Zeckendorf decomposition is **summand minimal**, because the Zeckendorf decomposition of any positive integer n uses the fewest summands out of any decomposition of n into Fibonacci numbers.

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Overall Question

What other recurrences are summand minimal?

Positive Linear Recurrence Sequences

Definition

A **positive linear recurrence sequence (PLRS)** is the sequence given by a recurrence a of the following form:

$$a_n = c_1 a_{n-1} + \cdots + c_t a_{n-t}$$

where all $c_i \geq 0$ and $c_1, c_t > 0$.

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Definition

We call $(c_1, \ldots, c_t)$ the **signature of the sequence.**

Representation Notation

Definition

For some sequence $\{a_i\}$, suppose we have:

$$n = b_k a_k + b_{k-1} a_{k-1} + \cdots + b_0 a_0$$

Then we call $[b_k, b_{k-1}, \ldots, b_0, \infty]$ a *representation* of n over $\{a_0\}$.

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Example

$$18 = F_6 + F_4 \quad \text{and} \quad F_6 + F_3 + F_2$$

We denote these two representations of 18 by:

$$[1, 0, 1, 0, 0, 0, 0, \infty] \quad \text{and} \quad [1, 0, 0, 1, 1, 0, 0, \infty]$$

Allowable Blocks

Definition

Given a PLRS with signature $(c_1, \dots, c_t)$, we say that $[b_1, \dots, b_k]$ is an **allowable block** if $k \leq t$ and $b_i = c_i$ for $i < k$ and $0 \leq b_k < c_k$.

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Example

Signature: (1, 2, 1)
Legal blocks: $[0], [1, 0], [1, 1], [1, 2, 0]$

Generalized Zeckendorf Decompositions

Definition

Given a positive linear recurrence, a representation of a positive integer n is a **generalized Zeckendorf decomposition** (GZD) if it can be formed as a tiling of a set of allowable blocks.

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Theorem (Miller et. al., Hamlin)

Given any positive linear recurrence, each positive integer n has a unique generalized Zeckendorf decomposition.

Not All Recurrences are Summand Minimal

Recurrence: $f_n = f_{n-1} + 2f_{n-2} + f_{n-3}$

Signature: (1, 2, 1)
Sequence: 1, 1, 3, 6, 13, ...
Allowable blocks: [0], [1, 0], [1, 1], [1, 2, 0]

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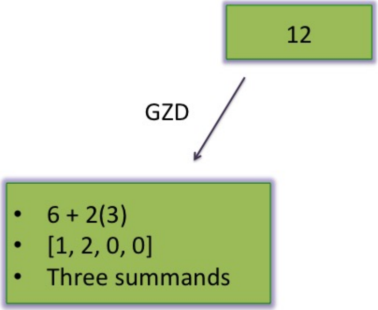

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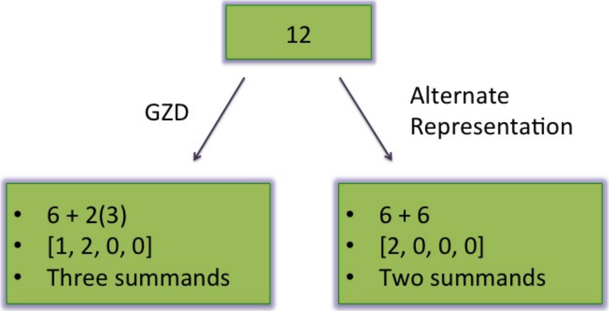


Main Result

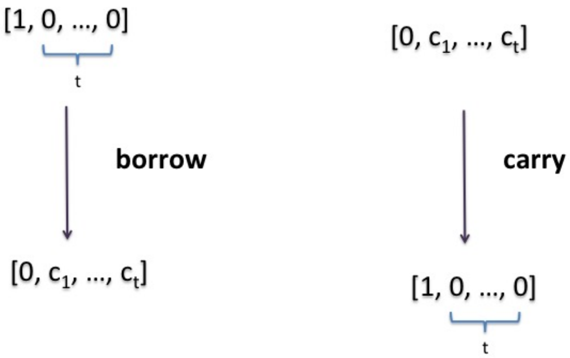
Theorem

Suppose we have a PLRS with signature $(c_1, c_2, \ldots, c_t)$ The corresponding GZD of each positive integer n is summand minimal if and only if:

$$c_1 \geq c_2 \geq \cdots \geq c_t.$$

Borrow and Carry

Signature: $(c_1, c_2, \ldots, c_t)$



Proof Sketch

Theorem ($\Longrightarrow$)

If the signature of a recurrence is weakly decreasing, then the recurrence is summand minimal.

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Proof.

Start with any representation and use successive borrows and carries to reach the GZD.
Because after borrowing we can always carry, we never increase the number of summands. □

Sufficency Example

Signature: (3, 2, 1)
Sequence: 1, 3, 11, 40, ...
Start with: 15

[5, 0, ∞]

$15 = 5 \cdot 3$

Sufficiency Example

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Sequence: 1, 3, 11, 40, ...
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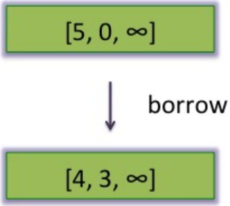

$15 = 5 \cdot 3$

$15 = 4 \cdot 3 + 3 \cdot 1$

Sufficiency Example

Signature: (3, 2, 1)
Sequence: 1, 3, 11, 40, ...
Start with: 15

[5, 0, ∞] — $15 = 5 \cdot 3$

↓ borrow

[4, 3, ∞] — $15 = 4 \cdot 3 + 3 \cdot 1$

↓ carry

[1, 1, 1, ∞] — $15 = 1 \cdot 11 + 1 \cdot 3 + 1 \cdot 1$

Theorem ($\Longleftarrow$)

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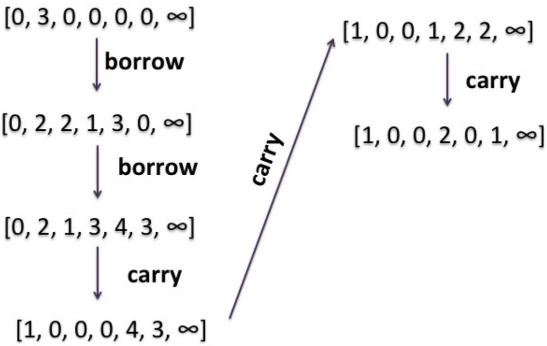
Proof.

Two cases: $c_1 > 1$ or $c_1 = 1$.
If $c_1 > 1$, then the general idea is to construct a non-legal representation and show that the corresponding legal representation uses more summands.
If $c_1 = 1$, then we use a growth rate argument to demonstrate the existence of a non-legal representation. □

Case 1: $c_1 > 1$

Further subcases. One example:

Signature: (2, 1, 3)
Sequence: 0, 0, 1, 2, 5, 15, 41, 112, ...
Blocks: [0], [1], [2, 0], [2, 1, 0], [2, 1, 1], [2, 1, 2]



Case 2: $c_1 = 1$

Assume $c_1 = 1$. What are good subcases? Let $k > 1$.

- $\{1, k, \cdots\}$
- $\{\underbrace{1, \cdots, 1}_{a}, k, \cdots\}$, $a \geq 2$
- $\{\underbrace{1, \ldots, 1}_{a_1}, \underbrace{0, \ldots, 0}_{b_1}, \cdots, \underbrace{1, \ldots, 1}_{a_t}, \underbrace{0, \ldots, 0}_{b_t}, k, \cdots\}$ $a_1 > 1$
- $\{1, \underbrace{0, \ldots, 0}_{a}, k, \ldots\}$

Example Construction

Signature: $(\underbrace{1, \dots, 1}_{a_1}, \underbrace{0, \dots, 0}_{b_1}, \dots, \underbrace{1, \dots, 1}_{a_t}, k, \dots)$, $a_1 > 1$

Non-legal representation:

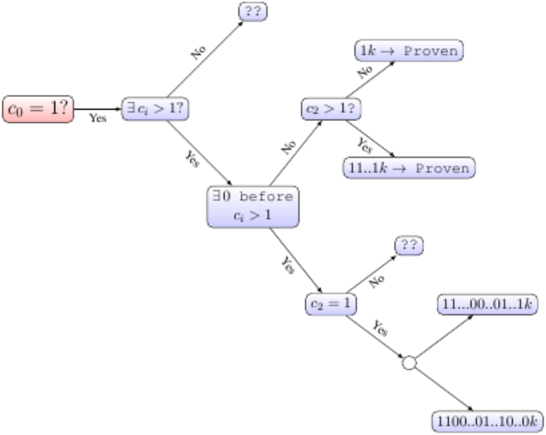
$$[\underbrace{1, \dots, 1}_{a_1}, \underbrace{0, \dots, 0}_{b_1}, \dots, \underbrace{1, \dots, 1}_{a_t - 1}, 2, \underbrace{0, \dots, 0}_{a_1 + 1}, \infty]$$

↓ **borrow**

$$[\underbrace{1, \dots, 1}_{a_1}, \underbrace{0, \dots, 0}_{b_1}, \dots, \underbrace{1, \dots, 1}_{a_t - 1}, \underbrace{1, \dots, 1}_{a_1 - 1}, 0, \infty]$$

Change in summands: $a_1 - 1$

Subcases Start to get Overwhelming



Solution: Growth Rates!

Definition

The characteristic polynomial of a recurrence with signature $(c_1, c_2, \ldots, c_k)$ is

$$x^k - c_1 x^{k-1} - \cdots - c_k.$$

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Theorem

Given a PLRS with a signature of the form $(1, c_2, \ldots)$ *, the characteristic polynomial has a unique largest positive root* $\alpha > 1$*. For large n,*

$$a_n \approx C\alpha^n$$

.

Counterexample: Trying to represent $2a_n$

Theorem

For every non-weakly-decreasing signature with $c_1 = 1$, then there exists some n for which the GZD of $2a_n$ has at least 3 summands.

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Theorem

For every non-weakly-decreasing signature with $c_1 = 1$, then there exists some n for which the GZD of $2a_n$ has at least 3 summands.

If summand minimal, the GZD of $2a_n$ must have only 1 or 2 summands:

- $[1, 0, 0, \dots]$
- $[1, 0, \dots, 1, 0, \dots]$

Representations of $2a_n$

Growth rate arguments give three specific forms:

- $2a_n = a_{n+r}$
- $2a_n = a_{n+r} + a_{n-s}$
- $2a_n = a_{n+r} + a_{n-s+1}$

for some fixed r, s.

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for some fixed r, s.

These recurrences have different growth rates; only one can correspond to our sequence.

For all $n > N$, every representation of $2a_n$ must be of the same form.

Last Step

The characteristic polynomial of a truncated sequence must divide exactly one of the following characteristic polynomials:

- $x^r - 2$
- $x^r - 2x^s - 1$
- $x^r - 2x^{s-1} - 1$

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- $x^r - 2x^{s-1} - 1$

By a result of Schinzel on the factorization of these polynomials, this cannot be the case.

Acknowledgments

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Selected References

- M. Catral, P. Ford, P. E. Harris, S. J. Miller, and D. Nelson, *Notes on Monoinvariants and Minimal Fibonacci Representations*, preprint 2016.
- N. Hamlin, *Representing Positive Integers as a Sum of Linear Recurrence Sequences*, Fibonacci Quarterly **50** (2012), no. 2, 99–10
- S. J. Miller and Y. Wang, *From Fibonacci numbers to Central Limit Type Theorems*, Journal of Combinatorial Theory, Series A **119** (2012), no. 7, 1398–1413.
- A. Schinzel, *Solution d'un probleme de K. Zarankiewicz sur les suites de puissances consécutives de nombres irrationnels*, Colloquium Mathematicae, **9** (1972), no. 2, pages 291–296.
- E. Zeckendorf, *Représentation des nombres naturels par une somme des nombres de Fibonacci ou de nombres de Lucas*, Bulletin de la Société Royale des Sciences de Liége **41** (1972), pages 179–182.

Sufficency Example

Signature: (3, 2, 1)
Sequence: 1, 3, 11, 40, 145 ...
Start with: 160

[4, 0, 0, 0, ∞]

$160 = 4 \cdot 40$

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Sequence: 1, 3, 11, 40, 145 ...
Start with: 160

[4, 0, 0, 0, ∞]

↓ borrow

[3, 3, 2, 1, ∞]

$160 = 4 \cdot 40$

Sufficency Example

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Sequence: 1, 3, 11, 40, 145 ...
Start with: 160

[4, 0, 0, 0, ∞] $160 = 4 \cdot 40$

↓ borrow

[3, 3, 2, 1, ∞]

↓ carry

[1, 0, 1, 1, 1, ∞]

Check: 1 + 3 + 11 + 145 = 160

Case 1: $c_1 > 1$ Further example

Further subcases. One example:

Signature: $(c_1, c_2, \ldots, c_t)$, some $c_i > c_1$
Non-legal representation: $[0, c_1, \ldots, c_{i-2}, c_{i-1} + 1, 0, \infty]$

$$[0, c_1, \ldots, c_{i-2}, c_{i-1} + 1, 0, \infty]$$

$$\Big\downarrow \quad \textbf{borrow}$$

$$[0, c_1, \ldots, c_{i-2}, c_{i-1}, c_1, \infty]$$

Net change in summands: $c_1 - 1 \geq 1$