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A Symplectic Test of the L-Functions Ratios Conjecture

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Recently Conrey, Farmer and Zirnbauer [8, 9] conjectured formulas for the averages over a family of ratios of products of shifted L-functions. Their L-functions Ratios Conjecture predicts both the main and lower order terms for many problems, ranging from n-level correlations and densities to mollifiers and moments to vanishing at the central point. There are now many results showing agreement between the main terms of number theory and random matrix theory; however, there are very few families where the lower order terms are known. These terms often depend on subtle arithmetic properties of the family, and provide a way to break the universality of behavior. The L-functions Ratios Conjecture provides a powerful and tractable way to predict these terms. We test a specific case here, that of the 1-level density for the symplectic family of quadratic Dirichlet characters arising from even fundamental discriminants $d \leq X$. For test functions supported in $(-1/3, 1/3)$ we calculate all the lower order terms up to size $O(X^{-1/2+\epsilon})$ and observe perfect agreement with the conjecture (for test functions supported in $(-1, 1)$ we show agreement up to errors of size $O(X^{-\epsilon})$ for any ϵ). Thus for this family and suitably restricted test functions, we completely verify the Ratios Conjecture's prediction for the 1-level density.

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1 Introduction

Montgomery's [35] analysis of the pair correlation of zeros of $\zeta(s)$ revealed a striking similarity to the behavior of eigenvalues of ensembles of random matrices. Since then, this connection has been a tremendous predictive aid to researchers in number theory in modeling the behavior of zeros and values of L-functions, ranging from spacings between adjacent zeros [35, 19, 36, 37, 44] to moments of L-functions [5, 10]. Katz and Sarnak [26, 27] conjectured that, in the limit as the conductors tend to infinity, the behavior of the normalized zeros near the central point agree with the $N \to \infty$ scaling limit of the normalized eigenvalues near 1 of a subgroup of $U(N)$. One way to test this correspondence is through the n-level density of a family $\mathcal{F}$ of L-functions $L(s,f)$; we concentrate on this statistic in this article. The n-level density is

$$D_{n,\mathcal{F}}(\phi) := \frac{1}{|\mathcal{F}|} \sum_{f\in\mathcal{F}} \sum_{\substack{\ell_1,\dots,\ell_n \\ \ell_i \neq \pm \ell_k}} \phi_1\left(\gamma_{f,\ell_1}\frac{\log Q_f}{2\pi}\right)\cdots\phi_n\left(\gamma_{f,\ell_n}\frac{\log Q_f}{2\pi}\right), \tag{1.1}$$

where the ϕ_i are even Schwartz test functions whose Fourier transforms have compact support, $\frac{1}{2} + i\gamma_{f,\ell}$ runs through the nontrivial zeros of $L(s,f)$, and Q_f is the analytic conductor of f. As the ϕ_i are even Schwartz functions, most of the contribution to $D_{n,\mathcal{F}}(\phi)$ arises from the zeros near the central point; thus this statistic is well-suited to investigating the low-lying zeros.

There are now many examples where the main term in number theory agrees with the Katz–Sarnak conjectures (at least for suitably restricted test functions), such as all Dirichlet characters, quadratic Dirichlet characters, $L(s,\psi)$ with ψ a character of the ideal class group of the imaginary quadratic field $\mathbb{Q}(\sqrt{-D})$, families of elliptic curves, weight k level N cuspidal newforms, symmetric powers of GL(2) L-functions, and certain families of GL(4) and GL(6) L-functions (see [13, 16, 18, 20–22, 27, 32, 39–42, 47]).

For families of L-functions over function fields, the corresponding classical compact group can be identified through the monodromy. While the situation is less clear for L-functions over number fields, there has been some recent progress. Dueñez and Miller [12] show that for sufficiently nice families and sufficiently small support, the main term in the 1-level density is determined by the first and second moments of the Satake parameters, and a symmetry constant (which identifies the corresponding classical compact group) may be associated to any nice family such that the symmetry constant of the Rankin-Selberg convolution of two families is the product of the symmetry constants.

There are two avenues for further research. The first is to increase the support of the test functions, which often leads to questions of arithmetic interest (see for example Hypothesis S in [22]). Another is to identify lower order terms in the 1-level density, which is the subject of this article. The main term in the 1-level density is independent of the arithmetic of the family, which surfaces in the lower order terms. This is very similar to the Central Limit Theorem. For nice densities the distribution of the normalized sample mean converges to the standard normal. The main term is controlled by the first two moments (the mean and the variance of the density) and the higher moments surface in the rate of convergence. This is similar to our situation, where the universal main terms arise from the first and second moments of the Satake parameters.

There are now several families where lower order terms have been isolated in the 1-level density [16, 33, 34, 46]; see also [3], where the Hardy–Littlewood conjectures are related to lower order terms in the pair correlation of zeros of $\zeta(s)$ (see for example [1, 2, 7, 28] for more on lower terms of correlations of Riemann zeros). Recently Conrey, Farmer and Zirnbauer [8, 9] formulated conjectures for the averages over families of L-functions of ratios of products of shifted L-functions, such as

$$\sum_{d\le X}\frac{L\left(\frac12+\alpha,\chi_d\right)}{L\left(\frac12+\gamma,\chi_d\right)}=\sum_{d\le X}\left[\frac{\zeta(1+2\alpha)}{\zeta(1+\alpha+\gamma)}A_D(\alpha;\gamma)\right.$$
$$\left.+\left(\frac{d}{\pi}\right)^{-\alpha}\frac{\Gamma\left(\frac14-\frac{\alpha}{2}\right)}{\Gamma\left(\frac14+\frac{\alpha}{2}\right)}\frac{\zeta(1-2\alpha)}{\zeta(1-\alpha+\gamma)}A_D(-\alpha;\gamma)\right]+O(X^{1/2+\epsilon}) \quad (1.2)$$

(here d ranges over even fundamental discriminants, $-1/4<\Re(\alpha)<1/4$, $1/\log X\ll\Re(\gamma)<1/4$, and A_D (we only give the definition for $\alpha=\gamma$, as that is the only instance that occurs in our applications) is defined in (1.4)). Their L-functions Ratios Conjecture arises from using the approximate functional equation, integrating term by term, and retaining only the diagonal pieces (which they then 'complete'); they also assume uniformity in the parameters so that the resulting expressions may be differentiated (this is an essential ingredient for 1-level density calculations). It is worth noting the incredible detail of the conjecture, predicting all terms down to $O(X^{1/2+\epsilon})$.

There are many difficult computations whose answers can easily be predicted through applications of the L-functions Ratios Conjecture, ranging from n-level correlations and densities to mollifiers and moments to vanishing at the central point (see [6]). While these are not proofs, it is extremely useful for researchers to have a sense of what the answer should be. One common difficulty in the subject is that often the number theory and random matrix theory answers appear different at first, and much effort must be spent on combinatorics to prove agreement (see for example [17, 21, 42, 44]); the analysis is significantly easier if one knows what the final answer should be. Further, the Ratios Conjecture often suggest a more enlightening way to group terms (see for instance Remark 1.4).

Our goal in this article is to test the predictions of the Ratios Conjecture for a specific family, that of quadratic Dirichlet characters. We let d be a fundamental discriminant. This means (see Section 5 of [11]) that either d is a square-free number congruent to 1 modulo 4, or $d/4$ is square-free and congruent to 2 or 3 modulo 4. If χ_d is the quadratic character associated to the fundamental discriminant d, then if $\chi_d(-1) = 1$ (resp., -1) we say d is even (resp., odd). If d is a fundamental discriminant then it is even (resp., odd) if $d > 0$ (resp., $d < 0$). We concentrate on even fundamental discriminants below, though with very few changes our arguments hold for odd discriminants (for example, if d is odd there is an extra $1/2$ in certain Gamma factors in the explicit formula).

For notational convenience we adopt the following conventions throughout the article:

- Let X^* denote the number of even fundamental discriminants at most X; thus $X^* = 3X/\pi^2 + O(X^{1/2})$, and $X/\pi^2 + O(X^{1/2})$ of these have $4|d$ (see Lemma B.1 for a proof).
- In any sum over d, d will range over even fundamental discriminants unless otherwise specified.

The goal of these notes is to calculate the lower order terms (on the number theory side) as much as possible, as unconditionally as possible, and then compare our answer to the prediction from the L-functions Ratios Conjecture, given in the theorem below.

THEOREM 1.1 (One-level density from the Ratios Conjecture [6]). *Let g be an even Schwartz test function such that $\widehat{g}$ has finite support. Let X^* denote the number of even fundamental discriminants at most X, and let d denote a typical even fundamental discriminant. Assuming the Ratios Conjecture for $\sum_{d\le X} L(\frac{1}{2}+\alpha,\chi_d)/L(\frac{1}{2}+\gamma,\chi_d)$, we have*

$$\frac{1}{X^*}\sum_{d\le X}\sum_{\gamma_d} g\left(\gamma_d\frac{\log X}{2\pi}\right)$$
$$= \frac{1}{X^*\log X}\int_{-\infty}^{\infty} g(\tau)\sum_{d\le X}\left[\log\frac{d}{\pi}+\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{i\pi\tau}{\log X}\right)+\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}-\frac{i\pi\tau}{\log X}\right)\right]d\tau$$

$$+\frac{2}{X^*\log X}\sum_{d\le X}\int_{-\infty}^{\infty} g(\tau)\left[\frac{\zeta'}{\zeta}\left(1+\frac{4\pi i\tau}{\log X}\right)+A_D'\left(\frac{2\pi i\tau}{\log X};\frac{2\pi i\tau}{\log X}\right)\right.$$
$$\left.-e^{-2\pi i\tau\log(d/\pi)/\log X}\frac{\Gamma\left(\frac14-\frac{\pi i\tau}{\log X}\right)}{\Gamma\left(\frac14+\frac{\pi i\tau}{\log X}\right)}\zeta\left(1-\frac{4\pi i\tau}{\log X}\right)A_D\left(-\frac{2\pi i\tau}{\log X};\frac{2\pi i\tau}{\log X}\right)\right]d\tau$$
$$+O(X^{-\frac12+\epsilon}), \tag{1.3}$$

with

$$A_D(-r,r)=\prod_p\left(1-\frac{1}{(p+1)p^{1-2r}}-\frac{1}{p+1}\right)\cdot\left(1-\frac1p\right)^{-1}$$
$$A_D'(r;r)=\sum_p\frac{\log p}{(p+1)(p^{1+2r}-1)}. \tag{1.4}$$

The above is

$$\frac{1}{X^*}\sum_{d\le X}\sum_{\gamma_d} g\left(\gamma_d\frac{\log X}{2\pi}\right)=\int_{-\infty}^{\infty}g(x)\left(1-\frac{\sin(2\pi x)}{2\pi x}\right)dx+O\left(\frac{1}{\log X}\right), \tag{1.5}$$

which is the 1*-level density for the scaling limit of* $\mathrm{USp}(2N)$*. If* $\mathrm{supp}(\widehat{g})\subset(-1,1)$*, then the integral of* $g(x)$ *against* $-\sin(2\pi x)/2\pi x$ *is* $-g(0)/2$*.*

If we assume the Riemann Hypothesis, for $\mathrm{supp}(\widehat{g})\subset(-\sigma,\sigma)\subset(-1,1)$ *we have*

$$\frac{-2}{X^*\log X}\sum_{d\le X}\int_{-\infty}^{\infty}g(\tau)\,e^{-2\pi i\tau\frac{\log(d/\pi)}{\log X}}\frac{\Gamma\left(\frac14-\frac{\pi i\tau}{\log X}\right)}{\Gamma\left(\frac14+\frac{\pi i\tau}{\log X}\right)}\zeta\left(1-\frac{4\pi i\tau}{\log X}\right)A_D\left(-\frac{2\pi i\tau}{\log X};\frac{2\pi i\tau}{\log X}\right)d\tau$$
$$=-\frac{g(0)}{2}+O(X^{-\frac34(1-\sigma)+\epsilon}); \tag{1.6}$$

the error term may be absorbed into the $O(X^{-1/2+\epsilon})$ *error in* (1.3) *if* $\sigma<1/3$*.* □

The conclusions of the above theorem are phenomenal, and demonstrate the power of the Ratios Conjecture. Not only does its main term agree with the Katz–Sarnak conjectures for arbitrary support, but it calculates the lower order terms up to size $O(X^{-1/2+\epsilon})$. While Theorem 1.1 is conditional on the Ratios Conjecture, the following theorem is not, and provides highly nontrivial support for the Ratios Conjecture.

THEOREM 1.2 (One-level density for quadratic Dirichlet characters)**.** *Let the notation be as in Theorem* 1.1*, with* $\mathrm{supp}(\widehat{g})\subset(-\sigma,\sigma)$*.*

(1) *Up to terms of size* $O(X^{-(1-\sigma)/2+\epsilon})$, *the* 1*-level density for the family of quadratic Dirichlet characters with even fundamental discriminants at most* X *agrees with (1.3) (the prediction from the Ratios Conjecture).*

(2) *If we instead consider the family* $\{8d : 0 < d \le X, d$ *an odd, positive square-free fundamental discriminant*$\}$, *then the* 1*-level density agrees with the prediction from the Ratios Conjecture up to terms of size* $O(X^{-1/2} + X^{-(1-\frac{3}{2}\sigma)+\epsilon} + X^{-\frac{3}{4}(1-\sigma)+\epsilon})$. *In particular, if* $\sigma < 1/3$ *then the number theory calculation agrees with the Ratios Conjecture up to errors at most* $O(X^{-1/2+\epsilon})$. □

REMARK 1.3. The above theorem indicates that, at least for the family of quadratic Dirichlet characters and suitably restricted test functions, the Ratios Conjecture *is* predicting all lower order terms up to size $O(X^{-1/2+\epsilon})$. This is phenomenal agreement between theory and conjecture. Previous investigations of lower order terms in 1-level densities went as far as $O(\log^N X)$ for some N; here we are getting square-root agreement, and strong evidence in favor of the Ratios Conjecture.

REMARK 1.4 (Influence of zeros of $\zeta(s)$ on lower order terms). From the expansion in (1.3) we see that one of the lower order terms (arising from the integral of $g(\tau)$ against $\zeta'(1 + 4\pi i\tau/\log X)/\zeta(1 + 4\pi i\tau/\log X)$) in the 1-level density for the family of quadratic Dirichlet characters is controlled by the nontrivial zeros of $\zeta(s)$. This phenomenon has been noted by other researchers (Bogomolny, Conrey, Keating, Rubinstein, Snaith); see [6, 3, 43] for more details, especially [43] for a plot of the influence of zeros of $\zeta(s)$ on zeros of L-functions of quadratic Dirichlet characters.

The proof of Theorem 1.2 starts with the Explicit Formula, which relates sums over zeros to sums over primes (for completeness a proof is given in Appendix A). For convenience to researchers interested in odd fundamental discriminants, we state it in more generality than we need.

THEOREM 1.5 (Explicit Formula for a family of Quadratic Dirichlet Characters)**.** *Let* g *be an even Schwartz test function such that* $\widehat{g}$ *has finite support. For* d *a fundamental discriminant let* $a(\chi_d) = 0$ *if* d *is even* ($\chi_d(-1) = 1$) *and* 1 *otherwise. Consider a family* $\mathcal{F}(X)$ *of fundamental discriminants at most* X *in absolute value. We have*

$$\frac{1}{|\mathcal{F}(X)|} \sum_{d\in\mathcal{F}(X)} \sum_{\gamma_d} g\left(\gamma_d \frac{\log X}{2\pi}\right)$$

$$= \frac{1}{|\mathcal{F}(X)|\log X}\int_{-\infty}^{\infty} g(\tau) \sum_{d\in\mathcal{F}(X)} \left[\log\frac{|d|}{\pi} + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{a(\chi_d)}{2}+\frac{i\pi\tau}{\log X}\right)\right.$$
$$\left. + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{a(\chi_d)}{2}-\frac{i\pi\tau}{\log X}\right)\right] d\tau - \frac{2}{|\mathcal{F}(X)|}\sum_{d\in\mathcal{F}(X)}\sum_{k=1}^{\infty}\sum_{p}\frac{\chi_d(p)^k\log p}{p^{k/2}\log X}\,\widehat{g}\left(\frac{\log p^k}{\log X}\right). \tag{1.7}$$

□

As our family has only even fundamental discriminants, all $a(\chi_d) = 0$. The terms arising from the conductors (the $\log(|d|/\pi)$ and the Γ'/Γ terms) agree with the Ratios Conjecture. We are reduced to analyzing the sums of $\chi_d(p)^k$ and showing they agree with the remaining terms in the Ratios Conjecture. As our characters are quadratic, this reduces to understanding sums of $\chi_d(p)$ and $\chi_d(p)^2$. We first analyze the terms from the Ratios Conjecture in Section 2 and then we analyze the character sums in Section 3. We proceed in this order as one of the main uses of the Ratios Conjecture is in predicting simple forms of the answer; in particular, it suggests nonobvious simplifications of the number theory sums.

2 Analysis of the Terms from the Ratios Conjecture.

We analyze the terms in the 1-level density from the Ratios Conjecture (Theorem 1.1). The first piece (involving $\log(d/\pi)$ and Γ'/Γ factors) is already matched with the terms in the Explicit Formula arising from the conductors and Γ-factors in the functional equation. In Section 3 we match the next two terms (the integral of $g(\tau)$ against ζ'/ζ and A_D') to the contributions from the sum over $\chi_d(p)^k$ for k even; we do this for test functions with arbitrary support. The number theory is almost equal to this; the difference is the presence of a factor $-g(0)/2$ from the even k terms, which we match to the remaining piece from the Ratios Conjecture.

This remaining piece is the hardest to analyze. We denote it by

$$R(g;X) = -\frac{2}{X^*\log X}\sum_{d\le X}\int_{-\infty}^{\infty} g(\tau) e^{-2\pi i\tau\frac{\log(d/\pi)}{\log X}} \frac{\Gamma\left(\frac{1}{4}-\frac{\pi i\tau}{\log X}\right)}{\Gamma\left(\frac{1}{4}+\frac{\pi i\tau}{\log X}\right)}$$
$$\times\,\zeta\left(1-\frac{4\pi i\tau}{\log X}\right) A_D\left(-\frac{2\pi i\tau}{\log X};\frac{2\pi i\tau}{\log X}\right) d\tau, \tag{2.1}$$

with (see (1.4))

$$A_D(-r,r) = \prod_p\left(1-\frac{1}{(p+1)p^{1-2r}}-\frac{1}{p+1}\right)\cdot\left(1-\frac{1}{p}\right)^{-1}. \tag{2.2}$$

There is a contribution to $R(g;X)$ from the pole of $\zeta(s)$. The other terms are at most $O(1/\log X)$; however, if the support of $\widehat{g}$ is sufficiently small then these terms contribute significantly less.

LEMMA 2.1. *Assume the Riemann Hypothesis. If* $\operatorname{supp}(\widehat{g}) \subset (-\sigma, \sigma)$ *then*

$$R(g;X) = -\frac{g(0)}{2} + O(X^{-\frac{3}{4}(1-\sigma)+\epsilon}). \tag{2.3}$$

In particular, if $\sigma < 1/3$ *then* $R(g;X) = -\frac{1}{2}g(0) + O(X^{-\frac{1}{2}+\epsilon})$. □

REMARK 2.2. If we do not assume the Riemann Hypothesis we may prove a similar result. The error term is replaced with $O(X^{-(1-\frac{\theta}{2})(1-\sigma)+\epsilon})$, where θ is the supremum of the real parts of zeros of $\zeta(s)$. As $\theta \leq 1$, we may always bound the error by $O(X^{-(1-\sigma)/2+\epsilon})$. Interestingly, this is the error we get in analyzing the number theory terms $\chi(p)^k$ with k odd by applying Jutila's bound (see Section 3.2.1); we obtain a better bound of $O(X^{-(1-\frac{3}{2}\sigma)})$ by using Poisson summation to convert long character sums to shorter ones (see Section 3.2.2).

REMARK 2.3. The proof of Lemma 2.1 follows from shifting contours and keeping track of poles of ratios of Gamma and zeta functions. We can prove a related result with significantly less work. Specifically, if for $\operatorname{supp}(\widehat{g}) \subset (-1,1)$ we are willing to accept error terms of size $O(\log^{-N} X)$ for any N then we may proceed as follows:

(1) modify Lemma B.2 to replace the d-sum with $X^* e^{-2\pi i\left(1-\frac{\log \pi}{\log X}\right)\tau}\left(1-\frac{2\pi i\tau}{\log X}\right)^{-1} + O(X^{1/2})$;

(2) use the decay properties of g to restrict the τ sum to $|\tau| \leq \log X$ and then Taylor expand everything but g, which gives a small error term and

$$\begin{aligned}\int_{|\tau|\leq \log X} g(\tau) \sum_{n=-1}^{N} \frac{a_n}{\log^n X}(2\pi i\tau)^n e^{-2\pi i\left(1-\frac{\log \pi}{\log X}\right)\tau} d\tau \\ = \sum_{n=-1}^{N} \frac{a_n}{\log^n X} \int_{|\tau|\leq \log X} (2\pi i\tau)^n g(\tau) e^{-2\pi i\left(1-\frac{\log \pi}{\log X}\right)\tau} d\tau;\end{aligned} \tag{2.4}$$

(3) use the decay properties of g to extend the τ-integral to all of $\mathbb{R}$ (it is essential here that N is fixed and finite!) and note that for $n \geq 0$ the above is the Fourier transform of $g^{(n)}$ (the n^{th} derivative of g) at $1-\frac{\pi}{\log X}$, and this is zero if $\operatorname{supp}(\widehat{g}) \subset (-1,1)$.

We prove Lemma 2.1 in Section 2.1; this completes our analysis of the terms from the Ratios Conjecture. We analyze the lower order term of size $1/\log X$ (present only if $\operatorname{supp}(\widehat{g}) \not\subset (-1,1)$) in Lemma 2.6 of Section 2.2. We explicitly calculate this contribution because in many applications all that is required are the main and first lower order terms. One example of this is that zeros at height T are modeled not by the $N \to \infty$ scaling limits of a classical compact group but by matrices of size $N \sim \log(T/2\pi)$ [31, 32, 33]. In fact, even better agreement is obtained by changing N slightly due to the first lower order term (see [4, 14]).

2.1 Analysis of $R(g;X)$

Before proving Lemma 2.1 we collect several useful facts.

Lemma 2.4. *In all statements below $r = 2\pi i\tau/\log X$ and $\operatorname{supp}(\widehat{g}) \subset (-\sigma,\sigma) \subset (-1,1)$.*

(1) *$A_D(-r,r) = \zeta(2)/\zeta(2-2r)$.*
(2) *If $|r| \geq \epsilon$ then $|\zeta(-3-2r)/\zeta(-2-2r)| \ll_\epsilon (1+|r|)$.*
(3) *For $w \geq 0$, $g\left(\tau - iw\frac{\log X}{2\pi}\right) \ll X^{\sigma w}\left(\tau^2 + (w\frac{\log X}{2\pi})^2\right)^{-B}$ for any $B \geq 0$.*
(4) *For $0 < a < b$ we have $|\Gamma(a \pm iy)/\Gamma(b \pm iy)| = O_{a,b}(1)$.* □

Proof. (1): From simple algebra, as we may rewrite each factor as

$$\frac{p}{p+1}\left(1 - \frac{1}{p^{2-2r}}\right)\frac{p}{p-1} = \left(1 - \frac{1}{p^2}\right)^{-1}\left(1 - \frac{1}{p^{2-2r}}\right). \tag{2.5}$$

(2): By the functional equations of the Gamma and zeta functions $\Gamma(s/2)\pi^{-s/2}\zeta(s) = \Gamma((1-s)/2)\pi^{-(1-s)/2}\zeta(1-s)$ and $\Gamma(1+x) = x\Gamma(x)$ gives

$$\frac{\zeta(-3-2r)}{\zeta(-2-2r)} = \frac{\Gamma(1-(-1-r))\pi^{-2-r}\Gamma(-1-r)\pi^{1+r}\zeta(4+2r)}{\Gamma(-\frac{3}{2}-r)\pi^{\frac{3}{2}+r}\Gamma(1-(-\frac{3}{2}-r))(\frac{3}{2}+r)^{-1}\pi^{-\frac{3}{2}+r}\zeta(3+2r)}. \tag{2.6}$$

Using

$$\Gamma(x)\Gamma(1-x) = \pi/\sin\pi x = 2\pi i/(e^{i\pi x} - e^{-i\pi x}), \tag{2.7}$$

we see the ratio of the Gamma factors have the same growth as $|r| \to \infty$ (if $r = 0$ then there is a pole from the zero of $\zeta(s)$ at $s = -2$), and the two zeta functions are bounded away from 0 and infinity.

(3): As $g(\tau) = \int \widehat{g}(\xi) e^{2\pi i \xi \tau} d\xi$, we have

$$\begin{aligned} g(\tau - iy) &= \int_{-\infty}^{\infty} \widehat{g}(\xi) e^{2\pi i(\tau - iy)\xi} d\xi \\ &= \int_{-\infty}^{\infty} \widehat{g}^{(2n)}(\xi)(2\pi i(\tau - iy))^{-n} e^{2\pi i(\tau - iy)\xi} d\xi \\ &\ll e^{2\pi y \sigma}(\tau - iy))^{-2n}; \end{aligned} \tag{2.8}$$

the claim follows by taking $y = (w \log X)/2\pi$.

(4): As $|\Gamma(x - iy)| = |\Gamma(x + iy)|$, we may assume all signs are positive. The claim follows from the definition of the Beta function:

$$\frac{\Gamma(a+iy)\Gamma(b-a)}{\Gamma(b+iy)} = \int_0^1 t^{a+iy-1}(1+t)^{b-a-1} = O_{a,b}(1); \tag{2.9}$$

see [15] for additional estimates of the size of ratios of Gamma functions. ■

PROOF OF LEMMA 2.1. By Lemma 2.4 we may replace $A_D(-2\pi i\tau/\log X, 2\pi i\tau/\log X)$ with $\zeta(2)/\zeta(2 - 4\pi i\tau/\log X)$. We replace τ with $\tau - iw\frac{\log X}{2\pi}$ with $w = 0$ (we will shift the contour in a moment). Thus

$$\begin{aligned} R(g;X) = -\frac{2}{X^* \log X} \sum_{d \le X} \int_{-\infty}^{\infty} g\left(\tau - iw\frac{\log X}{2\pi}\right) e^{-2\pi i\left(\tau - iw\frac{\log X}{2\pi}\right)\frac{\log(d/\pi)}{\log X}} \\ \times \frac{\Gamma\left(\frac{1}{4} - \frac{w}{2} - \frac{\pi i\tau}{\log X}\right)}{\Gamma\left(\frac{1}{4} + \frac{w}{2} + \frac{\pi i\tau}{\log X}\right)} \frac{\zeta(2)\zeta\left(1 - w - \frac{4\pi i\tau}{\log X}\right)}{\zeta\left(2 - 2w - \frac{4\pi i\tau}{\log X}\right)} d\tau. \end{aligned} \tag{2.10}$$

We now shift the contour to $w = 2$. There are two different residue contributions as we shift (remember we are assuming the Riemann Hypothesis, so that if $\zeta(\rho) = 0$ then either $\rho = \frac{1}{2} + i\gamma$ for some $\gamma \in \mathbb{R}$ or ρ is a negative even integer), arising from

- the pole of $\zeta\left(1 - w - \frac{4\pi i\tau}{\log X}\right)$ at $w = \tau = 0$;
- the zeros of $\zeta\left(2 - 2w - \frac{4\pi i\tau}{\log X}\right)$ when $w = 3/4$ and $\tau = \gamma\frac{\log X}{4\pi}$

(while potentially there is a residue from the pole of $\Gamma\left(\frac{1}{4} - \frac{w}{2} - \frac{\pi i\tau}{\log X}\right)$ when $w = 1/2$ and $\tau = 0$, this is canceled by the pole of $\zeta\left(2 - 2w - \frac{4\pi i\tau}{\log X}\right)$ in the denominator).

We claim the contribution from the pole of $\zeta\left(1 - w - \frac{4\pi i\tau}{\log X}\right)$ at $w = \tau = 0$ is $-g(0)/2$. As $w = \tau = 0$, the d-sum is just X^*. As the pole of $\zeta(s)$ is $1/(s-1)$, since $s = 1 - \frac{4\pi i\tau}{\log X}$ the $1/\tau$ term from the zeta function has coefficient $-\frac{\log X}{4\pi i}$. We lose the factor of $1/2\pi i$ when we apply the residue theorem, there is a minus sign outside the integral and another from the direction we integrate (we replace the integral from $-\epsilon$ to ϵ with a semicircle oriented clockwise; this gives us a minus sign as well as a factor of $1/2$ since we only have half the contour), and everything else evaluated at $\tau = 0$ is $g(0)$.

We now analyze the contribution from the zeros of $\zeta(s)$ as we shift w to 2. Thus $w = 3/2$ and we sum over $\tau = \gamma\frac{\log X}{4\pi}$ with $\zeta(\frac{1}{2} + i\gamma) = 0$. We use Lemma B.2 (with $z = \tau - iw\frac{\log X}{2\pi}$) to replace the d-sum with

$$X^{*} e^{-2\pi i\left(1-\frac{\log\pi}{\log X}\right)\tau}\left(\frac{1}{4} - \frac{2\pi i\tau}{\log X}\right)^{-1} X^{-\frac{3}{4}} X^{\frac{2\log\pi}{\log X}} + O(\log X). \tag{2.11}$$

The contribution from the $O(\log X)$ term is dwarfed by the main term (which is of size $X^{1/4+\epsilon}$). From (3) of Lemma 2.4 we have

$$g\left(\gamma\frac{\log X}{4\pi} - i\frac{3}{4}\frac{\log X}{2\pi}\right) \ll X^{3\sigma/4}(\tau^2+1)^{-B} \tag{2.12}$$

for any $B > 0$. From (4) of Lemma 2.4, we see that the ratio of the Gamma factors is bounded by a power of $|\tau|$ (the reason it is a power is that we may need to shift a few times so that the conditions are met; none of these factors will ever vanish as we are not evaluating at integral arguments). Finally, the zeta function in the numerator is bounded by $|\tau|^2$. Thus the contribution from the critical zeros of $\zeta(s)$ is bounded by

$$\sum_{\substack{\gamma \\ \zeta\left(\frac{1}{2}+i\gamma\right)=0}} \frac{1}{X^{*}\log X} \cdot X^{1/4} \cdot \frac{X^{3\sigma/4}}{(\gamma^2+1)^B} \cdot (|\gamma\log X| + 1)^n. \tag{2.13}$$

For sufficiently large B the sum over γ will converge. This term is of size $O(X^{-\frac{3}{4}(1-\sigma)+\epsilon})$. This error is $O(X^{-\epsilon})$ whenever $\sigma < 1$, and if $\sigma < 1/3$ then the error is at most $O(X^{-1/2+\epsilon})$.

The proof is completed by showing that the integral over $w = 2$ is negligible. We use Lemma B.2 (with $z = \tau - i2\frac{\log X}{2\pi}$) to show the d-sum is $O(X^{*}X^{-2+\epsilon})$. Arguing as above shows the integral is bounded by $O(X^{-2+2\sigma+\epsilon})$. (Note: some care is required, as there is a pole when $w = 2$ coming from the trivial zero of $\zeta(s)$ at $s = -2$. The contribution from the residue here is negligible; we could also adjust the contour to include a semicircle around $w = 2$ and use the residue theorem.) ■

REMARK 2.5. We sketch an alternate start of the proof of Lemma 2.1. One difficulty is that $R(g;X)$ is defined as an integral and there is a pole on the line of integration. We may write

$$\zeta(s) = (s-1)^{-1} \; + \; \left(\zeta(s) - (s-1)^{-1}\right). \tag{2.14}$$

For us $s = 1 - \frac{4\pi i\tau}{\log X}$, so the first factor is just $-\frac{\log X}{4\pi i\tau}$. As $g(\tau)$ is an even function, the main term of the integral of this piece is

$$\begin{aligned}\int_{-\infty}^{\infty} g(\tau)\frac{e^{-2\pi i\tau}}{2\pi i\tau}\, d\tau &= \int_{-\infty}^{\infty} g(\tau)\left(\frac{e^{-2\pi i\tau}}{4\pi i\tau} - \frac{e^{2\pi i\tau}}{4\pi i\tau}\right) d\tau \\ &= -\int_{-\infty}^{\infty} g(\tau)\frac{\sin(2\pi\tau)}{2\pi\tau}\, d\tau = -\frac{g(0)}{2},\end{aligned} \tag{2.15}$$

where the last equality is a consequence of $\operatorname{supp}(\widehat{g}) \subset (-1,1)$. The other terms from the $(s-1)^{-1}$ factor and the terms from the $\zeta(s) - (s-1)^{-1}$ piece are analyzed in a similar manner as the terms in the proof of Lemma 2.1.

2.2 Secondary term (of size $1/\log X$) of $R(g;X)$

LEMMA 2.6. *Let* $\operatorname{supp}(\widehat{g}) \subset (-\sigma,\sigma)$; *we do not assume* $\sigma < 1$. *Then the* $1/\log X$ *term in the expansion of* $R(g;X)$ *is*

$$\frac{1 - \frac{\Gamma'(\frac14)}{\Gamma(\frac14)} + 2\frac{\zeta'(2)}{\zeta(2)} - 2\gamma + 2\log\pi}{\log X}\,\widehat{g}(1). \tag{2.16}$$

It is important to note that this piece is only *present if the support of* $\widehat{g}$ *exceeds* $(-1,1)$ *(i.e. if* $\sigma > 1$*).* □

PROOF. We sketch the determination of the main and secondary terms of $R(g;X)$. We may restrict the integrals to $|\tau| \le \log^{1/4} X$ with negligible error; this will allow us to Taylor expand certain expressions and maintain good control over the errors. As g is a Schwartz function, for any $B > 0$ we have $g(\tau) \ll (1+\tau^2)^{-4B}$. The ratio of the Gamma factors is of absolute value 1, and $A_D(-r;r) = \zeta(2)/\zeta(2-2r) = O(1)$. Thus the contribution from $|\tau| \ge \log^{1/4} X$ is bounded by

$$\ll \int_{|\tau|\ge \log^{1/4} X} (1+\tau^2)^{-4B} \cdot \max\left(\frac{\log X}{\tau}, \frac{\tau^C}{\log^C \tau}\right) d\tau \;\ll\; (\log X)^{-B} \tag{2.17}$$

for B sufficiently large.

We use Lemma B.2 to evaluate the d-sum in (2.1) for $|\tau| \le \log^{1/4} X$; the error term is negligible and may be absorbed into the $O(\log^{-B} X)$ error. We now Taylor expand the three factors in (2.1). The main contribution comes from the pole of ζ; the other pieces contribute at the $1/\log X$ level.

We first expand the Gamma factors. We have

$$\frac{\Gamma\left(\frac{1}{4} - \frac{\pi i \tau}{\log X}\right)}{\Gamma\left(\frac{1}{4} + \frac{\pi i \tau}{\log X}\right)} = 1 - \frac{\Gamma'(\frac{1}{4})}{\Gamma(\frac{1}{4})}\frac{2\pi i \tau}{\log X} + O\left(\frac{\tau^2}{\log^2 X}\right). \tag{2.18}$$

As $A_D(-r; r) = \zeta(2)/\zeta(2-2r)$,

$$A_D\left(-\frac{2\pi i \tau}{\log X}; \frac{2\pi i \tau}{\log X}\right) = 1 + 2\frac{\zeta'(2)}{\zeta(2)}\frac{2\pi i \tau}{\log X} + O\left(\frac{\tau^2}{\log^2 X}\right). \tag{2.19}$$

Finally we expand the ζ-piece. We have (see [11]) that

$$\zeta(1 + iy) = \frac{1}{iy} + \gamma + O(y), \tag{2.20}$$

where γ is Euler's constant. Thus

$$\zeta\left(1 - \frac{4\pi i \tau}{\log X}\right) = -\frac{\log X}{4\pi i \tau} + \gamma + O\left(\frac{\tau}{\log X}\right). \tag{2.21}$$

We combine the Taylor expansions for the three pieces (the ratio of the Gamma factors, the ζ-function and A_D), and keep only the first two terms:

$$-\frac{\log X}{4\pi i \tau} + \left[\frac{1}{2}\frac{\Gamma'(\frac{1}{4})}{\Gamma(\frac{1}{4})} - \frac{\zeta'(2)}{\zeta(2)} + \gamma\right] + O\left(\frac{\tau}{\log X}\right). \tag{2.22}$$

Finally, we Taylor expand the d-sum, which was evaluated in Lemma B.2. We may ignore the error term there because it is $O(X^{1/2})$. The main term is

$$X^* e^{-2\pi i\left(1 - \frac{\log \pi}{\log X}\right)\tau}\left(1 - \frac{2\pi i \tau}{\log X}\right)^{-1} = X^* e^{-2\pi i\left(1 - \frac{\log \pi}{\log X}\right)\tau}\left(1 + \frac{2\pi i \tau}{\log X} + O\left(\frac{\tau^2}{\log^2 X}\right)\right). \tag{2.23}$$

Thus

$$\begin{aligned}
R(g;X) &= \frac{-2}{X^* \log X}\int_{-\log^{1/4} X}^{\log X} g(\tau)\cdot X^* e^{-2\pi i\left(1-\frac{\log \pi}{\log^{1/4} X}\right)\tau}\left(1+\frac{2\pi i\tau}{\log X}+O\left(\frac{\tau^2}{\log^2 X}\right)\right)\\
&\quad \times\left[-\frac{\log X}{4\pi i\tau}+\left(\frac{1}{2}\frac{\Gamma'(\frac14)}{\Gamma(\frac14)}-\frac{\zeta'(2)}{\zeta(2)}+\gamma\right)+O\left(\frac{\tau}{\log X}\right)\right]d\tau+O\left(\frac{1}{\log^B X}\right)\\
&= \frac{2}{\log X}\int_{-\log^{1/4} X}^{\log^{1/4} X} g(\tau)\cdot e^{-2\pi i\left(1-\frac{\log\pi}{\log X}\right)\tau}\cdot\left[\frac{\log X}{4\pi i\tau}+\left(\frac12-\frac12\frac{\Gamma'(\frac14)}{\Gamma(\frac14)}+\frac{\zeta'(2)}{\zeta(2)}-\gamma\right)\right]d\tau\\
&\quad +O\left(\frac{1}{\log^{5/4} X}\right).
\end{aligned}\tag{2.24}$$

We may write

$$e^{-2\pi i\left(1-\frac{\log\pi}{\log X}\right)\tau}=e^{-2\pi i\tau}\cdot\left(1+\frac{2\pi i\tau\log\pi}{\log X}+O\left(\frac{\tau^2}{\log^2 X}\right)\right).\tag{2.25}$$

The effect of this expansion is to change the $1/\log X$ term above by adding $\frac{\log\pi}{2}$.

Because g is a Schwartz function, we may extend the integration to all τ and absorb the error into our error term. The main term is from $(\log X)/4\pi i\tau$; it equals $-g(0)/2$ (see the analysis in Section 2.1). The secondary term is easily evaluated, as it is just the Fourier transform of g at 1. Thus

$$R(g;X)=-\frac{g(0)}{2}+\frac{1-\frac{\Gamma'(\frac14)}{\Gamma(\frac14)}+2\frac{\zeta'(2)}{\zeta(2)}-2\gamma+2\log\pi}{\log X}\widehat{g}(1)+O\left(\frac{1}{\log^{5/4} X}\right).\tag{2.26}$$

■

3 Analysis of the Terms from Number Theory

We now prove Theorem 1.2. The starting point is the Explicit Formula (Theorem 1.5, with each d an even fundamental discriminant). As the $\log(d/\pi)$ and the Γ'/Γ terms already appear in the expansion from the Ratios Conjecture (Theorem 1.1), we need only study the sums of $\chi_d(p)^k$. The analysis splits depending on whether or not k is even. Set

$$\begin{aligned}
S_{\text{even}} &= -\frac{2}{X^*}\sum_{d\le X}\sum_{\ell=1}^{\infty}\sum_p \frac{\chi_d(p)^2\log p}{p^\ell\log X}\,\widehat{g}\left(2\frac{\log p^\ell}{\log X}\right)\\
S_{\text{odd}} &= -\frac{2}{X^*}\sum_{d\le X}\sum_{\ell=0}^{\infty}\sum_p \frac{\chi_d(p)\log p}{p^{(2\ell+1)/2}\log X}\,\widehat{g}\left(\frac{\log p^{2\ell+1}}{\log X}\right).
\end{aligned}\tag{3.1}$$

On the basis of our analysis of the terms from the Ratios Conjecture, the proof of Theorem 1.2 is completed by the following lemma.

Lemma 3.1. *Let* $\mathrm{supp}(\widehat{g}) \subset (-\sigma, \sigma) \subset (-1, 1)$. *Then*

$$\begin{aligned} S_{\text{even}} &= -\frac{g(0)}{2} + \frac{2}{\log X}\int_{-\infty}^{\infty} g(\tau)\frac{\zeta'}{\zeta}\left(1 + \frac{4\pi i\tau}{\log X}\right) d\tau \\ &\quad + \frac{2}{\log X}\int_{-\infty}^{\infty} g(\tau)A_D'\left(\frac{2\pi i\tau}{\log X}; \frac{2\pi i\tau}{\log X}\right) + O(X^{-\frac{1}{2}+\epsilon}) \\ S_{\text{odd}} &= O(X^{-\frac{1-\sigma}{2}}\log^6 X). \end{aligned} \tag{3.2}$$

If instead we consider the family of characters χ_{8d} *for odd, positive square-free* $d \in (0, X)$ *(d a fundamental discriminant), then*

$$S_{\text{odd}} = O(X^{-1/2+\epsilon} + X^{-(1-\frac{3}{2}\sigma)+\epsilon}). \tag{3.3}$$

□

We prove Lemma 3.1 by analyzing S_{even} in Section 3.1 (in Lemmas 3.2 and 3.3) and S_{odd} in Section 3.2 (in Lemmas 3.4, 3.5 and 3.6).

3.1 Contribution from k even

The contribution from k even from the Explicit Formula is

$$S_{\text{even}} = -\frac{2}{X^*}\sum_{d\le X}\sum_{\ell=1}^{\infty}\sum_{p}\frac{\chi_d(p)^2\log p}{p^\ell\log X}\,\widehat{g}\left(2\frac{\log p^\ell}{\log X}\right), \tag{3.4}$$

where $\sum_{d\le X} 1 = X^*$, the cardinality of our family. Each $\chi_d(p)^2 = 1$ except when $p|d$. We replace $\chi_d(p)^2$ with 1, and subtract off the contribution from when $p|d$. We find

$$\begin{aligned} S_{\text{even}} &= -2\sum_{\ell=1}^{\infty}\sum_{p}\frac{\log p}{p^\ell\log X}\,\widehat{g}\left(2\frac{\log p^\ell}{\log X}\right) \\ &\quad + \frac{2}{X^*}\sum_{d\le X}\sum_{\ell=1}^{\infty}\sum_{p|d}\frac{\log p}{p^\ell\log X}\,\widehat{g}\left(2\frac{\log p^\ell}{\log X}\right) \\ &= S_{\text{even};1} + S_{\text{even};2}. \end{aligned} \tag{3.5}$$

In the next subsections we prove the following lemmas, which completes the analysis of the even k terms.

LEMMA 3.2. *Notation as above,*

$$S_{\text{even};1} = -\frac{g(0)}{2} + \frac{2}{\log X}\int_{-\infty}^{\infty} g(\tau)\frac{\zeta'}{\zeta}\left(1+\frac{4\pi i\tau}{\log X}\right)d\tau. \tag{3.6}$$

□

LEMMA 3.3. *Notation as above,*

$$S_{\text{even};2} = \frac{2}{\log X}\int_{-\infty}^{\infty} g(\tau)A_D'\left(\frac{2\pi i\tau}{\log X};\frac{2\pi i\tau}{\log X}\right) + O(X^{-\frac{1}{2}+\epsilon}). \tag{3.7}$$

□

3.1.1 Analysis of $S_{\text{even};1}$.

PROOF OF LEMMA 3.2. We have

$$S_{\text{even};1} = \frac{-2}{\log X}\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n}\widehat{g}\left(2\frac{\log n}{\log X}\right). \tag{3.8}$$

We use Perron's formula to rewrite $S_{\text{even};1}$ as a contour integral. For any $\epsilon > 0$ set

$$I_1 = \frac{1}{2\pi i}\int_{\Re(z)=1+\epsilon} g\left(\frac{(2z-2)\log A}{4\pi i}\right)\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^z}\,dz; \tag{3.9}$$

we will later take $A = X^{1/2}$. We write $z = 1+\epsilon+iy$ and use (A.6) (replacing ϕ with g) to write $g(x+iy)$ in terms of the integral of $\widehat{g}(u)$. We have

$$\begin{aligned} I_1 &= \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{1+\epsilon}}\frac{1}{2\pi i}\int_{-\infty}^{\infty} g\left(\frac{y\log A}{2\pi}-\frac{i\epsilon\log A}{2\pi}\right)e^{-iy\log n}i\,dy \\ &= \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{1+\epsilon}}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\left[\widehat{g}(u)e^{\epsilon u\log A}\right]e^{-2\pi i\frac{-y\log A}{2\pi}u}du\right]e^{-iy\log n}dy. \end{aligned} \tag{3.10}$$

We let $h_\epsilon(u) = \widehat{g}(u)e^{\epsilon u\log A}$. Note that h_ϵ is a smooth, compactly supported function and $\widehat{\widehat{h}}_\epsilon(w) = h_\epsilon(-w)$. Thus

$$\begin{aligned} I_1 &= \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{1+\epsilon}}\frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{h}_\epsilon\left(-\frac{y\log A}{2\pi}\right)e^{-iy\log n}dy \\ &= \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{1+\epsilon}}\frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{h}_\epsilon(y)e^{-2\pi i\frac{-y\log n}{\log A}}\frac{2\pi dy}{\log A} \\ &= \sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^{1+\epsilon}}\frac{1}{\log A}\widehat{\widehat{h}}_\epsilon\left(-\frac{\log n}{\log A}\right) \end{aligned}$$

$$= \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^{1+\epsilon}} \frac{1}{\log A} \widehat{g}\left(\frac{\log n}{\log A}\right) e^{\epsilon \log n}$$
$$= \frac{1}{\log A} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n} \widehat{g}\left(\frac{\log n}{\log A}\right). \tag{3.11}$$

By taking $A = X^{1/2}$ we find

$$S_{\text{even};1} = \frac{-2}{\log X} \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n} \widehat{g}\left(2\frac{\log n}{\log X}\right) = -I_1. \tag{3.12}$$

We now rewrite I_1 by shifting contours; we will not pass any poles as we shift. For each $\delta > 0$ we consider the contour made up of three pieces: $(1 - i\infty, 1 - i\delta]$, C_δ, and $[1-i\delta, 1+i\infty)$, where $C_\delta = \{z : z-1 = \delta e^{i\theta}, \theta \in [-\pi/2, \pi/2]\}$ is the semicircle going counter-clockwise from $1 - i\delta$ to $1 + i\delta$. By Cauchy's residue theorem, we may shift the contour in I_1 from $\Re(z) = 1 + \epsilon$ to the three curves above. Noting that $\sum_n \Lambda(n) n^{-z} = -\zeta'(z)/\zeta(z)$, we find that

$$I_1 = \frac{1}{2\pi i} \left[\int_{1-i\infty}^{1-i\delta} + \int_{C_\delta} + \int_{1+i\delta}^{1+i\infty} g\left(\frac{(2z-2)\log A}{4\pi i}\right) \frac{-\zeta'(z)}{\zeta(z)}\, dz\right]. \tag{3.13}$$

The integral over C_δ is easily evaluated. As $\zeta(s)$ has a pole at $s = 1$, it is just half the residue of $g\left(\frac{(2z-2)\log A}{4\pi i}\right)$ (the minus sign in front of $\zeta'(z)/\zeta(z)$ cancels the minus sign from the pole). Thus the C_δ piece is $g(0)/2$. We now take the limit as $\delta \to 0$:

$$I_1 = \frac{g(0)}{2} - \lim_{\delta \to 0} \frac{1}{2\pi} \left[\int_{-\infty}^{-\delta} + \int_{\delta}^{\infty} g\left(\frac{y \log A}{2\pi}\right) \frac{\zeta'(1+iy)}{\zeta(1+iy)}\, dy\right]. \tag{3.14}$$

As g is an even Schwartz function, the limit of the integral above is well-defined (for large y this follows from the decay of g, while for small y it follows from the fact that $\zeta'(1+iy)/\zeta(1+iy)$ has a simple pole at $y = 0$ and g is even). We again take $A = X^{1/2}$, and change variables to $\tau = \frac{y \log A}{2\pi} = \frac{y \log X}{4\pi}$. Thus

$$I_1 = \frac{g(0)}{2} - \frac{2}{\log X} \int_{-\infty}^{\infty} g(\tau) \frac{\zeta'}{\zeta}\left(1 + \frac{4\pi i \tau}{\log X}\right) d\tau, \tag{3.15}$$

which completes the proof of Lemma 3.2. ∎

3.1.2 Analysis of $S_{\text{even};2}$.

PROOF OF LEMMA 3.3. Recall

$$S_{\text{even};2} = \frac{2}{X^*} \sum_{d \le X} \sum_{\ell=1}^{\infty} \sum_{p|d} \frac{\log p}{p^\ell \log X} \widehat{g}\left(2 \frac{\log p^\ell}{\log X}\right). \tag{3.16}$$

We may restrict the prime sum to $p \le X^{1/2}$ at a cost of $O(\log\log X/X)$. We sketch the proof of this claim. Since $\widehat{g}$ has finite support, $p \le X^\sigma$ and thus the p-sum is finite. Since $d \le X$ and $p \ge X^{1/2}$, there are at most 2 primes which divide a given d. Thus

$$\begin{aligned} \frac{2}{X^*} \sum_{d \le X} \sum_{\ell=1}^{\infty} \sum_{\substack{p=X^{1/2} \\ p|d}}^{X^\sigma} \frac{\log p}{p^\ell \log X} \widehat{g}\left(2 \frac{\log p^\ell}{\log X}\right) &\ll \frac{1}{X^*} \sum_{\ell=1}^{\infty} \sum_{p=X^{1/2}}^{X^\sigma} \frac{1}{p^\ell} \sum_{\substack{d \le X \\ p|d}} 1 \\ &\ll \frac{1}{X^*} \sum_{p > X^{1/2}}^{X^\sigma} \frac{2}{p} \ll \frac{\log\log X}{X}. \end{aligned} \tag{3.17}$$

In Lemma B.1 we show that

$$X^* = \frac{3}{\pi^2} X + O(X^{1/2}) \tag{3.18}$$

and that for $p \le X^{1/2}$ we have

$$\sum_{\substack{d \le X \\ p|d}} 1 = \frac{X^*}{p+1} + O(X^{1/2}). \tag{3.19}$$

Using these facts we may complete the analysis of $S_{\text{even};2}$:

$$\begin{aligned} S_{\text{even};2} &= \frac{2}{X^*} \sum_{d \le X} \sum_{\ell=1}^{\infty} \sum_{\substack{p \le X^{1/2} \\ p|d}} \frac{\log p}{p^\ell \log X} \widehat{g}\left(2 \frac{\log p^\ell}{\log X}\right) + O\left(\frac{\log\log X}{X}\right) \\ &= \frac{2}{X^*} \sum_{\ell=1}^{\infty} \sum_{p \le X^{1/2}} \frac{\log p}{p^\ell \log X} \widehat{g}\left(2 \frac{\log p^\ell}{\log X}\right) \sum_{d \le X,\, p|d} 1 + O\left(\frac{\log\log X}{X}\right) \\ &= 2 \sum_{\ell=1}^{\infty} \sum_{p \le X^{1/2}} \frac{\log p}{p^\ell \log X} \cdot \frac{1}{p+1} \widehat{g}\left(2 \frac{\log p^\ell}{\log X}\right) \\ &\quad + O\left(\frac{X^{1/2}}{X} \sum_{\ell=1}^{\infty} \sum_{p \le X^{1/2}} \frac{1}{p^\ell} + \frac{\log\log X}{X}\right) \\ &= 2 \sum_{\ell=1}^{\infty} \sum_{p \le X^{1/2}} \frac{\log p}{p^\ell \log X} \cdot \frac{1}{p+1} \widehat{g}\left(2 \frac{\log p^\ell}{\log X}\right) + O(X^{-\frac{1}{2}+\epsilon}). \end{aligned} \tag{3.20}$$

We rewrite $\widehat{g}(2\log p^\ell/\log X)$ by expanding the Fourier transform.

$$\begin{aligned}
S_{\text{even};2} &= 2\sum_{\ell=1}^{\infty}\sum_{p\le X^{1/2}} \frac{\log p}{(p+1)p^\ell \log X}\int_{-\infty}^{\infty} g(\tau)e^{-2\pi i\tau\cdot 2\log p^\ell/\log X}d\tau + O(X^{-\frac{1}{2}+\epsilon})\\
&= 2\sum_{p\le X^{1/2}} \frac{\log p}{(p+1)\log X}\int_{-\infty}^{\infty} g(\tau)\sum_{\ell=1}^{\infty} p^{-\ell}\cdot p^{-2\pi i\tau\cdot 2\ell/\log X}d\tau + O(X^{-\frac{1}{2}+\epsilon})\\
&= 2\sum_{p\le X^{1/2}} \frac{\log p}{(p+1)\log X}\int_{-\infty}^{\infty} g(\tau)p^{-(1+2\cdot\frac{2\pi i\tau}{\log X})}\left(1-p^{-(1+2\cdot\frac{2\pi i\tau}{\log X})}\right)^{-1}d\tau + O(X^{-\frac{1}{2}+\epsilon}).
\end{aligned} \tag{3.21}$$

We may extend the p-sum to be over all primes at a cost of $O(X^{-1/2+\epsilon})$; this is because the summands are $O(\log p/p^2)$ and g is Schwartz. Recalling the definition of $A_D'(r;r)$ in (1.4), we see that the resulting p-sum is just $A_D'(2\pi i\tau/\log X; 2\pi i\tau/\log X)$; this completes the proof of Lemma 3.3. ■

3.2 Contribution from k odd

As k is odd, $\chi_d(p)^k = \chi_d(p)$. Thus we must analyze the sum

$$S_{\text{odd}} = -\frac{2}{X^*}\sum_{d\le X}\sum_{\ell=0}^{\infty}\sum_{p}\frac{\chi_d(p)\log p}{p^{(2\ell+1)/2}\log X}\,\widehat{g}\left(\frac{\log p^{2\ell+1}}{\log X}\right). \tag{3.22}$$

If $\text{supp}(\widehat{g}) \subset (-1,1)$, Rubinstein [42] showed (by applying Jutila's bound [23–25] for quadratic character sums) that if our family is all discriminants then $S_{\text{odd}} = O(X^{-\epsilon/2})$. In his dissertation Gao [17] extended these results to show that the odd terms do not contribute to the main term provided that $\text{supp}(\widehat{g}) \subset (-2,2)$. His analysis proceeds by using Poisson summation to convert long character sums to shorter ones. We shall analyze S_{odd} using both methods: Jutila's bound gives a self-contained presentation, but a much weaker result; the Poisson summation approach gives a better bound but requires a careful book-keeping of many of Gao's lemmas (as well as an improvement of one of his estimates).

3.2.1 Analyzing S_{odd} with Jutila's bound.

LEMMA 3.4. *Let* $\text{supp}(\widehat{g}) \subset (-\sigma,\sigma)$. *Then* $S_{\text{odd}} = O(X^{-\frac{1-\sigma}{2}}\log^6 X)$. □

PROOF. Jutila's bound (see (3.4) of [25]) is

$$\sum_{\substack{1<n\le N\\ n \text{ nonsquare}}} \left| \sum_{\substack{0<d\le X\\ d \text{ fund. disc.}}} \chi_d(n) \right|^2 \ll NX \log^{10} N \tag{3.23}$$

(note the d-sum is over even fundamental discriminants at most X). As $2\ell+1$ is odd, $p^{2\ell+1}$ is never a square. Thus Jutila's bound gives

$$\left(\sum_{\ell=0}^{\infty} \sum_{p^{(2\ell+1)/2}\le X^{\sigma}} \left| \sum_{d\le X} \chi_d(p) \right|^2 \right)^{1/2} \ll X^{\frac{1+\sigma}{2}} \log^5 X. \tag{3.24}$$

Recall

$$S_{\text{odd}} = -\frac{2}{X^*} \sum_{\ell=0}^{\infty} \sum_{p} \frac{\log p}{p^{(2\ell+1)/2} \log X}\, \widehat{g}\left(\frac{\log p^{2\ell+1}}{\log X} \right) \sum_{d\le X} \chi_d(p). \tag{3.25}$$

We apply Cauchy–Schwartz, and find

$$\begin{aligned} |S_{\text{odd}}| &\le \frac{2}{X^*} \left(\sum_{\ell=0}^{\infty} \sum_{p^{2\ell+1}\le X^{\sigma}} \left| \frac{\log p}{p^{(2\ell+1)/2}\log X}\, \widehat{g}\left(\frac{\log p^{2\ell+1}}{\log X} \right) \right|^2 \right)^{1/2} \\ &\quad \times \left(\sum_{\ell=0}^{\infty} \sum_{p^{2\ell+1}\le X^{\sigma}} \left| \sum_{d\le X} \chi_d(p) \right|^2 \right)^{1/2} \\ &\ll \frac{2}{X^*} \left(\sum_{n\le X^{\sigma}} \frac{1}{n} \right)^{1/2} \cdot X^{\frac{1+\sigma}{2}} \log^5 X \\ &\ll X^{-\frac{1-\sigma}{2}} \log^6 X; \end{aligned} \tag{3.26}$$

thus there is a power savings if $\sigma < 1$. ■

3.2.2 Analyzing S_{odd} with Poisson summation. Gao analyzes the contribution from S_{odd} by applying Poisson summation to the character sums. The computations are simplified if the character $\chi_2(n) = \left(\frac{2}{n}\right)$ is not present. He therefore studies the family of odd, positive square-free d (where d is a fundamental discriminant). His family is

$$\{8d :\ X < d \le 2X,\ d \text{ an odd square} - \text{free fundamental discriminant}\}; \tag{3.27}$$

we discuss in Lemma 3.6 how to easily modify the arguments to handle the related family with $0 < d \leq X$. The calculation of the terms from the Ratios Conjecture proceeds similarly (the only modification is to X^*, which also leads to a trivial modification of Lemma B.2 which does not change any terms larger than $O(X^{-1/2+\epsilon})$ if $\mathrm{supp}(\widehat{g}) \subset (-1/3, 1/3)$), as does the contribution from $\chi(p)^k$ with k even. We are left with bounding the contribution from S_{odd}. The following lemma shows that we can improve on the estimate obtained by applying Jutila's bound.

LEMMA 3.5. *Let* $\mathrm{supp}(\widehat{g}) \subset (-\sigma, \sigma) \subset (-1, 1)$. *Then for the family given in* (3.27), $S_{\mathrm{odd}} = O(X^{-\frac{1}{2}+\epsilon} + X^{-(1-\frac{3}{2}\sigma)+\epsilon})$. *In particular, if* $\sigma < 1/3$ *then* $S_{\mathrm{odd}} = O(X^{-1/2+\epsilon})$. □

PROOF. Gao is only concerned with main terms for the n-level density (for any n) for all sums. As we only care about S_{odd} for the 1-level density, many of his terms are not present. We highlight the arguments. We concentrate on the $\ell = 0$ term in (3.22) (the other $\ell \ll \log X$ terms are handled similarly, and the finite support of $\widehat{g}$ implies that $S_{\mathrm{odd}}(\ell) = 0$ for $\ell \gg \log X$):

$$S_{\mathrm{odd}} = -\frac{2}{X^*} \sum_{d \leq X} \sum_{\ell=0}^{\infty} \sum_{p} \frac{\chi_d(p) \log p}{p^{(2\ell+1)/2} \log X} \widehat{g}\left(\frac{\log p^{2\ell+1}}{\log X}\right) = \sum_{\ell=0}^{\infty} S_{\mathrm{odd}}(\ell). \tag{3.28}$$

Let $Y = X^\sigma$, where $\mathrm{supp}(\widehat{g}) \subset (-\sigma, \sigma)$. Our sum $S_{\mathrm{odd}}(0)$ is $S(X, Y, \widehat{g})$ in Gao's thesis:

$$S(X, Y, \widehat{g}) = \sum_{\substack{X<d<2X \\ (2,d)=1}} \mu(d)^2 \sum_{p<Y} \frac{\log p}{\sqrt{p}} \chi_{8d}(p) \widehat{g}\left(\frac{\log p}{\log X}\right). \tag{3.29}$$

Let Φ be a smooth function supported on $(1, 2)$ such that $\Phi(t) = 1$ for $t \in (1 + U^{-1}, 2 - U^{-1})$ and $\Phi^{(j)}(t) \ll_j U^j$ for all $j \geq 0$. We show that $S(X, Y, \widehat{g})$ is well approximated by the smoothed sum $S(X, Y, \widehat{g}, \Phi)$, where

$$S(X, Y, \widehat{g}, \Phi) = \sum_{(d,2)=1} \mu(d)^2 \sum_{p<Y} \frac{\log p}{\sqrt{p}} \chi_{8d}(p) \widehat{g}\left(\frac{\log p}{\log X}\right) \Phi\left(\frac{d}{X}\right). \tag{3.30}$$

To see this, note the difference between the two involved summing $d \in (X, X + X/U)$ and $d \in (2X - X/U, 2X)$. We trivially bound the prime sum for each fixed d by $\log^7 X$ (see Proposition III.1 of [17]). As there are $O(X/U)$ choices of d and $\Phi(d/X) \ll 1$, we have

$$S(X, Y, \widehat{g}) - S(X, Y, \widehat{g}, \Phi) \ll \frac{X \log^7 X}{U}. \tag{3.31}$$

We will take $U = \sqrt{X}$. Thus upon dividing by $X^* \gg X$ (the cardinality of the family), this difference is $O(X^{-1/2+\epsilon})$. The proof is completed by bounding $S(X, Y, \widehat{g}, \Phi)$.

To analyze $S(X, Y, \widehat{g}, \Phi)$, we write it as $S_M(X, Y, \widehat{g}, \Phi) + S_R(X, Y, \widehat{g}, \Phi)$, with

$$\begin{aligned} S_M(X, Y, \widehat{g}, \Phi) &= \sum_{(d,2)=1} M_Z(d) \sum_{p<Y} \frac{\log p}{\sqrt{p}} \chi_{8d}(p) \widehat{g}\left(\frac{\log p}{\log X}\right) \Phi\left(\frac{d}{X}\right) \\ S_R(X, Y, \widehat{g}, \Phi) &= \sum_{(d,2)=1} R_Z(d) \sum_{p<Y} \frac{\log p}{\sqrt{p}} \chi_{8d}(p) \widehat{g}\left(\frac{\log p}{\log X}\right) \Phi\left(\frac{d}{X}\right), \end{aligned} \tag{3.32}$$

where

$$\begin{aligned} \mu(d)^2 &= M_Z(d) + R_Z(d) \\ M_Z(d) &= \sum_{\substack{\ell^2 | d \\ \ell \le Z}} \mu(\ell), \quad R_Z(d) = \sum_{\substack{\ell^2 | d \\ \ell > Z}} \mu(\ell); \end{aligned} \tag{3.33}$$

here Z is a parameter to be chosen later, and $S_M(X, Y, \widehat{g}, \Phi)$ will be the main term (for a general n-level density sum) and $S_R(X, Y, \widehat{g}, \Phi)$ the error term. In our situation, both will be small.

In Lemma III.2 of [17], Gao proves that $S_R(X, Y, \widehat{g}, \Phi) \ll (X \log^3 X)/Z$. We have not divided any of our sums by the cardinality of the family (which is of size X). Thus for this term to yield contributions of size $X^{-1/2+\epsilon}$, we need $Z \ge X^{1/2}$.

We now analyze $S_M(X, Y, \widehat{g}, \Phi)$. Applying Poisson summation we convert long character sums to short ones. We need certain Gauss-type sums:

$$\left(\frac{1+i}{2} + \left(\frac{-1}{k}\right)\frac{1-i}{2}\right) G_m(k) = \sum_{a \bmod k} \left(\frac{a}{k}\right) e^{2\pi i a m/k}. \tag{3.34}$$

For a Schwartz function F let

$$\widetilde{F}(\xi) = \frac{1+i}{2}\widehat{F}(\xi) + \frac{1-i}{2}\widehat{F}(-\xi). \tag{3.35}$$

Using Lemma 2.6 of [45], we have (see page 32 of [17])

$$\begin{aligned} S_M(X, Y, \widehat{g}, \Phi) = \frac{X}{2} \sum_{2<p<Y} \frac{\log p}{p^{3/2}} \widehat{g}\left(\frac{\log p}{\log X}\right) \\ \times \sum_{\substack{\alpha \le Z \\ (\alpha, 2p)=1}} \frac{\mu(\alpha)}{\alpha} \sum_{m=0}^{\infty} (-1)^m G_m(p) \widetilde{\Phi}\left(\frac{mX}{2\alpha^2 p}\right). \end{aligned} \tag{3.36}$$

We follow the arguments in Chapter 3 of [17]. The $m = 0$ term is analyzed in Section 3.3 for the general n-level density calculations. It is zero if n is odd, and we do not need to worry about this error term (thus we do not see the error terms of size $X \log^{n-1} X$ or $(X \log^n X)/Z$ which appear in his later estimates). In Section 3.4 Gao analyzes the contributions from the nonsquare m in (3.36). In his notation, we have $k = 1$, $k_2 = 0$, $k_1 = 0$, $\alpha_1 = 1$ and $\alpha_0 = 0$, and these terms' contribution is $\ll (U^2 Z\sqrt{Y}\log^7 X)/X$ (remember we have not divided by the cardinality of the family, which is of order X). This is too large for our purposes (we have seen that we must take $U = Z = \sqrt{X}$ and $Y = X^\sigma$). We perform a more careful analysis of these terms in Appendix C, and bound these terms' contribution by

$$\frac{UZ\sqrt{Y}\log^7 X}{X} + \frac{UZY^{3/2}\log^4 X}{X} + \frac{Z^3U^2Y^{7/2}\log^4 X}{X^{4018-2\epsilon}}. \tag{3.37}$$

Lastly, we must analyze the contribution from m a square in (3.36). From Lemma III.3 of [17] we have that $G_m(p) = 0$ if $p|m$. If $p \nmid m$ and m is a square, then $G_m(p) = \sqrt{p}$. Arguing as in [17], we are left with

$$\sum_{\substack{p<Y\\(p,2)=1}} \frac{\log p}{p}\widehat{g}\left(\frac{\log p}{\log X}\right) \sum_{\substack{\alpha\le Z\\(\alpha,2p)=1}} \frac{\mu(\alpha)}{\alpha^2}\left[\sum_{m=1}^{\infty}(-1)^m\widetilde{\Phi}\left(\frac{m^2X}{2\alpha^2 p}\right) - \sum_{\widetilde{m}=1}^{\infty}(-1)^{\widetilde{m}}\widetilde{\Phi}\left(\frac{p^2\widetilde{m}^2X}{2\alpha^2 p}\right)\right]. \tag{3.38}$$

If we assume $\mathrm{supp}(\widehat{g}) \subset (-1,1)$, then arguing as on page 41 of [17] we find the m-sum above is $\ll \alpha\sqrt{p/X}$, which leads to a contribution $\ll \sqrt{Y/X}\log X \log Z$; the $\widetilde{m}$-sum is $\ll \alpha/\sqrt{pX}$ and is thus dominated by the contribution from the m-sum.

Collecting all our bounds, we see a careful book-keeping leads to smaller errors than in Section 3.6 of [17] (this is because (1) many of the error terms only arise from n-level density sums with n even, where there are main terms and (2) we did a more careful analysis of some of the errors). We find that

$$S(X,Y,\widehat{g},\Phi) \ll \frac{X\log^3 X}{Z} + \frac{UZ\sqrt{Y}\log^7 X}{X} + \frac{UZY^{3/2}\log^4 X}{X} + \frac{\sqrt{Y}\log X\log Z}{\sqrt{X}}. \tag{3.39}$$

We divide this by $X^* \gg X$ (the cardinality of the family). By choosing $Z = X^{1/2}$, $Y = X^\sigma$ with $\sigma < 1$, and $U = \sqrt{X}$ (remember we need such a large U to handle the error from smoothing the d-sum, i.e. showing $|S(X,Y,\widehat{g}) - S(X,Y,\widehat{g},\Phi)|/X \ll X^{-1/2+\epsilon}$), we find

$$S(X,Y,\widehat{g},\Phi)/X \ll X^{-1/2+\epsilon} + X^{-(1-\frac{3}{2}\sigma)+\epsilon}, \tag{3.40}$$

which yields

$$S_{\text{odd}} \ll X^{-1/2+\epsilon} + X^{-(1-\frac{3}{2}\sigma)+\epsilon}. \tag{3.41}$$

Note that if $\sigma < 1/3$ then $S_{\text{odd}} \ll X^{-1/2+\epsilon}$. ■

LEMMA 3.6. *Let* $\text{supp}(\widehat{g}) \subset (-\sigma, \sigma) \subset (-1, 1)$. *Then for the family*

$$\{8d : 0 < d \leq X, \ d \text{ an odd square-free fundamental discriminant}\} \tag{3.42}$$

we have $S_{\text{odd}} = O(X^{-1/2+\epsilon} + X^{-(1-\frac{3}{2}\sigma)+\epsilon})$. *In particular, if* $\sigma < 1/3$ *then* $S_{\text{odd}} = O(X^{-1/2+\epsilon})$. □

PROOF. As the calculation is standard, we merely sketch the argument. We write

$$(0, X] = \bigcup_{i=1}^{\log_2 X} \left(\frac{2X}{2^{i+1}}, \frac{2X}{2^i} \right]. \tag{3.43}$$

Let $X_i = X/2^i$. For each i, in Lemma 3.5 we replace most of the X's with X_i, U with $U/\sqrt{2^i}$, Z with $Z/\sqrt{2^i}$; the X's we do not replace are the cardinality of the family (which we divide by in the end) and the $\log X$ which occurs when we evaluate the test function $\widehat{g}$ at $\log p / \log X$. We do not change Y, which controls the bounds for the prime sum. As we do not have any main terms, there is no loss in scaling the prime sums by $\log X$ instead of $\log X_i$. We do not use much about the test function $\widehat{g}$ in our estimates. All we use is that the prime sums are restricted to $p < Y$, and therefore we will still have bounds of Y (to various powers) for our sums.

We now finish the book-keeping. Expressions such as UZ/X in (3.39) are still $O(1)$, and expressions such as X/U and X/Z are now smaller. When we divide by the cardinality of the family we still have terms such as $Y^{3/2}/X$, and thus the support requirements are unchanged (i.e. $S_{\text{odd}} \ll X^{-1/2+\epsilon} + X^{-(1-\frac{3}{2}\sigma)+\epsilon}$). ■

A The Explicit Formula

We quickly review some needed facts about Dirichlet characters; see [11] for details. Let χ_d be a primitive quadratic Dirichlet character of modulus $|d|$. Let $c(d, \chi_d)$ be the Gauss sum

$$c(d, \chi_d) = \sum_{k=1}^{d-1} \chi_d(k) e^{2\pi i k/d}, \tag{A.1}$$

which is of modulus $\sqrt{d}$. Let

$$L(s,\chi_d) = \prod_p \left(1-\chi_d(p)p^{-s}\right)^{-1} \tag{A.2}$$

be the L-function attached to χ_d; the completed L-function is

$$\Lambda(s,\chi_d) = \pi^{-(s+a)/2}\Gamma\left(\frac{s+a}{2}\right) d^{-(s+a)/2} L(s,\chi_d) = (-1)^a \frac{c(d,\chi_d)}{\sqrt{d}} \Lambda(1-s,\overline{\chi_d}), \tag{A.3}$$

where

$$a = a(\chi_d) = \begin{cases} 0 & \text{if } \chi_d(-1) = 1 \\ 1 & \text{if } \chi_d(-1) = -1. \end{cases} \tag{A.4}$$

We write the zeros of $\Lambda(s,\chi_d)$ as $\frac{1}{2}+i\gamma$; if we assume GRH then $\gamma \in \mathbb{R}$. Let ϕ be an even Schwartz function and $\widehat{\phi}$ be its Fourier transform ($\widehat{\phi}(\xi) = \int \phi(x)e^{-2\pi i x\xi}dx$); we often assume $\operatorname{supp}(\widehat{\phi}) \subset (-\sigma,\sigma)$ for some $\sigma < \infty$. We set

$$H(s) = \phi\left(\frac{s-\frac{1}{2}}{i}\right). \tag{A.5}$$

While $H(s)$ is initially define only when $\Re(s) = 1/2$, because of the compact support of $\widehat{\phi}$ we may extend it to all of $\mathbb{C}$:

$$\begin{aligned} \phi(x) &= \int_{-\infty}^{\infty} \widehat{\phi}(\xi)e^{2\pi i x\xi}d\xi \\ \phi(x+iy) &= \int_{-\infty}^{\infty} \widehat{\phi}(\xi)e^{2\pi i(x+iy)\xi}d\xi \\ H(x+iy) &= \int_{-\infty}^{\infty} \left[\widehat{\phi}(\xi)e^{2\pi\left(x-\frac{1}{2}\right)}\right] \cdot e^{2\pi i y\xi}d\xi. \end{aligned} \tag{A.6}$$

Note that $H(x+iy)$ is rapidly decreasing in y (for a fixed x it is the Fourier transform of a nice function, and thus the claim follows from the Riemann–Lebesgue lemma). We now derive the Explicit Formula for quadratic characters; note the functional equation will always be even. We follow the argument given in [44].

Proof of the Explicit Formula, Theorem 1.5. We have

$$\begin{aligned}\Lambda(s,\chi) &= \pi^{-(s+a)/2}\Gamma\left(\frac{s+a}{2}\right)d^{(s+a)/2}L(s,\chi_d) = \Lambda(1-s,\chi_d)\\ \frac{\Lambda'(s,\chi_d)}{\Lambda(s,\chi_d)} &= -\frac{\log\pi}{2}+\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s+a}{2}\right)+\frac{\log d}{2}+\frac{L'(s,\chi_d)}{L(s,\chi_d)}\\ \frac{L'(s,\chi_d)}{L(s,\chi_d)} &= -\sum_p \frac{\chi_d(p)\log p}{1-\chi_d(p)p^{-s}} = -\sum_{k=1}^{\infty}\sum_p \frac{\chi_d(p)^k\log p}{p^{ks}}.\end{aligned} \tag{A.7}$$

We will *not* approximate any terms; we are keeping all lower order terms to facilitate comparison with the L-functions Ratios Conjecture. We set

$$I = \frac{1}{2\pi i}\int_{\Re(s)=3/2}\frac{\Lambda'(s,\chi_d)}{\Lambda(s,\chi_d)}H(s)ds. \tag{A.8}$$

We shift the contour to $\Re(s) = -1/2$. We pick up contributions from the zeros and poles of $\Lambda(s,\chi_d)$. As χ_d is not the principal character, there is no pole from $L(s,\chi_d)$. There is also no need to worry about a zero or pole from the Gamma factor $\Gamma\left(\frac{s+a}{2}\right)$ as $L(1,\chi_d)\neq 0$. Thus the only contribution is from the zeros of $\Lambda(s,\chi_d)$; the residue at a zero $\frac{1}{2}+i\gamma$ is $\phi(\gamma)$. Therefore

$$I = \sum_\gamma \phi(\gamma) + \frac{1}{2\pi i}\int_{\Re(s)=-1/2}\frac{\Lambda'(s,\chi_d)}{\Lambda(s,\chi_d)}H(s)ds. \tag{A.9}$$

As $\Lambda(1-s,\chi_d) = \Lambda(s,\chi_d)$, $-\Lambda'(1-s,\chi_d) = \Lambda(s,\chi_d)$ and

$$I = \sum_\gamma \phi(\gamma) - \frac{1}{2\pi i}\int_{\Re(s)=-1/2}\frac{\Lambda'(1-s,\chi_d)}{\Lambda(1-s,\chi_d)}H(s)ds. \tag{A.10}$$

We change variables (replacing s with $1-s$), and then use the functional equation:

$$I = \sum_\gamma \phi(\gamma) - \frac{1}{2\pi i}\int_{\Re(s)=3/2}\frac{\Lambda'(s,\chi_d)}{\Lambda(s,\chi_d)}H(1-s)ds. \tag{A.11}$$

Recalling the definition of I gives

$$\sum_\gamma \phi(\gamma) = \frac{1}{2\pi i}\int_{\Re(s)=3/2}\frac{\Lambda'(s,\chi_d)}{\Lambda(s,\chi_d)}\left[H(s)+H(1-s)\right]ds. \tag{A.12}$$

We expand $\Lambda'(s,\chi_d)/\Lambda(s,\chi_d)$ and shift the contours of all terms except $L'(s,\chi_d)/L(s,\chi_d)$ to $\Re(s) = 1/2$ (this is permissible as we do not pass through any zeros or poles of the other terms); note that if $s = \frac{1}{2} + iy$ then $H(s) = H(1-s) = \phi(y)$ (ϕ is even). Expanding the logarithmic derivative of $\Lambda(s, \chi_d)$ gives

$$\begin{aligned}\sum_\gamma \phi(\gamma) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\log\frac{d}{\pi} + \frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{a}{2}+\frac{iy}{2}\right)\right]\phi(y)dy \\ &\quad + \frac{1}{2\pi i}\int_{\Re(s)=3/2}\frac{L'(s,\chi_d)}{L(s,\chi_d)}\cdot[H(s)+H(1-s)]\,ds \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\log\frac{d}{\pi} + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{a}{2}+\frac{iy}{2}\right) + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{a}{2}-\frac{iy}{2}\right)\right]\phi(y)dy \\ &\quad + \frac{1}{2\pi i}\int_{\Re(s)=3/2}\frac{L'(s,\chi_d)}{L(s,\chi_d)}\cdot[H(s)+H(1-s)]\,ds, \end{aligned} \tag{A.13}$$

where the last line follows from the fact that ϕ is even.

We use (A.7) to expand L'/L. In the arguments below we shift the contour to $\Re s = 1/2$; this is permissible because of the compact support of $\widehat{\phi}$ (see (A.6)):

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\Re(s)=3/2}\frac{L'}{L}(s+iy)\cdot[H(s)+H(1-s)]\,dy \\ &= -\frac{1}{2\pi i}\sum_{k=1}^{\infty}\sum_p \chi_d(p)^k\log p\int_{\Re(s)=3/2}[H(s)+H(1-s)]\,e^{-ks\log p}dy \\ &= -\frac{2}{2\pi}\sum_{k=1}^{\infty}\sum_p\frac{\chi_d(p)^k\log p}{p^{k/2}}\int_{-\infty}^{\infty}\phi(y)e^{-2\pi i y\cdot\frac{\log p^k}{2\pi}}dy \\ &= -\frac{2}{2\pi}\sum_{k=1}^{\infty}\sum_p\frac{\chi_d(p)^k\log p}{p^{k/2}}\,\widehat{\phi}\left(\frac{\log p^k}{2\pi}\right). \end{aligned} \tag{A.14}$$

We therefore find that

$$\begin{aligned}\sum_\gamma \phi(\gamma) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\log\frac{d}{\pi} + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{a}{2}+\frac{iy}{2}\right) + \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{a}{2}-\frac{iy}{2}\right)\right]\phi(y)dy \\ &\quad -\frac{2}{2\pi}\sum_{k=1}^{\infty}\sum_p\frac{\chi_d(p)^k\log p}{p^{k/2}}\,\widehat{\phi}\left(\frac{\log p^k}{2\pi}\right). \end{aligned} \tag{A.15}$$

We replace $\phi(x)$ with $g(x) = \phi\left(x\cdot\frac{\log X}{2\pi}\right)$. A standard computation gives $\widehat{g}(\xi) = \frac{2\pi}{\log X}\,\widehat{\phi}\left(\xi\cdot\frac{2\pi}{\log X}\right)$. Summing over $d \in \mathcal{F}(X)$ completes the proof. ■

B Sums over Fundamental Discriminants

LEMMA B.1. *Let d denote an even fundamental discriminant at most X, and set $X^* = \sum_{d \leq X} 1$. Then*

$$X^* = \frac{3}{\pi^2} X + O(X^{1/2}) \tag{B.1}$$

and for $p \leq X^{1/2}$ we have

$$\sum_{\substack{d \leq X \\ p|d}} 1 = \frac{X^*}{p+1} + O(X^{1/2}). \tag{B.2}$$

□

PROOF. We first prove the claim for X^*, and then indicate how to modify the proof when $p|d$. We could show this by recognizing certain products as ratios of zeta functions or by using a Tauberian theorem; instead we shall give a straightforward proof suggested to us by Tim Browning (see also [38]).

We first assume that $d \equiv 1 \bmod 4$, so we are considering even fundamental discriminants $\{d \leq X : d \equiv 1 \bmod 4, \mu(d)^2 = 1\}$; it is trivial to modify the arguments below for d such that $d/4 \equiv 2$ or 3 modulo 4 and $\mu(d/4)^2 = 1$. Let $\chi_4(n)$ be the nontrivial character modulo 4: $\chi_4(2m) = 0$ and

$$\chi_4(n) = \begin{cases} 1 & \text{if } n \equiv 1 \bmod 4 \\ 0 & \text{if } n \equiv 3 \bmod 4. \end{cases} \tag{B.3}$$

We have

$$\begin{aligned} S(X) &= \sum_{\substack{d \leq X \\ \mu(d)^2 = 1,\ d \equiv 1 \bmod 4}} 1 \\ &= \sum_{\substack{d \leq X \\ 2 \nmid d}} \mu(d)^2 \cdot \frac{1 + \chi_4(d)}{2} \\ &= \frac{1}{2} \sum_{\substack{d \leq X \\ 2 \nmid d}} \mu(d)^2 + \frac{1}{2} \sum_{d \leq X} \mu(d)^2 \chi_4(d) = S_1(X) + S_2(X). \end{aligned} \tag{B.4}$$

By Möbius inversion

$$\sum_{m^2 | d} \mu(m) = \begin{cases} 1 & \text{if } d \text{ is square-free} \\ 0 & \text{otherwise.} \end{cases} \tag{B.5}$$

Thus

$$\begin{aligned}
S_1(X) &= \frac{1}{2}\sum_{\substack{d\le X\\ 2\nmid d}}\sum_{m^2|d}\mu(m)\\
&= \frac{1}{2}\sum_{\substack{m\le X^{1/2}\\ 2\nmid m}}\mu(m)\cdot\sum_{\substack{d\,\le\, X/m^2\\ 2\nmid d}}1\\
&= \frac{1}{2}\sum_{\substack{m\le X^{1/2}\\ 2\nmid m}}\mu(m)\left(\frac{X}{2m^2}+O(1)\right)\\
&= \frac{X}{4}\sum_{\substack{m=1\\ 2\nmid m}}^{\infty}\frac{\mu(m)}{m^2}+O(X^{1/2})\\
&= \frac{1}{4}\frac{6}{\zeta(2)}\cdot\left(1-\frac{1}{2^2}\right)^{-1}\cdot X+O(X^{1/2})\\
&= \frac{2}{\pi^2}X+O(X^{1/2})
\end{aligned}\tag{B.6}$$

(because we are missing the factor corresponding to 2 in $1/\zeta(2)$ above). Arguing in a similar manner shows $S_2(X) = O(X^{1/2})$; this is due to the presence of χ_4, giving us

$$S_2(X) = \frac{1}{2}\sum_{m\le X^{1/2}}\chi_4(m^2)\mu(m)\sum_{d\le X/m^2}\chi_4(d) \ll X^{1/2} \tag{B.7}$$

(because we are summing χ_4 at consecutive integers, and thus this sum is at most 1). A similar analysis shows that the number of even fundamental discriminants $d \le X$ with $d/4 \equiv 2$ or 3 modulo 4 is $X/\pi^2 + O(X^{1/2})$. Thus

$$\sum_{\substack{d\le X\\ d \text{ an even fund. disc.}}} 1 = X^* = \frac{3}{\pi^2}X + O(X^{1/2}). \tag{B.8}$$

We may trivially modify the above calculations to determine the number of even fundamental discriminants $d \le X$ with $p|d$ for a fixed prime p. We first assume $p \equiv 1 \bmod 4$. In (B.4) we replace $\mu(d)^2$ with $\mu(pd)^2$, $d \le X$ with $d \le X/p$, $2\nmid d$ and $(2p, d) = 1$. These imply that $d \le X$, $p|d$ and p^2 does not divide d. As d and p are relatively prime, $\mu(pd) = \mu(p)\mu(d)$ and the main term becomes

$$\begin{aligned}
S_{1;p}(X) &= \frac{1}{2} \sum_{\substack{d \le X/p \\ (2p,d)=1}} \sum_{m^2|d} \mu(m) \\
&= \frac{1}{2} \sum_{\substack{m \le (X/p)^{1/2} \\ (2p,m)=1}} \mu(m) \cdot \sum_{\substack{d \,\le\, (X/p)/m^2 \\ (2p,d)=1}} 1 \\
&= \frac{1}{2} \sum_{\substack{m \le (X/p)^{1/2} \\ (2p,m)=1}} \mu(m) \left(\frac{X/p}{m^2} \cdot \frac{p-1}{2p} + O(1) \right) \\
&= \frac{(p-1)X}{4p^2} \sum_{\substack{m=1 \\ (2p,m)=1}}^{\infty} \frac{\mu(m)}{m^2} + O(X^{1/2}) \\
&= \frac{1}{4} \frac{6}{\zeta(2)} \cdot \left(1 - \frac{1}{2^2}\right)^{-1} \cdot \left(1 - \frac{1}{p^2}\right)^{-1} \cdot \frac{(p-1)X}{p^2} + O(X^{1/2}) \\
&= \frac{2X}{(p+1)\pi^2} + O(X^{1/2}),
\end{aligned} \tag{B.9}$$

and the cardinality of this piece is reduced by $(p+1)^{-1}$ (note above we used $\#\{n \le Y : (2p,n) = 1\} = \frac{p-1}{2p} Y + O(1)$). A similar analysis holds for $S_{2;p}(X)$, as well as the even fundamental discriminants d with $d/4 \equiv 2$ or 3 modulo 4).

We need to trivially modify the above arguments if $p \equiv 3 \bmod 4$. If for instance we require $d \equiv 1 \bmod 4$ then instead of replacing $\mu(d)^2$ with $\mu(d)^2(1+\chi_4(d))/2$ we replace it with $\mu(pd)^2(1-\chi_4(d))/2$, and the rest of the proof proceeds similarly.

It is a completely different story if $p = 2$. Note if $d \equiv 1 \bmod 4$ then 2 *never* divides d, while if $d/4 \equiv 2$ or 3 modulo 4 then 2 *always* divides d. There are $3X/\pi^2 + o(X^{1/2})$ even fundamental discriminants at most X, and $X/\pi^2 + O(x^{1/2})$ of these are divisible by 2. Thus, if our family is all even fundamental discriminants, we do get the factor of $1/(p+1)$ for $p = 2$, as one-third (which is $1/(2+1)$ of the fundamental discriminants in this family are divisible by 2. ■

In our analysis of the terms from the L-functions Ratios Conjecture, we shall need a partial summation consequence of Lemma B.1.

LEMMA B.2. *Let d denote an even fundamental discriminant at most X and $X^* = \sum_{d \le X} 1$ and let $z = \tau - iw\frac{\log X}{2\pi}$ with $w \ge 1/2$. Then*

$$\sum_{d \le X} e^{-2\pi i z \frac{\log(d/\pi)}{\log X}} = X^* e^{-2\pi i \left(1 - \frac{\log \pi}{\log X}\right) z} \left(1 - \frac{2\pi i z}{\log X}\right)^{-1} + O(\log X). \tag{B.10}$$

□

PROOF. By Lemma B.1 we have

$$\sum_{d\le u} 1 = \frac{3u}{\pi^2} + O(u^{1/2}). \tag{B.11}$$

Therefore by partial summation we have

$$\begin{aligned}
&\sum_{d\le X} e^{-2\pi i z \frac{\log(d/\pi)}{\log X}} \\
&= e^{2\pi i \frac{\log \pi}{\log X}} \sum_{d\le X} d^{-2\pi i z/\log X} \\
&= e^{2\pi i z \frac{\log \pi}{\log X}} \left[\frac{3X + O(X^{1/2})}{\pi^2} X^{-\frac{2\pi i z}{\log X}} - \int^{X} \left(\frac{3u}{\pi^2} + O(u^{1/2}) \right) \cdot u^{-\frac{2\pi i z}{\log X}} \frac{-2\pi i z}{\log X} \frac{du}{u} \right].
\end{aligned} \tag{B.12}$$

As we are assuming $w \ge 1/2$, the first error term is of size $O(X^{1/2}X^{-w}) = O(1)$. The second error term (from the integral) is $O(\log X)$ for such w. This is because the integration begins at 1 and the integrand is bounded by $u^{-\frac{1}{2}-w}$. Thus

$$\begin{aligned}
&\sum_{d\le X} e^{-2\pi i z \frac{\log(d/\pi)}{\log X}} \\
&= e^{2\pi i z \frac{\log \pi}{\log X}} \left[\frac{3X}{\pi^2} e^{-2\pi i z} + \frac{\frac{3}{\pi^2} \cdot 2\pi i z}{\log X} \int^{X} u^{-2\pi i z/\log X} du \right] + O(\log X) \\
&= e^{2\pi i z \frac{\log \pi}{\log X}} \left[\frac{3X}{\pi^2} e^{-2\pi i z} + \frac{\frac{3}{\pi^2} \cdot 2\pi i z}{\log X} \frac{X^{1-2\pi i z/\log X}}{1 - 2\pi i z/\log X} \right] + O(\log X) \\
&= X^* e^{2\pi i z \frac{\log \pi}{\log X}} e^{-2\pi i z} \left[1 + \frac{2\pi i z}{\log X} \sum_{\nu=0}^{\infty} \left(\frac{2\pi i z}{\log X} \right)^{\nu} \right] + O(\log X) \\
&= X^* e^{-2\pi i \left(1 - \frac{\log \pi}{\log X}\right) z} \left(1 - \frac{2\pi i z}{\log X} \right)^{-1} + O(\log X).
\end{aligned} \tag{B.13}$$

■

C Improved Bound for Nonsquare m Terms in $S_M(X, Y, \widehat{g}, \Phi)$

Gao [17] proves that the nonsquare m-terms contribute $\ll (U^2 Z \sqrt{Y} \log^7 X)/X$ to $S_M(X, Y, \widehat{g}, \Phi)$. As this bound is just a little too large for our applications, we perform a more careful analysis below. Denoting the sum of interest by R,

$$R = \sum_{\substack{\alpha\le Z \\ (\alpha,2)=1}} \sum_{\substack{p\le Y \\ (2\alpha,p)=1}} \frac{\log p}{p} \widehat{g}\left(\frac{\log p}{\log X} \right) \sum_{m\ne 0,\square} (-1)^m \widetilde{\Phi}\left(\frac{mX}{2\alpha^2 p} \right) \left(\frac{m}{p} \right), \tag{C.1}$$

Gao shows that

$$R \ll \sum_{\alpha\le Z} \frac{\log^3 X}{\alpha^2}(R_1+R_2+R_3), \tag{C.2}$$

with

$$\begin{aligned} R_1, R_2 &\ll \frac{U\alpha^2\sqrt{Y}\log^4 X}{X} \\ R_3 &\ll \frac{U^2\alpha^2\sqrt{Y}\log^7 X}{X}. \end{aligned} \tag{C.3}$$

The bounds for R_1 and R_2 suffice for our purpose, leading to contributions bounded by $(UZ\sqrt{Y}\log^4 X)/X$; however, the R_3 bound gives too crude a bound—we need to save a power of U.

We have (see page 36 of [17], with $k=1$, $k_2=0$, $k_1=0$, $\alpha_1=1$ and $\alpha_0=0$) that

$$R_3 \ll \int_1^Y \frac{X}{\alpha^2 V^{5/2}} \sum_{p<Y} \frac{\log p}{p^2} \sum_{m=1}^{\infty} (\log^3 m) m \widetilde{\Phi}'\left(\frac{mX}{2\alpha^2 pV}\right) dV. \tag{C.4}$$

We have (see (3.10) of [17]) that

$$\widetilde{\Phi}'(\xi) \ll U^{j-1}|\xi|^{-j} \text{ for any integer j} \ge 1. \tag{C.5}$$

Letting $M = X^{2008}$, we break the m-sum in R_3 into $m\le M$ and $m>M$. For $m\le M$ we use (C.5) with $j=2$ while for $m>M$ we use (C.5) with $j=3$. (Gao uses $j=3$ for all m. While we save a bit for small m by using $j=2$, we cannot use this for all m as the resulting m sum does not converge.)

Thus the small m contribute

$$\begin{aligned} &\ll \int_1^Y \frac{X}{\alpha^2 V^{5/2}} \sum_{p<Y} \frac{\log p}{p^2} \sum_{m\le M} (\log^3 m) m \frac{U 2^2\alpha^4 p^2 V^2}{m^2 X^2} dV \\ &\ll \frac{U\alpha^2}{X} \sum_{p<Y} \log p \sum_{m\le M} \frac{\log^3 m}{m} \int_1^Y \frac{dV}{\sqrt{V}} \\ &\ll \frac{UY^{3/2}\alpha^2\log^4 X}{X} \end{aligned} \tag{C.6}$$

(since $M = X^{2008}$ the m-sum is $O(\log^4 X)$). The large m contribute

$$\begin{aligned}
&\ll \int_1^Y \frac{X}{\alpha^2 V^{5/2}} \sum_p \frac{\log p}{p^2} \sum_{m>M} (\log^3 m) m \frac{U^2 2^3 \alpha^6 p^3 V^3}{m^3 X^3} dV \\
&\ll \frac{U^2\alpha^4}{X^2} \sum_{p<Y} p \log p \sum_{m>M} \frac{\log^3 m}{m^3} \int_1^Y V^{1/2} dV \\
&\ll \frac{\alpha^4 U^2 Y^{3/2} Y^2 \log X}{X^2 M^{2-\epsilon}}.
\end{aligned} \tag{C.7}$$

For our choices of U, Y and Z, the contribution from the large m will be negligible (due to the $M^{2-\epsilon} = X^{4016-2\epsilon}$ in the denominator). Thus for these choices

$$\begin{aligned}
R &\ll \sum_{\alpha \le Z} \frac{\log^3 X}{\alpha^2} (R_1 + R_2 + R_3) \\
&\ll \frac{UZ\sqrt{Y} \log^7 X}{X} + \frac{UZY^{3/2} \log^4 X}{X} + \frac{Z^3 U^2 Y^{7/2} \log^4 X}{X^{4018-2\epsilon}}.
\end{aligned} \tag{C.8}$$

The last term is far smaller than the first two. In the first term we save a power of U from Gao's bound, and in the second we replace U with Y. As $Y = X^\sigma$, for σ sufficiently small there is a significant savings.

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