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LOW-LYING ZEROS OF L-FUNCTIONS AND RANDOM MATRIX THEORY

MICHAEL RUBINSTEIN

Abstract
By looking at the average behavior (n-level density) of the low-lying zeros of certain families of L-functions, we find evidence, as predicted by function field analogs, in favor of a spectral interpretation of the nontrivial zeros in terms of the classical compact groups.

1. Introduction

In this paper, a connection is made between the low-lying zeros of L-functions and the eigenvalues of large matrices from the classical compact groups. The Langlands program (see [2], [10], [7]) predicts that all L-functions can be written as products of $\zeta(s)$ and L-functions attached to automorphic cuspidal representations of GL_M over $\mathbb{Q}$. Such an L-function is given intially (for $\Re s$ sufficiently large) as an Euler product of the form

$$L(s,\pi) = \prod_p L(s,\pi_p) = \prod_p \prod_{j=1}^{M} (1-\alpha_\pi(p,j)p^{-s})^{-1}. \tag{1.1}$$

Basic properties of such L-functions are described in [15]. The L-functions that arise in the $m = 1$ case are the Riemann zeta-function $\zeta(s)$ and Dirichlet L-functions $L(s,\chi)$, χ a primitive character. For $m = 2$, the L-functions in question are associated to cusp forms or Maass forms of congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$.

The Riemann hypothesis (RH) for $L(s,\pi)$ asserts that the nontrivial zeros of $L(s,\pi)$, $\{1/2 + i\gamma_\pi\}$ all have $\gamma_\pi \in \mathbb{R}$. *(Our L-functions are always normalized so that the critical line is through $\Re s = 1/2$.)*

A vague suggestion of G. Pólya and D. Hilbert suggests an approach that one might take in establishing RH. They hypothesized (for $\zeta(s)$) that one might be able to associate the nontrivial zeros of ζ to the eigenvalues of some operator acting on some Hilbert space, thus (depending on the properties of the operator) forcing the zeros to lie on a line.

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The first evidence in favor of this approach was obtained by H. Montgomery [9], who derived (under certain restrictions) the pair correlation of the zeros of $\zeta(s)$. Together with an observation of Freeman Dyson, who pointed out that the Gaussian Unitary Ensemble (GUE), consisting of $N \times N$ random Hermitian matrices (see [8] for a more precise definition), has the same pair correlation (as $N \to \infty$), it seems to suggest that the relevant operator, at least for $\zeta(s)$, might be Hermitian. Extensive computations of A. Odlyzko [11], [12] further seem to bolster the Hermitian nature of the zeros of $\zeta(s)$, as might the work of Z. Rudnick and P. Sarnak [15], where, under certain restrictions, the n-level correlations of $\zeta(s)$ and $L(s,\pi)$ are found to be the same as those of the GUE.

However, recent developments suggest that, rather than being Hermitian, the relevant operators for L-functions belong to the classical compact groups. (This is consistent with the above work of Montgomery, Odlyzko, and Rudnick and Sarnak since all the classical compact groups have the same n-level correlations as the GUE (as $N \to \infty$).) First, analogs with function field zeta-functions, where there is a spectral interpretation of the zeros in terms of Frobenius on cohomology, point towards the classical compact groups (see [6]). Second, even though all the mentioned families of matrices have the same n-level correlations, there is another statistic, called n-level density, which is sensitive to the particular family. By looking at this statistic for zeros of L-functions, one finds the fingerprints of the classical compact groups. For $n = 1$ this was done, for quadratic twists of $\zeta(s)$, and with certain restrictions, by A. Özlük and C. Snyder [13]. A stronger result (which takes into account certain nondiagonal contributions and allows one to choose test functions whose Fourier transform is supported in $(-2/M, 2/M)$) which applies for $\zeta(s)$ as well as all $L(s,\pi)$ was obtained by N. Katz and Sarnak [6]. The general case, $n \geq 1$, is worked out (again, with some restrictions) in this paper.

2. n-level density

For an $(N \times N)$-matrix A in one of the classical compact groups, write its eigenvalues as $\lambda_j = e^{i\theta_j}$, with

$$0 \leq \theta_1 \leq \cdots \leq \theta_N < 2\pi. \tag{2.1}$$

Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is bounded, Borel measurable, and compactly supported. Then, letting

$$H^{(n)}(A, f) = \sum_{\substack{1 \leq j_1, \ldots, j_n \leq N \\ \text{distinct}}} f\left(\theta_{j_1} N/(2\pi), \ldots, \theta_{j_n} N/(2\pi)\right),$$

Katz and Sarnak [5, Appendix] obtain the following family dependent result:

$$\lim_{N\to\infty} \int_{G(N)} H^{(n)}(A, f)\, dA = \int_{\mathbb{R}^n_{\geq 0}} W_G^{(n)}(x) f(x)\, dx \tag{2.2}$$

for the following families:

G	$W_G^{(n)}$
$\mathrm{U}(N), \mathrm{U}_\kappa(N)$	$\det\left(K_0(x_j, x_k)\right)_{\substack{1\le j\le n\\ 1\le k\le n}}$
$\mathrm{USp}(N)$	$\det\left(K_{-1}(x_j, x_k)\right)_{\substack{1\le j\le n\\ 1\le k\le n}}$
$\mathrm{SO}(2N)$, $\mathrm{O}^-(2N+1)$	$\det\left(K_1(x_j, x_k)\right)_{\substack{1\le j\le n\\ 1\le k\le n}}$
$\mathrm{SO}(2N+1)$, $\mathrm{O}^-(2N)$	$\det\left(K_{-1}(x_j, x_k)\right)_{\substack{1\le j\le n\\ 1\le k\le n}} + \sum_{\nu=1}^n \delta(x_\nu) \det\left(K_{-1}(x_j, x_k)\right)_{\substack{1\le j\ne\nu\le n\\ 1\le k\ne\nu\le n}}$

with

$$K_\varepsilon(x, y) = \frac{\sin(\pi(x-y))}{\pi(x-y)} + \varepsilon \frac{\sin(\pi(x+y))}{\pi(x+y)}.$$

In the above, dA is the Haar measure on $G(N)$ (normalized so that $\int_{G(N)} dA = 1$), and

$$\begin{aligned}
\mathrm{U}_\kappa(N) &= \left\{A \in \mathrm{U}(N) : \det^\kappa(A) = 1\right\}, \\
\mathrm{SO}(N) &= \{A \in \mathrm{O}(N) : \det A = 1\}, \\
\mathrm{O}^-(N) &= \{A \in \mathrm{O}(N) : \det A = -1\}.
\end{aligned}$$

The delta functions in the $\mathrm{SO}(2N+1)$, $\mathrm{O}^-(2N)$ case are accounted for by the eigenvalue $\lambda_1 = 1$. (Notice, for $\mathrm{O}(N)$, that $\lambda = 1$ is an eigenvalue if N is even and $\det A = -1$ (i.e., $A \in \mathrm{O}^-(2N)$) or if N is odd and $\det A = 1$ (i.e., if $A \in \mathrm{SO}(2N+1)$.) Removing this zero from (2.2) would yield the same $W_G^{(n)}$ as for USp. For ease of notation, we refer to the third $W_G^{(n)}$ above (i.e., $\det\left(K_1(x_j, x_k)\right)$) as the scaling density of O^+ and the fourth $W_G^{(n)}$ as the scaling density of O^-. (We use this notation because the former comes from orthogonal matrices with even functional equations $p(z) = z^N p(1/z)$, while the latter comes from orthogonal matrices with odd functional equations $p(z) = -z^N p(1/z)$.)

One could also form a similar statistic for the eigenvalues of the GUE (where we would normalize the eigenvalues according to the Wigner semicircle law), and one could obtain the same answer (as $N \to \infty$) as for $\mathrm{U}(N)$.

The function $W_G^{(n)}(x)$ is called the n-level scaling density of the group $G(N)$, and its nonuniversality can be used to detect which group lies behind which family of L-functions.

Notice that the normalization by $N/(2\pi)$ is such that the mean spacing is 1 and that only the low-lying eigenvalues (those with $\theta \le c/N$ for some constant c) contribute to $H^{(n)}(A, f)$. So, (2.2) measures how the low-lying eigenvalues of matrices in $G(N)$ fall near the point 1 (as $N \to \infty$).

3. Results

In this section, we consider the analog of (2.2) for the zeros of families of L-functions. One looks at the average behavior of the low-lying nontrivial zeros (i.e., those close to the real axis) of a family of L-functions hoping to find evidence (as predicted by functional field analogs (see [6])) in favor of a spectral interpretation in terms of the classical compact groups.

Indeed, if we take quadratic twists of $\zeta(s)$, $\{L(s,\chi_d)\}$, as our family of L-functions, where $\chi_d(n) = \left(\frac{d}{n}\right)$ is Kronecker's symbol and we restrict ourselves to primitive χ_d, we find evidence of a $\mathrm{USp}(\infty)$ symmetry. This is Theorem 3.1.

More generally, we take a self-contragredient automorphic cuspidal representation of GL_M over $\mathbb{Q}$, $\pi = \tilde{\pi}$, that is, one whose L-function has real coefficients, $\alpha_\pi(p,j) \in \mathbb{R}$, and we look at the family of quadratic twists, $\{L(s,\pi\otimes\chi_d)\}$. The low-lying zeros of this family behave as if they are coming either from $\mathrm{USp}(\infty)$ or from $\mathrm{O}_\pm(\infty)$. (Here the $\pm$ is to indicate that we need to consider separately the $L(s,\pi\otimes\chi_d)$'s with even (resp., odd) functional equations.) We describe this result in Theorem 3.2. It confirms the connection to the classical compact groups, and it gives an answer that cannot be confused with the corresponding statistic for the GUE.

Numerical experiments that further support the connection to classical compact groups are described in the author's thesis [14] and in Katz and Sarnak [6].

3.1. Main theorem

Write the nontrivial zeros of $L(s,\chi_d)$ as

$$1/2 + i\gamma_d^{(j)}, \quad j = \pm 1, \pm 2, \ldots,$$

where

$$0 \le \Re\gamma_d^{(1)} \le \Re\gamma_d^{(2)} \le \Re\gamma_d^{(3)} \ldots$$

and

$$\gamma_d^{(-j)} = -\gamma_d^{(j)}. \tag{3.4}$$

Here $\chi_d(n) = \left(\frac{d}{n}\right)$ is Kronecker's symbol, and we restrict ourselves to primitive χ_d. Let D denote the set of such d's, and let $D(X) = \{d \in D : X/2 \le |d| < X\}$.

Notice that we are *not* assuming the Riemann hypothesis for $L(s,\chi_d)$ since we allow that the $\gamma_d^{(j)}$'s be complex.

THEOREM 3.1

Let

$$f(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} f_i(x_i), \tag{3.5}$$

where each f_i *is even and in* $S(\mathbb{R})$ *(i.e., smooth and rapidly decreasing). Assume further that* $\hat{f}(u_1,\dots,u_n)=\prod_{i=1}^n \hat{f}_i(u_i)$ *is supported in* $\sum_{i=1}^n |u_i|<1$*, where*

$$\hat{f}(u) \stackrel{\text{def}}{=} \int_{\mathbb{R}^n} f(x)e^{2\pi i x\cdot u}\,dx. \tag{3.6}$$

Then

$$\begin{aligned}\lim_{X\to\infty}\frac{1}{|D(X)|}\sum_{d\in D(X)}\sideset{}{^*}\sum_{j_1,\dots,j_n} f\left(L\gamma_d^{(j_1)}, L\gamma_d^{(j_2)},\dots,L\gamma_d^{(j_n)}\right)\\ = \int_{\mathbb{R}^n} f(x)W_{\mathrm{USp}}^{(n)}(x)\,dx,\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}L &= \frac{\log X}{2\pi},\\ W_{\mathrm{USp}}^{(n)}(x_1,\dots,x_n) &= \det\left(K_{-1}(x_j,x_k)\right)_{\substack{1\le j\le n\\ 1\le k\le n}},\\ K_{-1}(x,y) &= \frac{\sin(\pi(x-y))}{\pi(x-y)}-\frac{\sin(\pi(x+y))}{\pi(x+y)},\end{aligned}$$

and where $\sum^*_{j_1,\dots,j_n}$ *is over* $j_k=\pm1,\pm2,\dots$*, with* $j_{k_1}\neq \pm j_{k_2}$ *if* $k_1\neq k_2$.

Plan. We first use the explicit formula to study the l.h.s. (left-hand side) of (3.7), and we end up expressing it in terms of the $\hat{f}_i$'s. Parseval's formula is then applied to the r.h.s. (right-hand side) of (3.7), and terms are matched with the l.h.s.

Remark. The condition f_i even is not essential to the proof, nor is the assumption that f be of the form $\prod f_i$. At the expense of more cumbersome writing, these can be removed.

3.2. l.h.s.

By (3.4), (3.5), and since $f_i(-x)=f_i(x)$,

$$\begin{aligned}\frac{1}{|D(X)|}\sum_{d\in D(X)}\sideset{}{^*}\sum_{j_1,\dots,j_n} f\left(L\gamma_d^{(j_1)}, L\gamma_d^{(j_2)},\dots,L\gamma_d^{(j_n)}\right)\\ = \frac{2^n}{|D(X)|}\sum_{d\in D(X)}\sum_{\substack{j_1,\dots,j_n\\ \text{positive}\\ \text{and}\\ \text{distinct}}} \tilde{f}_d(j_1,\dots,j_n),\end{aligned} \tag{3.8}$$

where

$$\tilde{f}_d(j_1,\dots,j_n)=\prod_{i=1}^n f_i\left(L\gamma_d^{(j_i)}\right). \tag{3.9}$$

In order to apply the explicit formula to (3.8), we need to circumvent the fact that the j_i's are distinct. By combinatorial sieving, as in [15, p. 305], the r.h.s. of (3.8) is

$$\frac{2^n}{|D(X)|} \sum_{d \in D(X)} \sum_{\underline{F}} (-1)^{n-\nu(\underline{F})} \left(\prod_{\ell=1}^{\nu(\underline{F})} (|F_\ell| - 1)! \right) w_{\underline{F}},$$

where $\underline{F}$ ranges over all ways of decomposing $\{1, 2, \ldots, n\}$ into disjoint subsets $[F_1, \ldots, F_\nu]$, and where

$$w_{\underline{F}} = \sum_{\substack{j_1, \ldots, j_\nu \\ \text{positive}}} \tilde{f}_d(\ell_{\underline{F}}(j_1, \ldots, j_\nu)).$$

Here $\ell_{\underline{F}} : \mathbb{R}^\nu \to \mathbb{R}^n$, $\ell_{\underline{F}}(x_1, \ldots, x_\nu) = (y_1, \ldots, y_n)$ with $y_i = x_j$ if $i \in \underline{F}_\ell$.

For example, for $n = 3$, the possible $\underline{F}$'s are $[\{1, 2, 3\}]$, $[\{1, 2\}, \{3\}]$, $[\{1, 3\}, \{2\}]$, $[\{2, 3\}, \{1\}]$, $[\{1\}, \{2\}, \{3\}]$, and $\ell_{[\{1,3\},\{2\}]}(x_1, x_2) = (x_1, x_2, x_1)$.

Thus, (3.8) is

$$\frac{2^n}{|D(X)|} \sum_{d \in D(X)} \sum_{\underline{F}} (-1)^{n-\nu(\underline{F})} \left(\prod_{\ell=1}^{\nu(\underline{F})} (|F_\ell| - 1)! \right) \sum_{\substack{j_1, \ldots, j_{\nu(\underline{F})} \\ \text{positive}}} \tilde{f}_d(\ell_{\underline{F}}(j_1, \ldots, j_{\nu(\underline{F})})),$$

which, by (3.9), equals

$$\frac{2^n}{|D(X)|} \sum_{d \in D(X)} \sum_{\underline{F}} \frac{(-1)^{n-\nu(\underline{F})}}{2^{\nu(\underline{F})}} \prod_{\ell=1}^{\nu(\underline{F})} \left((|F_\ell| - 1)! \sum_{\gamma_d} \prod_{i \in F_\ell} f_i\,(L\gamma_d) \right). \tag{3.10}$$

In the innermost sum, we are going over all $\gamma_d^{(j)}$ (instead of $j > 0$) and hence the presence of the $1/2^{\nu(\underline{F})}$. This is justified by (3.4) and because we are assuming that the f_i's are even.

Let

$$F_\ell(x) = \prod_{i \in F_\ell} f_i(x). \tag{3.11}$$

By the explicit formula (see [15, (2.16)], with, in the notation of that paper, $h(r) = F_\ell(Lr)$, $g(y) = (1/\log X)\hat{F}_\ell(-y/\log X)$),

$$\sum_{\gamma_d} F_\ell\left(L\gamma_d^{(j)}\right) = \int_{\mathbb{R}} F_\ell(x)\,dx + O(1/\log X)$$
$$- \frac{2}{\log X} \sum_{m=1}^{\infty} \frac{\Lambda(m)}{m^{1/2}} \chi_d(m) \hat{F}_\ell\left(\frac{\log m}{\log X}\right). \tag{3.12}$$

(Note that $\hat{F}_\ell(x)$ is even since each f_i is even. We have also used the facts that $F_\ell(x)$ is rapidly decreasing and that $\Gamma'(s)/\Gamma(s) = O(\log|s|)$ to replace the Γ'/Γ-terms in [15, (2.16)] by $O(1/\log X)$. Note further that $\hat{F}_\ell$ is compactly supported (see Claim 1).)

Plugging (3.12) into (3.10) (without the $O(1/\log X)$-term, a step that is justified in Lemma 2), we see, on multiplying out the product over ℓ in (3.10), that (3.10) is

$$\frac{1}{|D(X)|}\sum_{d\in D(X)}\sum_{\underline{F}}(-2)^{n-\nu(\underline{F})}\prod_{\ell=1}^{\nu(\underline{F})}(|F_\ell|-1)!(C_\ell+D_\ell),$$

where

$$C_\ell = \int_{\mathbb{R}} F_\ell(x)\,dx,$$

$$D_\ell = -\frac{2}{\log X}\sum_{m=1}^{\infty}\frac{\Lambda(m)}{m^{1/2}}\chi_d(m)\hat{F}_\ell\left(\frac{\log m}{\log X}\right).$$

When we expand the product over ℓ, we obtain $2^{\nu(\underline{F})}$ terms, each a product of C_ℓ's and D_ℓ's. A typical term can be written as

$$\prod_{\ell\in S^c} C_\ell \prod_{\ell\in S} D_\ell$$

for some subset S of $\{1, 2, \ldots, \nu(\underline{F})\}$. (Empty products are taken to be 1.) The product of the C_ℓ's contributes to (3.10) a factor of

$$\prod_{\ell\in S^c}\int_{\mathbb{R}} F_\ell(x)\,dx.$$

The product of the D_ℓ's equals

$$\left(\frac{-2}{\log X}\right)^{|S|}\prod_{\ell\in S}\sum_{m=1}^{\infty}\frac{\Lambda(m)}{m^{1/2}}\chi_d(m)\hat{F}_\ell\left(\frac{\log m}{\log X}\right),$$

which, by Lemma 1, contributes a factor of

$$\sum_{\substack{S_2\subseteq S\\ |S_2|\text{ even}}}\left(\left(\frac{-1}{2}\right)^{|S_2^c|}\prod_{\ell\in S_2^c}\int_{\mathbb{R}}\hat{F}_\ell(u)\,du\right)\left(\sum_{(A;B)}2^{|S_2|/2}\prod_{j=1}^{|S_2|/2}\int_{\mathbb{R}}|u|\,\hat{F}_{a_j}(u)\hat{F}_{b_j}(u)\,du\right),$$

from which we find that (3.10) (and hence (3.8)) tends, as $X\to\infty$, to

$$\boxed{\begin{aligned}\sum_{\underline{F}}(-2)^{n-\nu(\underline{F})}\left(\prod_{\ell=1}^{\nu(\underline{F})}(|F_\ell|-1)!\right)\sum_{S}\left(\prod_{\ell\in S^c}\int_{\mathbb{R}}F_\ell(x)\,dx\right)\\ \cdot\sum_{\substack{S_2\subseteq S\\ |S_2|\text{ even}}}\left(\left(\frac{-1}{2}\right)^{|S_2^c|}\prod_{\ell\in S_2^c}\int_{\mathbb{R}}\hat{F}_\ell(u)\,du\right)\\ \cdot\left(\sum_{(A;B)}2^{|S_2|/2}\prod_{j=1}^{|S_2|/2}\int_{\mathbb{R}}|u|\,\hat{F}_{a_j}(u)\hat{F}_{b_j}(u)\,du\right).\end{aligned}} \tag{3.13}$$

Here S ranges over all $2^{\nu(\underline{F})}$ subsets of $\{1, 2, \ldots, \nu(\underline{F})\}$, and S^c denotes the complement of S. The rest of the notation is as in Lemma 1.

LEMMA 1
We have

$$\begin{aligned}\lim_{X\to\infty}\frac{1}{|D(X)|}\sum_{d\in D(X)}\left(\frac{-2}{\log X}\right)^k\prod_{j=1}^{k}\quad\cdot\sum_{m=1}^{\infty}\frac{\Lambda(m)}{m^{1/2}}\chi_d(m)\hat{F}_{\ell_j}\left(\frac{\log m}{\log X}\right)\\ =\sum_{\substack{S_2\subseteq S\\ |S_2|\text{ even}}}\left(\left(\frac{-1}{2}\right)^{|S_2^c|}\prod_{\ell\in S_2^c}\int_{\mathbb{R}}\hat{F}_\ell(u)\,du\right)\\ \cdot\left(\sum_{(A;B)}2^{|S_2|/2}\prod_{j=1}^{|S_2|/2}\int_{\mathbb{R}}|u|\,\hat{F}_{a_j}(u)\hat{F}_{b_j}(u)\,du\right),\end{aligned} \tag{3.14}$$

where $S = \{l_1, \ldots, l_k\}$. $\sum_{\substack{S_2\subseteq S\\ |S_2|\text{ even}}}$ is over all subsets S_2 of S whose size is even. $\sum_{(A;B)}$ is over all ways of pairing up the elements of S_2. $F_\ell(x)$ is defined in (3.11).

For example, if $S = \{1, 2, 5, 7\}$, the possible S_2's are $\emptyset$, $\{1, 2\}$, $\{1, 5\}$, $\{1, 7\}$, $\{2, 5\}$, $\{2, 7\}$, $\{5, 7\}$, $\{1, 2, 5, 7\}$.

And if $S_2 = \{1, 2, 5, 7\}$, then the possible $(A; B)$'s are $(1, 2; 5, 7)$, $(1, 2; 7, 5)$, $(1, 5; 2, 7)$. These correspond, respectively, to matching 1 with 5 and 2 with 7, 1 with 7 and 2 with 5, 1 with 2 and 5 with 7. Note that our notation is not unique. For example, $(1, 2; 5, 7) \equiv (7, 1; 2, 5)$.

Proof
Lemma 1 is obtained in a sequence of claims.

CLAIM 1
Suppose that $\prod_{i=1}^{n} \hat{f}_i(u_i)$ *is supported in* $\sum_{i=1}^{n} |u_i| \le \alpha$. *Then* $\prod_{j=1}^{k} \hat{F}_{\ell_j}(u_j)$ *is supported in* $\sum_{j=1}^{k} \left|u_j\right| \le \alpha$.

Proof
By (3.11),

$$\begin{aligned} \hat{F}_\ell(u) &= \int_{\mathbb{R}} \prod_{i \in F_\ell} f_i(x) e^{2\pi i u x}\, dx \\ &= \int_{\mathbb{R}^{|F_\ell|}} \left(\prod_{i \in F_\ell} dx_i\, f_i(x_i) \right) e^{2\pi i u \sum_{i \in F_\ell} x_i / |F_\ell|} \prod_{m=2}^{|F_\ell|} \delta(x_{i_m} - x_{i_1}) \\ &= \int_{\mathbb{R}^{|F_\ell|}} \left(\prod_{i \in F_\ell} dv_i\, \hat{f}_i(v_i) \right) \delta\left(u - \sum_{i \in F_\ell} v_i \right), \end{aligned} \tag{3.15}$$

the last step following from Parseval's formula. (Note: If $|F_\ell| = 1$, then the product over m is taken to be 1.) Hence,

$$\prod_{j=1}^{k} \hat{F}_{\ell_j}(u_j) = \int_{\mathbb{R}^{\sum_1^k \left|F_{\ell_j}\right|}} \left(\prod_{i \in \bigcup F_{\ell_j}} dv_i\, \hat{f}_i(v_i) \right) \prod_{j=1}^{k} \delta\left(u_j - \sum_{i \in F_{\ell_j}} v_i \right). \tag{3.16}$$

In the integrand, the δ's restrict us to

$$\sum_{j=1}^{k} \left|u_j\right| = \sum_{j=1}^{k} \left| \sum_{i \in F_{\ell_j}} v_i \right| \le \sum_{i \in \bigcup F_{\ell_j}} |v_i|.$$

So, if $\sum_{j=1}^{k} \left|u_j\right| > \alpha$, then $\sum_{i \in \cup F_{\ell_j}} |v_i| > \alpha$. But, by the support condition on $\prod_{i=1}^{n} \hat{f}_i(v_i)$, $\prod_{i \in \cup F_{\ell_j}} \hat{f}_i(v_i) = 0$ if $\sum_{i \in \cup F_{\ell_j}} |v_i| > \alpha$. Hence (3.16) is zero if $\sum_{j=1}^{k} \left|u_j\right| > \alpha$; thus the claim is proved. □

CLAIM 2

Suppose that $\prod_{i=1}^{n} \hat{f}_i(u_i)$ *is supported in* $\sum_{i=1}^{n} |u_i| \le \alpha < 1$. *Then*

$$\lim_{X\to\infty} \frac{1}{|D(X)|} \sum_{d\in D(X)} \left(\frac{-2}{\log X}\right)^k \sum_{\substack{m_i\ge 1\\ i=1,\dots,k\\ m_1\cdot\ldots\cdot m_k\ne\square}} \left(\prod_{j=1}^{k} \frac{\Lambda(m_j)}{m_j^{1/2}} \chi_d(m_j) \hat{F}_{\ell_j}\left(\frac{\log m_j}{\log X}\right)\right)$$

$$= 0. \tag{3.17}$$

Here we are summing over all k*-tuples* $(m_1, \dots, m_k)$ *of positive integers with* $\prod_1^k m_i \notin \{1, 4, 9, 16, \dots\}$, *and* $S = \{l_1, \dots, l_k\}$.

Remark. This claim tells us that the only contributions to (3.14) come from perfect squares. (This is dealt with in Claim 3.)

Proof

Changing the order of summation and applying Claim 1 and the Cauchy-Schwarz inequality, we find that the l.h.s. of (3.17) is

$$\ll \lim_{X\to\infty} \frac{1}{|D(X)|} \frac{1}{\log^k X} \left(\sum_{\substack{m_i\ge 1\\ \sum \log m_i \le \alpha \log X\\ m_1\cdot\ldots\cdot m_k\ne\square}} \frac{\Lambda^2(m_1)\cdot\ldots\cdot\Lambda^2(m_k)}{m_1\cdot\ldots\cdot m_k} \right)^{1/2}$$

$$\cdot \left(\sum_{\substack{m_i\ge 1\\ \sum \log m_i \le \alpha \log X\\ m_1\cdot\ldots\cdot m_k\ne\square}} \left| \sum_{d\in D(X)} \chi_d(m_1\cdot\ldots\cdot m_k) \right|^2 \right)^{1/2}. \tag{3.18}$$

The first bracketed term is less than

$$\left(\sum_{m\le X^\alpha} \frac{\Lambda^2(m)}{m} \right)^{k/2} \ll \log^k X. \tag{3.19}$$

Next, the number of times we may write $m = m_1\cdot\ldots\cdot m_k$, $m_i \ge 1$, is $O\big(\sigma_0^{k-1}(m)\big) = O_\varepsilon(m^\varepsilon)$ for any $\varepsilon > 0$ ($\sigma_0(m)$ being the number of divisors of m), so that the second bracketed term is

$$\ll_\varepsilon \left(X^\varepsilon \sum_{m\le X^\alpha} \left| \sum_{d\in D(X)} \chi_d(m) \right|^2 \right)^{1/2}. \tag{3.20}$$

Applying the methods of M. Jutila [4], we find that the above is

$$\ll_\varepsilon \left(X^{\varepsilon+1+\alpha}\log^A X\right)^{1/2} \quad \text{for some constant } A \ (A = 10 \text{ is admissable}),$$

which, combined with (3.19), shows that (3.18) is

$$\ll_\varepsilon \lim_{X\to\infty} \frac{1}{|D(X)|} X^{\varepsilon+(1+\alpha)/2},$$

But, for ε small enough, this limit equals zero (because $|D(X)| \sim cX$ for some constant c, and we are assuming $\alpha < 1$). □

CLAIM 3

We have

$$\lim_{X\to\infty}\frac{1}{|D(X)|}\sum_{d\in D(X)}\left(\frac{-2}{\log X}\right)^k \sum_{\substack{m_i\geq 1\\ m_1\cdot\ldots\cdot m_k=\square}}\left(\prod_{j=1}^{k}\frac{\Lambda(m_j)}{m_j^{1/2}}\chi_d(m_j)\hat{F}_{\ell_j}\left(\frac{\log m_j}{\log X}\right)\right)$$
$$= \sum_{\substack{S_2\subseteq S\\ |S_2| \text{ even}}}\left(\left(\frac{-1}{2}\right)^{|S_2^c|}\prod_{\ell\in S_2^c}\int_{\mathbb{R}}\hat{F}_\ell(u)\,du\right)$$
$$\cdot\left(\sum_{(A;B)}2^{|S_2|/2}\prod_{j=1}^{|S_2|/2}\int_{\mathbb{R}}|u|\,\hat{F}_{a_j}(u)\hat{F}_{b_j}(u)\,du\right). \tag{3.21}$$

Here we are summing over all k-tuples $(m_1,\ldots,m_k)$ of positive integers with $\prod_1^k m_i \in \{1, 4, 9, 16, \ldots\}$.

Proof

First, the $\Lambda(m_i)$'s restrict us to prime powers, $m_i = p_i^{e_i}$, so the only way that $\prod_1^k m_i$ can equal a perfect square is if some of the e_i's are even, and the rest of the $p_i^{e_i}$'s match up to produce squares.

We can focus our attention on $e_i = 1$ or 2 since the sum over $e_i \geq 3$ contributes zero as $X \to \infty$.

Also, note, in (3.21), that $\chi_d(\prod_1^k m_i) = 1$ since $\prod_1^k m_i$ is restricted to perfect squares. Hence the l.h.s. of (3.21) is

$$\lim_{X\to\infty} \sum_{\substack{S_2\subseteq S\\ |S_2| \text{ even}}} \sum_{\substack{p_\ell\\ \ell\in S_2\\ \prod_{\ell\in S_2} p_\ell=\square}} \left(\frac{-2}{\log X}\right)^{|S_2|} \prod_{i\in S_2} \frac{\log(p_i)}{p_i^{1/2}} \hat{F}_i\left(\frac{\log p_i}{\log X}\right)$$
$$\cdot \sum_{\substack{p_\ell\\ \ell\in S_2^c}} \left(\frac{-2}{\log X}\right)^{|S_2^c|} \prod_{i\in S_2^c} \frac{\log(p_i)}{p_i} \hat{F}_i\left(\frac{2\log p_i}{\log X}\right).$$

(We have dropped the $(1/|D(X)|)\sum_{d\in D(X)}$ since the terms in the sum do not depend on d.) The sum over $\ell\in S_2$ corresponds to the e_ℓ's that are equal to 1 (and that pair up to produce squares), while the sum over $\ell\in S_2^c$ corresponds to the e_ℓ's that are equal to 2. To complete the proof of this claim and hence of Lemma 1, we establish the two subclaims below.

SUBCLAIM 3.1
We have

$$\lim_{X\to\infty} \sum_{\substack{p_\ell\\ \ell\in S_2^c}} \left(\frac{-2}{\log X}\right)^{|S_2^c|} \prod_{i\in S_2^c} \frac{\log(p_i)}{p_i} \hat{F}_i\left(\frac{2\log p_i}{\log X}\right)$$
$$= \left(\frac{-1}{2}\right)^{|S_2^c|} \prod_{\ell\in S_2^c} \int_{\mathbb{R}} \hat{F}_\ell(u)\,du. \tag{3.22}$$

Proof
The l.h.s. of (3.22) factors

$$\prod_{\ell\in S_2^c} \left(\frac{-2}{\log X} \sum_p \frac{\log(p)}{p} \hat{F}_\ell\left(\frac{2\log p}{\log X}\right)\right),$$

which, summing by parts, equals

$$\prod_{\ell\in S_2^c} \frac{2}{\log X} \int_1^\infty \sum_{p\le t} \frac{\log(p)}{p} \left(\hat{F}_\ell\left(\frac{2\log t}{\log X}\right)\right)' dt.$$

The sum $\sum_{p\le t}\log(p)/p$ can be evaluated elementarily (see [3, p. 22]), and the above becomes

$$\prod_{\ell \in S_2^c} \frac{2}{\log X} \int_1^{\infty} (\log t + O(1)) \left(\hat{F}_\ell \left(\frac{2 \log t}{\log X} \right) \right)' dt$$
$$= \prod_{\ell \in S_2^c} \left(\frac{-2}{\log X} \int_1^{\infty} \hat{F}_\ell \left(\frac{2 \log t}{\log X} \right) \frac{dt}{t} + O\left(\frac{1}{\log X} \right) \right), \tag{3.23}$$

the last step from integration by parts, and using the fact that $\hat{F}_\ell^{(1)}(u)$ is supported in $|u| \leq \alpha$. Changing variables $u = 2 \log t / \log X$ and noting that all the $\hat{F}_\ell$'s are even (since all the f_i's are), we thus find that the limit in (3.22) is

$$\left(\frac{-1}{2} \right)^{|S_2^c|} \prod_{\ell \in S_2^c} \int_{\mathbb{R}} \hat{F}_\ell(u)\, du.$$

□

SUBCLAIM 3.2
We have

$$\lim_{X \to \infty} \sum_{\substack{p_\ell \\ \ell \in S_2 \\ \prod_{\ell \in S_2} p_\ell = \square}} \left(\frac{-2}{\log X} \right)^{|S_2|} \prod_{i \in S_2} \frac{\log(p_i)}{p_i^{1/2}} \hat{F}_i \left(\frac{\log p_i}{\log X} \right)$$
$$= \sum_{(A;B)} 2^{|S_2|/2} \prod_{j=1}^{|S_2|/2} \int_{\mathbb{R}} |u|\, \hat{F}_{a_j}(u) \hat{F}_{b_j}(u)\, du. \tag{3.24}$$

Proof
In (3.24), $\prod_{\ell \in S_2} p_\ell = \square$ implies that the p_ℓ's pair up to produce squares. So, the l.h.s. of (3.24) equals

$$\lim_{X \to \infty} \sum_{(A;B)} \sum_{\substack{p_i \\ i=1,\ldots,|S_2|/2}} \prod_{j=1}^{|S_2|/2} \frac{4}{\log^2(X)} \frac{\log^2(p_j)}{p_j} \hat{F}_{a_j} \left(\frac{\log p_j}{\log X} \right) \hat{F}_{b_j} \left(\frac{\log p_j}{\log X} \right). \tag{3.25}$$

The sum over $(A; B)$ accounts for all ways of pairing up primes in (3.24). Note that there is a bit of overlap produced in (3.25), but this overlap contributes zero as $X \to \infty$. For example, if $S_2 = \{1, 2, 5, 7\}$, then the three ways of pairing up p_1, p_2, p_5, p_7 are: $p_1 = p_5$ and $p_2 = p_7$, $p_1 = p_7$ and $p_2 = p_5$, $p_1 = p_2$ and $p_5 = p_7$. So the sum over $p_1 = p_2 = p_5 = p_7$ is counted three times in (3.25), whereas it is counted only once in the l.h.s. of (3.24). Such diagonal sums do not bother us since there are $O_k(1)$

such sums, and a typical $p_{j_1} = p_{j_2} = \cdots = p_{j_{2r}}$, $r \geq 2$, contributes to (3.25) a term with a factor that is

$$\ll \lim_{X\to\infty} \frac{1}{\log^{2r} X} \sum_p \frac{\log^{2r} p}{p^r} \ll \lim_{X\to\infty} \frac{1}{\log^{2r} X} = 0.$$

Now, (3.25) can be written as

$$\lim_{X\to\infty} \sum_{(A;B)} \prod_{j=1}^{|S_2|/2} \left(\frac{4}{\log^2(X)} \sum_p \frac{\log^2(p)}{p} \hat{F}_{a_j}\left(\frac{\log p}{\log X}\right) \hat{F}_{b_j}\left(\frac{\log p}{\log X}\right)\right).$$

Summing by parts, we find that the bracketed term is

$$4\int_0^\infty u \hat{F}_{a_j}(u) \hat{F}_{b_j}(u)\, du + O\left(1/\log X\right).$$

Recalling that the $\hat{F}$'s are even, we obtain the subclaim. □

We thus obtain Claim 3 and Lemma 1. □

LEMMA 2
Let

$$a_\ell(d) = \sum_{\gamma_d} F_\ell\left(L\gamma_d^{(j)}\right),$$

where $F_\ell(x) = \prod_{i\in F_\ell} f_i(x)$, *and* f_i *is as in Theorem 3.1. Then*

$$\lim_{X\to\infty} \frac{1}{|D(X)|} \sum_{d\in D(X)} \prod_{\ell=1}^{\nu(\underline{F})} a_\ell(d) = \lim_{X\to\infty} \frac{1}{|D(X)|} \sum_{d\in D(X)} \prod_{\ell=1}^{\nu(\underline{F})} \left(a_\ell(d) + O(1/\log X)\right).$$

Remark. This lemma justifies dropping the $O(1/\log X)$ when plugging (3.12) into (3.10).

Proof
The proof is by induction. We consider

$$\lim_{X\to\infty} \frac{1}{|D(X)|} \sum_{d\in D(X)} \prod_{\ell=1}^{k} \left(a_\ell(d) + O(1/\log X)\right) \tag{3.26}$$

for $k = 1, 2, \ldots, \nu(\underline{F})$. When $k = 1$, this clearly equals

$$\lim_{X\to\infty} \frac{1}{|D(X)|} \sum_{d\in D(X)} a_\ell(d).$$

Now, consider the general case. Multiplying out the product in (3.26), we get

$$\lim_{X\to\infty} \frac{1}{|D(X)|} \sum_{d\in D(X)} \prod_{\ell=1}^{k} a_\ell(d) + \text{remainder},$$

where the remainder consists of $2^k - 1$ terms, each of which is of the form

$$O\left(\frac{1}{\log^r(X)} \frac{1}{|D(X)|} \sum_{d\in D(X)} \prod_{j=1}^{k_2} \left|a_{\ell_j}(d)\right|\right) \tag{3.27}$$

with $r \geq 1$, $k_2 < k$. Now, if $F_\ell(x) \geq 0$ for all x, then $\left|a_{\ell_j}(d)\right| = a_{\ell_j}(d)$, and, by our inductive hypothesis combined with Lemma 1, the O-term above tends to zero as $X \to \infty$.

If $F_\ell(x)$ is not greater than or equal to zero for all x, we can show that the O-term in (3.27) tends to zero as $X \to \infty$ by replacing each $f_i(x)$ $(i = 1, \ldots, n)$ with a function $g_i(x)$, which is positive and bigger in absolute value than $f_i(x)$, and which satisfies the conditions of Theorem 3.1; that is, we require that

- $g_i(x) \geq |f_i(x)|$,
- $g_i(x)$ be even and in $S(\mathbb{R})$,
- $\prod_{i=1}^n \hat{g}_i(u_i)$ be supported in $\sum_{i=1}^n |u_i| < 1$.

That there exist g_i's satisfying the required conditions can be seen as follows. Let

$$h(t) = \begin{cases} K\exp(-1/(1-t^2)), & |t| < 1, \\ 0, & |t| \geq 1, \end{cases}$$

where K is chosen so that

$$\int_{-1}^{1} h(t)\,dt = 1,$$

let

$$\theta_\beta(t) = \frac{1}{\beta} h(t/\beta) \tag{3.28}$$

(so that θ_β approximates the δ-function when β is small), and consider

$$\Psi_\beta(x) = (\theta_\beta * \theta_\beta)\hat{}\,(x) = (\hat{\theta}_\beta(x))^2. \tag{3.29}$$

Now

$$\hat{\theta}_\beta(x) = \frac{1}{\beta}\int_{-\beta}^{\beta} h(t/\beta)\cos(2\pi x t)\,dt$$
$$= \int_{-1}^{1} h(u)\cos(2\pi\beta u x)\,du. \tag{3.30}$$

But when $|x| \le 1/(8\beta)$, we have

$$\hat{\theta}_\beta(x) > \frac{\sqrt{2}}{2}\int_{-1}^{1} h(u)\,du = \frac{\sqrt{2}}{2}$$

(since, when $|x| \le 1/(8\beta)$, $|u| \le 1$, we get, $|2\pi\beta u x| \le \pi/4$). Hence

$$\Psi_\beta(x) > 1/2 \quad \text{when } |x| \le 1/(8\beta)$$

(so Ψ_β is bounded away from zero for long stretches when β is small), and, from (3.29),

$$\Psi_\beta(x) \ge 0 \quad \text{for all } x.$$

Also, note that Ψ_β is even and in $S(\mathbb{R})$ (since $h(t)$ enjoys these properties), and note that $\hat{\Psi}_\beta(t) = (\theta_\beta * \theta_\beta)(t)$ is supported in $[-2\beta, 2\beta]$. We use $\Psi_\beta(x)$'s to construct a $g_i(x)$ satisfying the three required properties.

Let

$$M_f(c,d) = \max_{c\le|x|\le d} |f(x)|,$$

and let

$$\beta_j^{-1} = \begin{cases} 2n+j, & j \ge 1, \\ 0, & j = 0. \end{cases}$$

(The $j=0$ case is only for notational convenience.) Then

$$g_i(x) = 2\sum_{j=0}^{\infty} M_{f_i}\left((8\beta_j)^{-1}, (8\beta_{j+1})^{-1}\right)\Psi_{\beta_{j+1}}(x)$$

has the required properties. □

3.3. r.h.s.

Our goal is to express

$$\int_{\mathbb{R}^n} f(x) W_{\text{USp}}^{(n)}(x)\,dx$$

in a manner that allows us to easily see how to match terms with (3.13).

We consider the more general

$$\int_{\mathbb{R}^n} f(x) W_\varepsilon(x)\,dx, \tag{3.31}$$

where $\varepsilon \in \{-1, 1\}$ and

$$W_\varepsilon(x_1, \ldots, x_n) = \det\big(K_\varepsilon(x_j, x_k)\big)_{\substack{1\le j\le n\\1\le k\le n}},$$

$$K_\varepsilon(x, y) = \frac{\sin(\pi(x-y))}{\pi(x-y)} + \varepsilon\frac{\sin(\pi(x+y))}{\pi(x+y)}$$

because it is needed when we study analogous questions for $\mathrm{GL}_M/\mathbb{Q}$.

Write

$$W_\varepsilon(x_1, \ldots, x_n) = \sum_\sigma \mathrm{sgn}(\sigma) \prod_{j=1}^n K_\varepsilon(x_j, x_{\sigma(j)}).$$

Here, σ is over all permutations of n elements. Express σ as a product of disjoint cycles

$$\sigma \in \bigsqcup_{\underline{F}} S^*(F_1) \times \cdots \times S^*(F_{\nu(\underline{F})}), \tag{3.32}$$

where $\underline{F}$ is over set partitions of $\{1, \ldots, n\}$ (as in Section 3.2) and $S^*(F_\ell)$ denotes the set of all $(|F_\ell| - 1)!$ cyclic permutations of the elements of F_ℓ. Notice that $\mathrm{sgn}(\sigma) = \prod_{\ell=1}^{\nu(\underline{F})}(-1)^{|F_\ell|-1}$.

For example, if $n = 7$ and $\underline{F} = [\{1, 3, 4, 6\}, \{2, 5, 7\}]$, then $S^*(\{1, 3, 4, 6\}) \times S^*(\{2, 5, 7\})$ is the set of 12 permutations:

$$\begin{aligned}\{&(1\,3\,4\,6)(2\,5\,7), (1\,3\,6\,4)(2\,5\,7), (1\,4\,3\,6)(2\,5\,7), (1\,4\,6\,3)(2\,5\,7),\\ &(1\,6\,3\,4)(2\,5\,7), (1\,6\,4\,3)(2\,5\,7), (1\,3\,4\,6)(2\,7\,5), (1\,3\,6\,4)(2\,7\,5),\\ &(1\,4\,3\,6)(2\,7\,5), (1\,4\,6\,3)(2\,7\,5), (1\,6\,3\,4)(2\,7\,5), (1\,6\,4\,3)(2\,7\,5)\}.\end{aligned}$$

We are applying Parseval's formula to (3.31), and thus we need to determine $\hat{W}_\varepsilon(u)$. So, for each cycle $(i_1, \ldots, i_m)$, we evaluate the Fourier transform

$$\int_{\mathbb{R}^m} K_\varepsilon(x_{i_1}, x_{i_2}) K_\varepsilon(x_{i_2}, x_{i_3}) \cdot \ldots \cdot K_\varepsilon(x_{i_m}, x_{i_1}) e^{2\pi i \sum_{j=1}^m u_{i_j} x_{i_j}}\, dx_{i_1} \cdots dx_{i_m}. \tag{3.33}$$

Expanding the product of K_ε's, we obtain 2^m terms

$$\int_{\mathbb{R}^m} \sum_{\boldsymbol{a}} \varepsilon^{\beta(\boldsymbol{a})} \frac{\sin(\pi(x_{i_1} - a_1 x_{i_2}))}{\pi(x_{i_1} - a_1 x_{i_2})} \cdots \frac{\sin(\pi(x_{i_m} - a_m x_{i_1}))}{\pi(x_{i_m} - a_m x_{i_1})} \cdot e^{2\pi i \sum_{j=1}^m u_{i_j} x_{i_j}}\, dx_{i_1} \cdots dx_{i_m}. \tag{3.34}$$

Here $\boldsymbol{a}$ ranges over all 2^m m-tuples $(a_1,\dots,a_m)$ with $a_j \in \{1,-1\}$, and $\beta(\boldsymbol{a}) = \#\left\{j \mid a_j = -1\right\}$.

According to Lemma 3, if $\sum \left|u_{i_j}\right| < 1$, then (3.34) is

$$2^{m-2}\varepsilon + \sum_{\boldsymbol{c}} \delta\left(\sum_{j=1}^{m} c_j u_{i_j}\right)\left(1 - V\left(c_1 u_{i_1},\dots,c_m u_{i_m}\right)\right), \tag{3.35}$$

where $\boldsymbol{c}$ is over all 2^{m-1} m-tuples $(c_1,\dots,c_m)$ with $c_j \in \{1,-1\}$, $c_m = 1$, and where

$$\begin{aligned} V(\boldsymbol{y}) &= M(\boldsymbol{y}) - m(\boldsymbol{y}), \\ M(\boldsymbol{y}) &= \max\{s_k(\boldsymbol{y}), k = 1,\dots,n\}, \\ m(\boldsymbol{y}) &= \min\{s_k(\boldsymbol{y}), k = 1,\dots,n\}, \\ s_j(\boldsymbol{y}) &= \sum_{j=1}^{k} y_j. \end{aligned} \tag{3.36}$$

Applying Parseval's formula to (3.31) and recalling the assumption that the support of $\prod_{i=1}^{n} \hat{f}_i(u_i)$ is in $\sum_{i=1}^{n} |u_i| < 1$ (so in the integral below, we are restricted to the region where Lemma 3 applies), we find that (3.31) equals

$$\begin{aligned} &\int_{\mathbb{R}^n} \left(\prod_{i=1}^{n} du_i\, \hat{f}_i(u_i)\right) \sum_{\underline{F}} \prod_{\ell=1}^{\nu(\underline{F})} (-1)^{|F_\ell|-1} \\ &\quad \cdot \sum_{\{i \mid i \in F_\ell\}}{}' \left(2^{|F_\ell|-2}\varepsilon + \sum_{\boldsymbol{c}} \delta\left(\sum_{j=1}^{|F_\ell|} c_j u_{i_j}\right)\left(1 - V\left(c_1 u_{i_1},\dots,c_{|F_\ell|} u_{i_{|F_\ell|}}\right)\right)\right), \end{aligned} \tag{3.37}$$

where $\sum'_{\{i \mid i \in F_\ell\}}$ is over all $(|F_\ell|-1)!$ cyclic permutations of the elements of F_ℓ.

Next, in the inner sum, change variables $w_{i_j} = c_j u_{i_j}$. Recalling that the $\hat{f}$'s are assumed to be even functions, we find that the above becomes

$$\begin{aligned} &\int_{\mathbb{R}^n} \left(\prod_{i=1}^{n} dw_i\, \hat{f}_i(w_i)\right) \sum_{\underline{F}} \prod_{\ell=1}^{\nu(\underline{F})} (-2)^{|F_\ell|-1} \\ &\quad \cdot \sum_{\{i \mid i \in F_\ell\}}{}' \left(\frac{\varepsilon}{2} + \delta\left(\sum_{j=1}^{|F_\ell|} w_{i_j}\right)\left(1 - V\left(w_{i_1},\dots,w_{i_{|F_\ell|}}\right)\right)\right). \end{aligned}$$

Applying the combinatorial identity [15, (4.35)], we get

$$\int_{\mathbb{R}^n}\left(\prod_{i=1}^{n} dw_i\, \hat{f}_i(w_i)\right)\sum_{\underline{F}}\prod_{\ell=1}^{\nu(\underline{F})}(-2)^{|F_\ell|-1}\cdot\left((|F_\ell|-1)!\frac{\varepsilon}{2}+\delta\left(\sum_{i\in F_\ell} w_i\right)\right.$$
$$\left.\cdot\left((|F_\ell|-1)! - \sum_{[H,H^c]}(|H|-1)!\,(|F_\ell|-1-|H|)!\left|\sum_{k\in H} w_k\right|\right)\right).$$

Here, $\left[H, H^c\right]$ runs over all $\left(2^{|F_\ell|}-2\right)/2$ ways of decomposing F_ℓ into two disjoint proper subsets: $H \cup H^c = F_\ell$, $H\cap H^c = \emptyset$, with $H \neq \emptyset, F_\ell$. Since $\sum |F_\ell| = n$, we can rewrite the above as

$$\boxed{\begin{aligned}&\int_{\mathbb{R}^n}\left(\prod_{i=1}^{n} du_i\, \hat{f}_i(u_i)\right)\sum_{\underline{F}}(-2)^{n-\nu(\underline{F})}\prod_{\ell=1}^{\nu(\underline{F})}\left((|F_\ell|-1)!\frac{\varepsilon}{2}+\delta\left(\sum_{i\in F_\ell} u_i\right)\right.\\ &\qquad\left.\left((|F_\ell|-1)! - \sum_{[H,H^c]}(|H|-1)!(|F_\ell|-1-|H|)!\left|\sum_{k\in H} u_k\right|\right)\right).\end{aligned}}$$

(3.38)

We now prove the lemma that was required in deriving the above.

LEMMA 3

Let $\sum_{j=1}^{m}\left|u_j\right| < 1$. *Then*

$$\int_{\mathbb{R}^m}\sum_{\boldsymbol{a}}\varepsilon^{\beta(\boldsymbol{a})}\frac{\sin(\pi(x_1-a_1x_2))}{\pi(x_1-a_1x_2)}\cdots\frac{\sin(\pi(x_m-a_mx_1))}{\pi(x_m-a_mx_1)}e^{2\pi i u\cdot x}\,dx$$
$$= 2^{m-2}\varepsilon + \sum_{\boldsymbol{c}}\delta\left(\sum_{j=1}^{m} c_ju_j\right)\left(1 - V\left(c_1u_1,\ldots,c_mu_m\right)\right). \qquad (3.39)$$

The notation here is defined between (3.34) and (3.36). Note: In the degenerate case $m = 1$, *the above should be read as*

$$\int_{\mathbb{R}}\left(1+\varepsilon\frac{\sin(2\pi x)}{2\pi x}\right)e^{2\pi iux}\,dx = \frac{1}{2}\varepsilon+\delta(u), \quad |u| < 1.$$

Proof

The $m = 1$ case is easy to check and follows from the fact that $(1/2)\chi_{[-1,1]}(u) = \int_{\mathbb{R}}(\sin(2\pi x)/(2\pi x))e^{2\pi iux}\,dx$. So, assume that $m \geq 2$, and consider a typical

$$\int_{\mathbb{R}^m}\frac{\sin(\pi(x_1-a_1x_2))}{\pi(x_1-a_1x_2)}\cdots\frac{\sin(\pi(x_m-a_mx_1))}{\pi(x_m-a_mx_1)}e^{2\pi i u\cdot x}\,dx. \qquad (3.40)$$

Let

$$\begin{aligned} t_i &= x_i - a_i x_{i+1}, \quad i = 1, \dots, m-1, \\ t_m &= x_m, \end{aligned} \tag{3.41}$$

so that

$$\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} 1 & a_1 & a_1 a_2 & a_1 a_2 a_3 & \dots & a_1 \cdot \dots \cdot a_{m-1} \\ 0 & 1 & a_2 & a_2 a_3 & \dots & a_2 \cdot \dots \cdot a_{m-1} \\ 0 & 0 & 1 & a_3 & \dots & a_3 \cdot \dots \cdot a_{m-1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & & 0 & 1 & a_{m-1} \\ 0 & \dots & & & 0 & 1 \end{pmatrix} \begin{pmatrix} t_1 \\ \vdots \\ t_m \end{pmatrix}.$$

Let

$$K(y) \stackrel{\text{def}}{=} \sin(\pi y)/(\pi y).$$

Changing variables, (3.40) is

$$\int_{\mathbb{R}^m} K(t_1) \cdots K(t_{m-1}) K(t_m - a_m(t_1 + a_1 t_2 + a_1 a_2 t_3 + \cdots + a_1 \cdot \dots \cdot a_{m-1} t_m)) \cdot e^{2\pi i (t_1 s_1 + \cdots + t_m s_m)} \, dt_1 \cdots dt_m, \tag{3.42}$$

where

$$\begin{aligned} s_1 &= u_1, \\ s_2 &= a_1 u_1 + u_2, \\ s_3 &= a_1 a_2 u_1 + a_2 u_2 + u_3, \\ &\vdots \\ s_k &= a_1 \cdot \dots \cdot a_{k-1} u_1 + a_2 \cdot \dots \cdot a_{k-1} u_2 + \cdots + a_{k-1} u_{k-1} + u_k, \\ &\vdots \end{aligned} \tag{3.43}$$

Now, $K(y) = K(-y)$, so, because $a_m \in \{1, -1\}$, we find that (3.42) equals

$$\int_{\mathbb{R}^m} K(t_1) \cdots K(t_{m-1}) K(a_m t_m - t_1 - a_1 t_2 - a_1 a_2 t_3 - \cdots - a_1 \cdot \dots \cdot a_{m-1} t_m)) \cdot e^{2\pi i (t_1 s_1 + \cdots + t_m s_m)} \, dt_1 \cdots dt_m.$$

Applying [15, (4.28)] (to the variable t_1 with $\tau = -a_m t_m + a_1 t_2 + a_1 a_2 t_3 + \cdots + a_1 \cdot \dots \cdot a_{m-1} t_m$), the above becomes

$$\int_{\mathbb{R}^m} \chi_{[-1/2,1/2]}(v) \, \chi_{[-1/2,1/2]}(v + s_1) \, e^{2\pi i v(-a_m t_m + a_1 t_2 + a_1 a_2 t_3 + \cdots + a_1 \cdot \dots \cdot a_{m-1} t_m)} \cdot K(t_2) \cdots K(t_{m-1}) e^{2\pi i (t_2 s_2 + \cdots + t_m s_m)} \, dv \, dt_2 \cdots dt_m.$$

Integrating over $t_2, \ldots, t_{m-1}$, we get

$$\int_{\mathbb{R}^2} \chi_{[-1/2,1/2]}(v)\, \chi_{[-1/2,1/2]}(v+s_1)\, \chi_{[-1/2,1/2]}(a_1 v + s_2) \cdot \chi_{[-1/2,1/2]}(a_1 a_2 v + s_3) \cdot \ldots \cdot \chi_{[-1/2,1/2]}(a_1 \cdot \ldots \cdot a_{m-2} v + s_{m-1}) \cdot e^{2\pi i t_m (s_m + v(a_1 \cdot \ldots \cdot a_{m-1} - a_m))}\, dv\, dt_m. \tag{3.44}$$

Now, if $\beta(\boldsymbol{a}) = \#\{i \mid a_i = -1\}$ is even, then $a_1 \cdot \ldots \cdot a_m = 1$, so $a_1 \cdot \ldots \cdot a_{m-1} = a_m$ and thus $a_1 \cdot \ldots \cdot a_{m-1} - a_m = 0$. Hence the integral over t_m pulls out a $\delta(s_m)$ from the integral.

Next, if $\beta(\boldsymbol{a})$ is odd, then $a_1 \cdot \ldots \cdot a_m = -1$, so $a_1 \cdot \ldots \cdot a_{m-1} = -a_m$ and thus $a_1 \cdot \ldots \cdot a_{m-1} - a_m = -2a_m$. Hence the integral over t_m gives us a $\delta(s_m - 2a_m v)$, which, when integrated over v, pulls out a product of characteristic functions.

Hence, we find that (3.44) (and hence that (3.40)) is

$$\delta(s_m) \int_{\mathbb{R}} \chi_{[-1/2,1/2]}(v)\, \chi_{[-1/2,1/2]}(v+s_1)\, \chi_{[-1/2,1/2]}(a_1 v + s_2) \cdot \ldots \cdot \chi_{[-1/2,1/2]}(a_1 \cdot \ldots \cdot a_{m-2} v + s_{m-1})\, dv \quad \text{if } \beta(\boldsymbol{a}) \text{ is even}, \tag{3.45}$$

$$\frac{1}{2} \chi_{[-1/2,1/2]}\left(\frac{s_m}{2a_m}\right) \chi_{[-1/2,1/2]}\left(\frac{s_m}{2a_m} + s_1\right) \chi_{[-1/2,1/2]}\left(a_1 \frac{s_m}{2a_m} + s_2\right) \cdot \ldots \cdot \chi_{[-1/2,1/2]}\left(a_1 \cdot \ldots \cdot a_{m-2} \frac{s_m}{2a_m} + s_{m-1}\right) \quad \text{if } \beta(\boldsymbol{a}) \text{ is odd}. \tag{3.46}$$

We require the following two claims.

CLAIM 4

Let $\beta(\boldsymbol{a})$ be odd, and assume that $\sum_{i=1}^m |u_i| < 1$. Then

$$\chi_{[-1/2,1/2]}\left(a_1 \cdot \ldots \cdot a_{k-1} \frac{s_m}{2a_m} + s_k\right) = 1, \quad k = 1, \ldots, m-1. \tag{3.47}$$

Thus, (3.46) equals $1/2$.

Proof

Because $a_k \in \{1, -1\}$, we have, from (3.43),

$$s_k = a_1 \cdot \ldots \cdot a_{k-1} (u_1 + a_1 u_2 + a_1 a_2 u_3 + \cdots + a_1 \cdot \ldots \cdot a_{k-1} u_k). \tag{3.48}$$

So the coefficient of u_j in (3.47) is

$$\frac{(a_1 \cdot \ldots \cdot a_{k-1})(a_1 \cdot \ldots \cdot a_{m-1})\left(a_1 \cdot \ldots \cdot a_{j-1}\right)}{2a_m} + (a_1 \cdot \ldots \cdot a_{k-1})\left(a_1 \cdot \ldots \cdot a_{j-1}\right). \tag{3.49}$$

When $\beta(\boldsymbol{a})$ is odd, $\prod_{i=1}^m a_i = -1$; hence (3.49) equals

$$\frac{(a_1 \cdot \ldots \cdot a_{k-1})\left(a_1 \cdot \ldots \cdot a_{j-1}\right)}{2} \in \left\{\frac{1}{2}, -\frac{1}{2}\right\}.$$

So

$$\left| a_1 \cdot \ldots \cdot a_{k-1} \frac{s_m}{2a_m} + s_k \right| < 1/2$$

(since we are assuming $\sum_{i=1}^m |u_i| < 1$), and hence the claim is proved. □

CLAIM 5
Let $\beta(\boldsymbol{a})$ be even, and assume that $\sum_{i=1}^m |u_i| < 1$. Then (3.45) equals

$$\delta(s_m)\left(1 - V(u_1, a_1 u_2, \ldots, a_1 \cdot \ldots \cdot a_{m-1} u_m)\right)$$

with $V(\boldsymbol{y})$ defined in (3.36).

Proof
In (3.45), we have, by (3.48),

$$\begin{aligned}&\chi_{[-1/2,1/2]}\left(a_1 \cdot \ldots \cdot a_{k-1} v + s_k\right)\\ &= \chi_{[-1/2,1/2]}\left(a_1 \cdot \ldots \cdot a_{k-1}\left(v + u_1 + a_1 u_2 + a_1 a_2 u_3 + \cdots + a_1 \cdot \ldots \cdot a_{k-1} u_k\right)\right),\end{aligned}$$

and we can drop the $a_1 \cdot \ldots \cdot a_{k-1} \in \{1, -1\}$ since $\chi_{[-1/2,1/2]}(y)$ is even. Furthermore, the $\delta(s_m)$ restricts us to $u_1 + a_1 u_2 + a_1 a_2 u_3 + \cdots + a_1 \cdot \ldots \cdot a_{m-1} u_m = 0$. And because we are assuming $\sum_{i=1}^m |u_i| < 1 < 2$, we may apply [15, Lemma 4.3], obtaining the claim. *Note: In [15, (4.32)], n could read $n - 1$ without affecting the truth of the equation since, in the notation of that paper, $f_2(v)\, f_2(v + u_1 + \cdots + u_n) = f_2(v)$.* □

We are now ready to complete the proof of this lemma. By Claim 4, the contribution to (3.39) from $\boldsymbol{a}$ with $\beta(\boldsymbol{a})$ odd is

$$\sum_{\substack{\boldsymbol{a} \\ \beta(\boldsymbol{a}) \text{ odd}}} \frac{1}{2} \varepsilon^{\beta(\boldsymbol{a})}.$$

But we are assuming $\varepsilon \in \{1, -1\}$, so the above is

$$2^{m-2} \varepsilon. \tag{3.50}$$

The contribution to (3.39) from $\boldsymbol{a}$ with $\beta(\boldsymbol{a})$ even is, by Claim 5,

$$\sum_{\substack{\boldsymbol{a} \\ \beta(\boldsymbol{a}) \text{ even}}} \delta(s_m)\left(1 - V(u_1, a_1 u_2, \ldots, a_1 \cdot \ldots \cdot a_{m-1} u_m)\right). \tag{3.51}$$

Now,

$$\begin{aligned} s_m &= a_1 \cdot \ldots \cdot a_{m-1} \left(u_1 + a_1 u_2 + a_1 a_2 u_3 + \cdots + a_1 \cdot \ldots \cdot a_{m-1} u_m\right) \\ &= a_m u_1 + a_m a_1 u_2 + a_m a_1 a_2 u_3 + \cdots + a_m a_1 \cdot \ldots \cdot a_{m-1} u_m \end{aligned}$$

because $\prod_{i=1}^{m} a_i = 1$ when $\beta(\boldsymbol{a})$ is even. Let

$$\boldsymbol{c} = (c_1, \ldots, c_m) = (a_m, a_m a_1, a_m a_1 a_2, \ldots, a_m a_1 \cdot \ldots \cdot a_{m-1}).$$

Now, because $\prod_{i=1}^{m} a_i = 1$, $\boldsymbol{c}$ ranges over all m-tuples with $c_j \in \{1, -1\}$ and $c_m = 1$. So, summing over such $\boldsymbol{c}$, we find that (3.51) equals

$$\sum_{\boldsymbol{c}} \delta \left(\sum_{j=1}^{m} c_j u_j \right) (1 - V(a_m c_1 u_1, \ldots, a_m c_m u_m)) . \tag{3.52}$$

But, because $V(-\boldsymbol{y}) = V(\boldsymbol{y})$, the above is (regardless of the value of $a_m = \pm 1$)

$$\sum_{\boldsymbol{c}} \delta \left(\sum_{j=1}^{m} c_j u_j \right) (1 - V(c_1 u_1, \ldots, c_m u_m)) . \tag{3.53}$$

This, in combination with (3.50), establishes the lemma. □

3.4. l.h.s. = r.h.s.

LEMMA 4

We have

$$\int_{\mathbb{R}^{|F_\ell|}} \prod_{i \in F_\ell} du_i \, \hat{f}_i(u_i) = \int_{\mathbb{R}} \hat{F}_\ell(u) \, du.$$

Proof

Both are equal, by Fourier inversion, to $\prod_{i \in F_\ell} f_i(0)$. □

LEMMA 5

We have

$$\int_{\mathbb{R}^{|F_\ell|}} \left(\prod_{i \in F_\ell} du_i \, \hat{f}_i(u_i) \right) \delta \left(\sum_{i \in F_\ell} u_i \right) = \int_{\mathbb{R}} F_\ell(x) \, dx.$$

Proof

We obtain the lemma by Parseval's formula. □

LEMMA 6
Let $H \subset F_\ell$, $H \neq \emptyset$. Then

$$\int_{\mathbb{R}^{|F_\ell|}} \left(\prod_{i \in F_\ell} du_i \, \hat{f}_i(u_i) \right) \delta \left(\sum_{i \in F_\ell} u_i \right) \left| \sum_{k \in H} u_k \right| = \int_{\mathbb{R}} \left(\widehat{\prod_{i \in H} f_i} \right)(u) \left(\widehat{\prod_{i \in H^c} f_i} \right)(u) \, |u| \, du.$$

Proof
We obtain the lemma by Parseval's formula. □

Now, $W^{(n)}_{\mathrm{USp}} = W_{-1}$, so we need to compare (3.38), with $\varepsilon = -1$, to (3.13). By Lemmas 4–6, write (3.38) as

$$\sum_{\underline{F}} (-2)^{n-\nu(\underline{F})} \prod_{\ell=1}^{\nu(\underline{F})} (P_\ell + Q_\ell + R_\ell) \tag{3.54}$$

with

$$P_\ell = (|F_\ell| - 1)! \left(\frac{-1}{2} \right) \int_{\mathbb{R}} \hat{F}_\ell(u) \, du,$$
$$Q_\ell = (|F_\ell| - 1)! \int_{\mathbb{R}} F_\ell(x) \, dx,$$
$$R_\ell = - \sum_{[H, H^c]} (|H| - 1)! \, (|F_\ell| - 1 - |H|)! \int_{\mathbb{R}} \left(\widehat{\prod_{i \in H} f_i} \right)(u) \left(\widehat{\prod_{i \in H^c} f_i} \right)(u) \, |u| \, du. \tag{3.55}$$

Expanding the product over ℓ, we get

$$\sum_{\underline{F}} (-2)^{n-\nu(\underline{F})} \sum_{S} \left(\prod_{\ell \in S^c} Q_\ell \right) \sum_{T \subseteq S} \left(\prod_{\ell \in T^c} P_\ell \right) \left(\prod_{\ell \in T} R_\ell \right), \tag{3.56}$$

where S ranges over all subsets of $\{1, \ldots, \nu(\underline{F})\}$. (We take empty products to be 1.) Expanding the product $\prod_{\ell \in T} R_\ell$, we find that (3.56) is

$$\sum_{\underline{F}} (-2)^{n-\nu(\underline{F})} \sum_{S} \left(\prod_{\ell \in S^c} Q_\ell \right) \sum_{T \subseteq S} \left(\prod_{\ell \in T^c} P_\ell \right) \cdot \Bigg((-1)^{|T|} \sum_{\boldsymbol{H}} \prod_{j=1}^{|T|} \left(|H_j| - 1 \right)! \cdot \left(|F_{\ell_j}| - 1 - |H_j| \right)! \int_{\mathbb{R}} \left(\widehat{\prod_{i \in H_j} f_i} \right)(u) \left(\widehat{\prod_{i \in H_j^c} f_i} \right)(u) |u| \, du \Bigg), \tag{3.57}$$

where $\sum_{\boldsymbol{H}}$ is over all $|T|$-tuples $\left(\left[H_1, H_1^c\right], \ldots, \left[H_{|T|}, H_{|T|}^c\right]\right)$ and where $T = \left\{\ell_1, \ldots, \ell_{|T|}\right\}$. (If $T = \emptyset$, we take the large bracketed factor to be 1. And if $T \neq \emptyset$, but $\sum_{\boldsymbol{H}}$ contains no terms, we take it to be zero.) We have thus expressed, in (3.57), the r.h.s. of (3.7) in a form that can easily be compared with the l.h.s., as expressed in (3.13).

More precisely, a typical term in (3.13) is specified by $\underline{F}_{\text{l.h.s.}}$, $S_{\text{l.h.s.}}$, S_2, $(A; B)$. The sum over $\underline{F}$ arises from combinatorial sieving, and the sum over $S \subseteq \left\{1, \ldots, \nu(\underline{F})\right\}$ arises from multiplying out the explicit formula (3.12). The sum over $S_2 \subseteq S$ comes from deciding which prime powers are paired up to produce squares and which are already squares (S_2^c). $(A; B)$ accounts for all ways of pairing up S_2. The contribution to (3.13) from a typical term is

$$
\begin{aligned}
&(-2)^{n-\nu(\underline{F}_{\text{l.h.s.}})} \left(\prod_{\ell \in S_{\text{l.h.s.}}^c} (|F_\ell| - 1)! \int_{\mathbb{R}} F_\ell(x)\, dx \right) \\
&\quad \cdot \left(\prod_{\ell \in S_2^c} (|F_\ell| - 1)! \left(\frac{-1}{2}\right) \int_{\mathbb{R}} \hat{F}_\ell(u)\, du \right) \\
&\quad \cdot \left(2^{|S_2|/2} \prod_{j=1}^{|S_2|/2} \left(\left|F_{a_j}\right| - 1\right)! \left(\left|F_{b_j}\right| - 1\right)! \int_{\mathbb{R}} \hat{F}_{a_j}(u) \hat{F}_{b_j}(u)\, |u|\, du \right) \\
&= (-2)^{n-\nu(\underline{F}_{\text{l.h.s.}})} \left(\prod_{\ell \in S_{\text{l.h.s.}}^c} Q_\ell \right) \left(\prod_{\ell \in S_2^c} P_\ell \right) \\
&\quad \cdot \left(2^{|S_2|/2} \prod_{j=1}^{|S_2|/2} \left(\left|F_{a_j}\right| - 1\right)! \left(\left|F_{b_j}\right| - 1\right)! \int_{\mathbb{R}} \hat{F}_{a_j}(u) \hat{F}_{b_j}(u)\, |u|\, du \right). \qquad (3.58)
\end{aligned}
$$

On the other hand, in (3.57), a typical term is specified by $\underline{F}_{\text{r.h.s.}}$, $S_{\text{r.h.s.}}$, T, $\left(\left[H_1, H_1^c\right], \ldots, \left[H_{|T|}, H_{|T|}^c\right]\right)$. Set

$$
\begin{aligned}
&\underline{F}_{\text{r.h.s.}} = \left\{F_\ell \mid \ell \in S_{\text{l.h.s.}}^c\right\} \bigcup \left\{F_\ell \mid \ell \in S_2^c\right\} \bigcup \left\{F_{a_j} \cup F_{b_j} \mid j = 1, \ldots, |S_2|/2\right\}, \\
&H_1 = F_{a_1}, \qquad H_1^c = F_{b_1}, \\
&\vdots \qquad\qquad \vdots \qquad\qquad (3.59) \\
&H_{|S_2|/2} = F_{a_{|S_2|/2}}, \qquad H_{|S_2|/2}^c = F_{b_{|S_2|/2}}.
\end{aligned}
$$

$S_{\text{r.h.s.}}$ and T are chosen in the obvious way (so that both products of Q's match, and both products of P's match). Notice that $|T| = |S_2|/2$ and that $\nu(\underline{F}_{\text{l.h.s.}}) = \nu(\underline{F}_{\text{r.h.s.}}) + |S_2|/2$.

The contribution to (3.57) from this term is thus

$$(-2)^{n-\nu(\underline{F}_{\text{l.h.s.}})+|S_2|/2}\left(\prod_{\ell\in S^c_{\text{l.h.s.}}} Q_\ell\right)\left(\prod_{\ell\in S^c_2} P_\ell\right)$$
$$\cdot\left((-1)^{|S_2|/2}\prod_{j=1}^{|S_2|/2}\left(\left|F_{a_j}\right|-1\right)!\left(\left|F_{b_j}\right|-1\right)!\int_{\mathbb{R}}\hat{F}_{a_j}(u)\hat{F}_{b_j}(u)\,|u|\,du\right), \tag{3.60}$$

which is equal, because $|S_2|$ is even, to (3.58).

So every term on the l.h.s. has a corresponding term on the r.h.s.

Conversely, this method of matching (i.e., (3.59)) produces for every term on the r.h.s. its corresponding term on the l.h.s. (with the convention that we disregard, on the r.h.s., any term with $|T| \geq 1$ but $\sum_{\boldsymbol{H}}$ empty; we can do so since these terms contribute nothing to (3.57)).

Thus (3.13) = (3.38) and Theorem 3.1 is proved. □

3.5. Examples

One term for $n = 17$

Let $n = 17$, and let

$$\begin{aligned}
\underline{F}_{\text{l.h.s.}} &= [F_1, F_2, F_3, F_4, F_5, F_6, F_7]\\
&= [\{1,2,13\}, \{4\}, \{3,6,7,9,17\}, \{8,10,11\}, \{5,12\}, \{14\}, \{15,16\}],\\
S_{\text{l.h.s.}} &= \{1,2,3,5,6\}, \qquad S^c_{\text{l.h.s.}} = \{4,7\},\\
S_2 &= \{1,2,5,6\}, \qquad S^c_2 = \{3\},\\
(A;B) &= (1,5;2,6).
\end{aligned} \tag{3.61}$$

This corresponds on the r.h.s. to

$$\begin{aligned}
&\underline{F}_{\text{r.h.s.}} = [\mathfrak{F}_1, \mathfrak{F}_2, \mathfrak{F}_3, \mathfrak{F}_4, \mathfrak{F}_5],\\
&\mathfrak{F}_1 = F_4, \qquad \mathfrak{F}_2 = F_7, \qquad \mathfrak{F}_3 = F_3,\\
&\mathfrak{F}_4 = F_1 \cup F_2, \qquad \mathfrak{F}_5 = F_5 \cup F_6,\\
&S_{\text{r.h.s.}} = \{3,4,5\}, \qquad S^c_{\text{r.h.s.}} = \{1,2\},\\
&T = \{4,5\}, \qquad T^c = \{3\},\\
&H_1 = F_1, \qquad H^c_1 = F_2,\\
&H_2 = F_5, \qquad H^c_2 = F_6.
\end{aligned} \tag{3.62}$$

Tables 3.1 and 3.2 show the correspondence between terms on the l.h.s. (as expressed in (3.58)) and the r.h.s. (as expressed in (3.60)).

3.6. Analogous results for $\mathrm{GL}_M/\mathbb{Q}$

Let $L(s,\pi)$ be the L-function attached to a self-contragredient ($\pi=\tilde{\pi}$) automorphic cuspidal representation of GL_M over $\mathbb{Q}$. Such an L-function is given initially (for $\Re s$ sufficiently large) as an Euler product of the form

$$L(s,\pi)=\prod_p L(s,\pi_p)=\prod_p\prod_{j=1}^{M}(1-\alpha_\pi(p,j)p^{-s})^{-1}.$$

The condition $\pi=\tilde{\pi}$ implies that $\alpha_\pi(p,j)\in\mathbb{R}$. The Rankin-Selberg L-function $L(s,\pi\otimes\tilde{\pi})$ factors as the product of the symmetric and exterior square L-functions (see [1]):

$$L(s,\pi\otimes\tilde{\pi})=L(s,\pi\otimes\pi)=L(s,\pi,\vee^2)L(s,\pi,\wedge^2)$$

and has a simple pole at $s=1$ which is carried by one of the two factors. Write the order of the pole of $L(s,\pi,\wedge^2)$ as $(\delta(\pi)+1)/2$ (so that $\delta(\pi)=\pm1$).

We desire to generalize Theorem 3.1 to the zeros of $L(s,\pi\otimes\chi_d)$ whose Euler product is given by

$$L(s,\pi\otimes\chi_d)=\prod_p\prod_{j=1}^{M}(1-\chi_d(p)\alpha_\pi(p,j)p^{-s})^{-1}.$$

Now, when $\pi=\tilde{\pi}$, $L(s,\pi\otimes\chi_d)$ has a functional equation of the form

$$\begin{aligned}\Phi(s,\pi\otimes\chi_d)&:=\pi^{-Ms/2}\prod_{j=1}^{M}\Gamma\big((s+\mu_{\pi\otimes\chi_d}(j))/2\big)L(s,\pi\otimes\chi_d)\\&=\varepsilon(s,\pi\otimes\chi_d)\Phi(1-s,\pi\otimes\chi_d),\end{aligned}$$

where the $\mu_{\pi\otimes\chi_d}(j)$'s are complex numbers that are known to satisfy

$$\Re\left(\mu_{\pi\otimes\chi_d}(j)\right)>-1/2$$

(and are conjectured to satisfy $\Re\left(\mu_{\pi\otimes\chi_d}(j)\right)\geq0$). We also have

$$\varepsilon(s,\pi\otimes\chi_d)=\varepsilon(\pi\otimes\chi_d)Q_{\pi\otimes\chi_d}^{-s+1/2}=\pm Q_{\pi\otimes\chi_d}^{-s+1/2}$$

with $\varepsilon(\pi\otimes\chi_d)=\chi'(d)$, where χ' is a quadratic character that depends only on π. When $\delta(\pi)=-1$, all twists have $\varepsilon(\pi\otimes\chi_d)=1$. If $\delta(\pi)=1$, then half the $L(s,\pi\otimes\chi_d)$'s have $\varepsilon(\pi\otimes\chi_d)=1$ and the other half have $\varepsilon(\pi\otimes\chi_d)=-1$ (with

Table 3.1. Matching the l.h.s. with the r.h.s. for $n = 1, 2, 3$. Here $S_{\text{l.h.s.}} \subseteq \{1, \dots, \nu(\underline{F}_{\text{l.h.s.}})\}$, $S_2 \subseteq S_{\text{l.h.s.}}$, with $|S_2|$ even. $(A; B)$ accounts for all ways of pairing up S_2. Further, $S_{\text{r.h.s.}} \subseteq \{1, \dots, \nu(\underline{F}_{\text{r.h.s.}})\}$, $T \subseteq S_{\text{r.h.s.}}$, and $\boldsymbol{H}$ is over all $|T|$-tuples $([H_1, H_1^c], \dots, [H_{|T|}, H_{|T|^c}])$. The matching is as described in (3.59).

n	$\underline{F}_{\text{l.h.s.}}$	$S_{\text{l.h.s.}}$	S_2	$(A; B)$	$\underline{F}_{\text{r.h.s.}}$	$S_{\text{r.h.s.}}$	T	$\boldsymbol{H}$
1	$[\{1\}]$	$\emptyset$	$\emptyset$	—	$[\{1\}]$	$\emptyset$	$\emptyset$	—
		$\{1\}$	$\emptyset$	—		$\{1\}$	$\emptyset$	—
2	$[\{1,2\}]$	$\emptyset$	$\emptyset$	—	$[\{1,2\}]$	$\emptyset$	$\emptyset$	—
		$\{1\}$	$\emptyset$	—		$\{1\}$	$\emptyset$	—
	$[\{1\},\{2\}]$	$\emptyset$	$\emptyset$	—	$[\{1\},\{2\}]$	$\emptyset$	$\emptyset$	—
		$\{1\}$	$\emptyset$	—		$\{1\}$	$\emptyset$	—
		$\{2\}$	$\emptyset$	—		$\{2\}$	$\emptyset$	—
		$\{1,2\}$	$\emptyset$	—		$\{1,2\}$	$\emptyset$	—
			$\{1,2\}$	$(1;2)$	$[\{1,2\}]$	$\{1\}$	$\{1\}$	$[\{1\},\{2\}]$
3	$[\{1,2,3\}]$	$\emptyset$	$\emptyset$	—	$[\{1,2,3\}]$	$\emptyset$	$\emptyset$	—
		$\{1\}$	$\emptyset$	—		$\{1\}$	$\emptyset$	—
	$[\{1,2\},\{3\}]$	$\emptyset$	$\emptyset$	—	$[\{1,2\},\{3\}]$	$\emptyset$	$\emptyset$	—
		$\{1\}$	$\emptyset$	—		$\{1\}$	$\emptyset$	—
		$\{2\}$	$\emptyset$	—		$\{2\}$	$\emptyset$	—
		$\{1,2\}$	$\emptyset$	—		$\{1,2\}$	$\emptyset$	—
			$\{1,2\}$	$(1;2)$	$[\{1,2,3\}]$	$\{1\}$	$\{1\}$	$[\{1,2\},\{3\}]$
	$[\{1,3\},\{2\}]$	$\emptyset$	$\emptyset$	—	$[\{1,3\},\{2\}]$	$\emptyset$	$\emptyset$	—
		$\{1\}$	$\emptyset$	—		$\{1\}$	$\emptyset$	—
		$\{2\}$	$\emptyset$	—		$\{2\}$	$\emptyset$	—
		$\{1,2\}$	$\emptyset$	—		$\{1,2\}$	$\emptyset$	—
			$\{1,2\}$	$(1;2)$	$[\{1,2,3\}]$	$\{1\}$	$\{1\}$	$[\{1,3\},\{2\}]$
	$[\{2,3\},\{1\}]$	$\emptyset$	$\emptyset$	—	$[\{2,3\},\{1\}]$	$\emptyset$	$\emptyset$	—
		$\{1\}$	$\emptyset$	—		$\{1\}$	$\emptyset$	—
		$\{2\}$	$\emptyset$	—		$\{2\}$	$\emptyset$	—
		$\{1,2\}$	$\emptyset$	—		$\{1,2\}$	$\emptyset$	—
			$\{1,2\}$	$(1;2)$	$[\{1,2,3\}]$	$\{1\}$	$\{1\}$	$[\{2,3\},\{1\}]$
	$[\{1\},\{2\},\{3\}]$	$\emptyset$	$\emptyset$	—	$[\{1\},\{2\},\{3\}]$	$\emptyset$	$\emptyset$	—
		$\{1\}$	$\emptyset$	—		$\{1\}$	$\emptyset$	—
		$\{2\}$	$\emptyset$	—		$\{2\}$	$\emptyset$	—
		$\{3\}$	$\emptyset$	—		$\{3\}$	$\emptyset$	—
		$\{1,2\}$	$\emptyset$	—		$\{1,2\}$	$\emptyset$	—
			$\{1,2\}$	$(1;2)$	$[\{1,2\},\{3\}]$	$\{1\}$	$\{1\}$	$[\{1\},\{2\}]$
		$\{1,3\}$	$\emptyset$	—	$[\{1\},\{2\},\{3\}]$	$\{1,3\}$	$\emptyset$	—
			$\{1,3\}$	$(1;3)$	$[\{1,3\},\{2\}]$	$\{1\}$	$\{1\}$	$[\{1\},\{3\}]$
		$\{2,3\}$	$\emptyset$	—	$[\{1\},\{2\},\{3\}]$	$\{2,3\}$	$\emptyset$	—
			$\{2,3\}$	$(2;3)$	$[\{2,3\},\{1\}]$	$\{1\}$	$\{1\}$	$[\{2\},\{3\}]$
		$\{1,2,3\}$	$\emptyset$	—	$[\{1\},\{2\},\{3\}]$	$\{1,2,3\}$	$\emptyset$	—
			$\{1,2\}$	$(1;2)$	$[\{1,2\},\{3\}]$	$\{1,2\}$	$\{1\}$	$[\{1\},\{2\}]$
			$\{1,3\}$	$(1;3)$	$[\{1,3\},\{2\}]$	$\{1,2\}$	$\{1\}$	$[\{1\},\{3\}]$
			$\{2,3\}$	$(2;3)$	$[\{2,3\},\{1\}]$	$\{1,2\}$	$\{1\}$	$[\{2\},\{3\}]$

Table 3.2. Terms on the r.h.s. that are discarded since they contribute nothing to (3.57).

n	$\underline{F}_{\text{r.h.s.}}$	$S_{\text{r.h.s.}}$	T	$\boldsymbol{H}$
1	[{1}]	{1}	{1}	none
2	[{1} , {2}]	{1}	{1}	none
	[{1} , {2}]	{2}	{2}	none
	[{1} , {2}]	{1, 2}	{1, 2}	none
3	[{1, 2} , {3}]	{2}	{2}	none
	[{1, 2} , {3}]	{1, 2}	{2}	none
	[{1, 2} , {3}]	{1, 2}	{1, 2}	none
	[{1, 3} , {2}]	{2}	{2}	none
	[{1, 3} , {2}]	{1, 2}	{2}	none
	[{1, 3} , {2}]	{1, 2}	{1, 2}	none
	[{2, 3} , {1}]	{2}	{2}	none
	[{2, 3} , {1}]	{1, 2}	{2}	none
	[{2, 3} , {1}]	{1, 2}	{1, 2}	none
	[{1} , {2} , {3}]	{1}	{1}	none
	[{1} , {2} , {3}]	{2}	{2}	none
	[{1} , {2} , {3}]	{3}	{3}	none
	[{1} , {2} , {3}]	{1, 2}	{1}	none
	[{1} , {2} , {3}]	{1, 2}	{2}	none
	[{1} , {2} , {3}]	{1, 2}	{1, 2}	none
	[{1} , {2} , {3}]	{1, 3}	{1}	none
	[{1} , {2} , {3}]	{1, 3}	{3}	none
	[{1} , {2} , {3}]	{1, 3}	{1, 3}	none
	[{1} , {2} , {3}]	{2, 3}	{2}	none
	[{1} , {2} , {3}]	{2, 3}	{3}	none
	[{1} , {2} , {3}]	{2, 3}	{2, 3}	none
	[{1} , {2} , {3}]	{1, 2, 3}	{1}	none
	[{1} , {2} , {3}]	{1, 2, 3}	{2}	none
	[{1} , {2} , {3}]	{1, 2, 3}	{3}	none
	[{1} , {2} , {3}]	{1, 2, 3}	{1, 2}	none
	[{1} , {2} , {3}]	{1, 2, 3}	{1, 3}	none
	[{1} , {2} , {3}]	{1, 2, 3}	{2, 3}	none
	[{1} , {2} , {3}]	{1, 2, 3}	{1, 2, 3}	none

the corresponding d's lying in fixed arithmetic progressions to the modulus of the character χ'). When $\varepsilon(\pi \otimes \chi_d) = 1$, we write the nontrivial zeros of $L(s, \pi \otimes \chi_d)$ as

$$1/2 + i\gamma_{\pi\otimes\chi_d}^{(j)}, \quad j = \pm 1, \pm 2, \pm 3, \dots,$$

with

$$\dots \Re\gamma_{\pi\otimes\chi_d}^{(-2)} \le \Re\gamma_{\pi\otimes\chi_d}^{(-1)} \le 0 \le \Re\gamma_{\pi\otimes\chi_d}^{(1)} \le \Re\gamma_{\pi\otimes\chi_d}^{(2)} \le \dots$$

and

$$\gamma_{\pi\otimes\chi_d}^{(-k)} = -\gamma_{\pi\otimes\chi_d}^{(k)}.$$

When $\varepsilon(\pi \otimes \chi_d) = -1$, $\gamma = 0$ is a zero of $L(s, \pi \otimes \chi_d)$, and we index the zeros as

$$1/2 + i\gamma_{\pi\otimes\chi_d}^{(j)}, \quad j \in \mathbb{Z},$$

with

$$\dots \Re\gamma_{\pi\otimes\chi_d}^{(-2)} \le \Re\gamma_{\pi\otimes\chi_d}^{(-1)} \le \gamma_{\pi\otimes\chi_d}^{(0)} = 0 \le \Re\gamma_{\pi\otimes\chi_d}^{(1)} \le \Re\gamma_{\pi\otimes\chi_d}^{(2)} \le \dots$$

and

$$\gamma_{\pi\otimes\chi_d}^{(-k)} = -\gamma_{\pi\otimes\chi_d}^{(k)}.$$

Next, let $D(X)$ be as in (3.4), and let

$$D_{\pi,+}(X) = \{d \in D(X) : \varepsilon(\pi \otimes \chi_d) = 1\},$$
$$D_{\pi,-}(X) = \{d \in D(X) : \varepsilon(\pi \otimes \chi_d) = -1\}.$$

Then, assuming, for $M \ge 4$, the Ramanujan conjecture

$$|\alpha_\pi(p, j)| \le 1,$$

we have the following theorem.

THEOREM 3.2
Let $f(x_1, \dots, x_n) = \prod_{i=1}^n f_i(x_i)$ be even in all its variables with each f_i in $S(\mathbb{R})$. Assume further that $\hat{f}(u_1, \dots, u_n)$ is supported in $\sum_{i=1}^n |u_i| < 1/M$. Then if $\delta(\pi) = 1$,

$$\lim_{X\to\infty} \frac{1}{|D_{\pi,\pm}(X)|} \sum_{d \in D_{\pi,\pm}(X)} \sum_{j_1,\dots,j_n}^{*} f\left(L_M\gamma_{\pi\otimes\chi_d}^{(j_1)}, L_M\gamma_{\pi\otimes\chi_d}^{(j_2)}, \dots, L_M\gamma_{\pi\otimes\chi_d}^{(j_n)}\right)$$
$$= \int_{\mathbb{R}^n} f(x) W_{\pm,O}^{(n)}(x)\,dx, \tag{3.63}$$

and if $\delta(\pi) = -1$ *(so that all twists have* $\varepsilon(\pi \otimes \chi_d) = 1$*),*

$$\lim_{X\to\infty} \frac{1}{|D(X)|} \sum_{d\in D(X)} \sum_{j_1,\ldots,j_n}{}^{*} f\left(L_M \gamma^{(j_1)}_{\pi\otimes\chi_d}, L_M \gamma^{(j_2)}_{\pi\otimes\chi_d}, \ldots, L_M \gamma^{(j_n)}_{\pi\otimes\chi_d}\right)$$
$$= \int_{\mathbb{R}^n} f(x) W^{(n)}_{\mathrm{USp}}(x)\, dx, \tag{3.64}$$

where

$$L_M = \frac{M \log X}{2\pi},$$
$$W^{(n)}_{\mathrm{USp}}(x_1, \ldots, x_n) = \det\left(K_{-1}(x_j, x_k)\right)_{\substack{1\le j\le n\\1\le k\le n}},$$
$$W^{(n)}_{+,O}(x_1, \ldots, x_n) = \det\left(K_{1}(x_j, x_k)\right)_{\substack{1\le j\le n\\1\le k\le n}},$$
$$W^{(n)}_{-,O}(x_1, \ldots, x_n) = \det\left(K_{-1}(x_j, x_k)\right)_{\substack{1\le j\le n\\1\le k\le n}} + \sum_{\nu=1}^{n} \delta(x_\nu) \det\left(K_{-1}(x_j, x_k)\right)_{\substack{1\le j\ne\nu\le n\\1\le k\ne\nu\le n}},$$
$$K_\varepsilon(x, y) = \frac{\sin(\pi(x-y))}{\pi(x-y)} + \varepsilon \frac{\sin(\pi(x+y))}{\pi(x+y)}$$

($W^{(1)}_{-,O}(x) = 1 - \sin(2\pi x)/(2\pi x) + \delta(x)$*) and where* $\sum^*_{j_1,\ldots,j_n}$ *is over* $j_k = (0), \pm1, \pm2, \ldots$*, with* $j_{k_1} \ne \pm j_{k_2}$ *if* $k_1 \ne k_2$*.*

Remark. Again, as in Theorem 3.1, the assumptions f_i even and f of the form $\prod f_i$ can be removed.

Proof

The proof is similar to that of Theorem 3.1. The main difference is in the explicit formula that, for $L(s, \pi \otimes \chi_d)$, reads

$$\sum_{\gamma_{\pi\otimes\chi_d}} F_\ell\left(L_M \gamma_{\pi\otimes\chi_d}\right) = \int_{\mathbb{R}} F_\ell(x)\, dx + O(1/\log X) - \frac{2}{M\log X} \sum_{m=1}^{\infty} \frac{\Lambda(m) a_\pi(m)}{m^{1/2}} \chi_d(m) \hat{F}_\ell\left(\frac{\log m}{M \log X}\right) \tag{3.65}$$

where

$$a_\pi(p^k) = \sum_{j=1}^{M} \alpha^k_\pi(p, j).$$

We consider the two cases, $\delta(\pi) = -1$ and $\delta(\pi) = 1$, separately.

For both cases we require the estimates

$$\begin{aligned} \sum_{m\le T} \left|a_\pi(m)\Lambda(m)\right|^2/m &\sim \log^2(T)/2, \\ \sum_{p\le T} a_\pi(p^2)\log p &\sim -\delta(\pi)T, \\ \sum_{p\le T} \left|a_\pi(p)\log p\right|^2/p &\sim \log^2(T)/2 \end{aligned} \tag{3.66}$$

(see [15] and [6]). For these estimates, and $M \ge 4$, the Ramanujan conjecture is assumed; these three are needed in the analogs of Claim 2, Subclaim 3.22, and Subclaim 3.24.

When $\delta(\pi) = -1$, all twists have $\varepsilon(\pi \otimes \chi_d) = 1$. The combinatorics work out exactly the same. The smaller support of $\hat{f}$ compensates for the presence of the M in the explicit formula.

When $\delta(\pi) = 1$, we need to examine the two subcases, $\varepsilon(\pi \otimes \chi_d) = 1$ and $\varepsilon(\pi \otimes \chi_d) = -1$, separately.

As the analog of Lemma 1, we have the following lemma.

LEMMA 7

When $\delta(\pi) = 1$,

$$\begin{aligned} \lim_{X\to\infty} \frac{1}{\left|D_{\pi,+}(X)\right|} \sum_{d\in D_{\pi,+}(X)} \left(\frac{-2}{M\log X}\right)^k \prod_{j=1}^{k} \sum_{m=1}^{\infty} \frac{\Lambda(m)a_\pi(m)}{m^{1/2}}\chi_d(m) \\ \cdot \hat{F}_{\ell_j}\left(\frac{\log m}{M\log X}\right) = \sum_{\substack{S_2\subseteq S\\ |S_2| \text{ even}}} \left(\left(\frac{1}{2}\right)^{|S_2^c|} \prod_{\ell\in S_2^c} \int_{\mathbb{R}} \hat{F}_\ell(u)\,du\right) \\ \cdot \left(\sum_{(A;B)} 2^{|S_2|/2} \prod_{j=1}^{|S_2|/2} \int_{\mathbb{R}} |u|\hat{F}_{a_j}(u)\hat{F}_{b_j}(u)\,du\right), \end{aligned} \tag{3.67}$$

where $S = \{l_1, \ldots, l_k\}$. $\sum_{\substack{S_2\subseteq S\\ |S_2| \text{ even}}}$ *is over all subsets* S_2 *of* S *whose size is even.* $\sum_{(A;B)}$ *is over all ways of pairing up the elements of* S_2. $F_\ell(x)$ *is defined in (3.11).*

Proof

Notice that the only difference in the r.h.s. of this lemma as compared to Lemma 1 is in the factor

$$\left(\frac{1}{2}\right)^{|S_2^c|} \prod_{\ell\in S_2^c} \int_{\mathbb{R}} \hat{F}_\ell(u)\,du.$$

The difference in sign is accounted for by the opposite sign in (3.66). □

So, we have that the l.h.s. of (3.63), for $D_{\pi,+}$, tends, as $X \to \infty$, to

$$\sum_{\underline{F}} (-2)^{n-\nu(\underline{F})} \left(\prod_{\ell=1}^{\nu(\underline{F})} (|F_\ell| - 1)! \right) \sum_S \left(\prod_{\ell \in S^c} \int_{\mathbb{R}} F_\ell(x)\, dx \right)$$
$$\cdot \sum_{\substack{S_2 \subseteq S \\ |S_2| \text{ even}}} \left(\left(\frac{1}{2}\right)^{|S_2^c|} \prod_{\ell \in S_2^c} \int_{\mathbb{R}} \hat{F}_\ell(u)\, du \right)$$
$$\cdot \left(\sum_{(A;B)} 2^{|S_2|/2} \prod_{j=1}^{|S_2|/2} \int_{\mathbb{R}} |u|\, \hat{F}_{a_j}(u) \hat{F}_{b_j}(u)\, du \right).$$

This expression matches (3.38) with $\varepsilon = 1$; that is, this equals (in the notation of Section 3.3)

$$\int_{\mathbb{R}^n} f(x) W_+(x)\, dx = \int_{\mathbb{R}^n} f(x) W_{+,\mathrm{O}}^{(n)}(x)\, dx.$$

For the $\varepsilon(\pi \otimes \chi_d) = -1$ case, there is always a zero at $s = 1/2$,

$$\gamma_{\pi \otimes \chi_d}^{(0)} = 0,$$

and, before applying the combinatorial sieving of Section 3.2, we need to isolate this zero. Now

$$\sum_{j_1,\ldots,j_n}^{*} f\left(L_M \gamma_{\pi\otimes\chi_d}^{(j_1)}, L_M \gamma_{\pi\otimes\chi_d}^{(j_2)}, \ldots, L_M \gamma_{\pi\otimes\chi_d}^{(j_n)} \right)$$
$$= \sum_{j_1 \neq 0,\ldots,j_n \neq 0}^{*} f\left(L_M \gamma_{\pi\otimes\chi_d}^{(j_1)}, L_M \gamma_{\pi\otimes\chi_d}^{(j_2)}, \ldots, L_M \gamma_{\pi\otimes\chi_d}^{(j_n)} \right)$$
$$+ \sum_{\nu=1}^{n} \sum_{\substack{j_\nu = 0 \\ j_k \neq 0, k \neq \nu}}^{*} f\left(L_M \gamma_{\pi\otimes\chi_d}^{(j_1)}, \ldots, L_M \gamma_{\pi\otimes\chi_d}^{(j_{\nu-1})}, 0, L_M \gamma_{\pi\otimes\chi_d}^{(j_{\nu+1})}, \ldots, L_M \gamma_{\pi\otimes\chi_d}^{(j_n)} \right).$$

We only focus on the first sum on the r.h.s. above. The same technique applies to the remaining sums.

By combinatorial sieving and the explicit formula, we find that

$$\sum_{j_1\neq 0,\dots,j_n\neq 0}^{*} f\left(L_M\gamma^{(j_1)}_{\pi\otimes\chi_d}, L_M\gamma^{(j_2)}_{\pi\otimes\chi_d}, \dots, L_M\gamma^{(j_n)}_{\pi\otimes\chi_d}\right)$$
$$= \sum_{\underline{F}} (-2)^{n-\nu(\underline{F})} \prod_{\ell=1}^{\nu(\underline{F})} (|F_\ell|-1)!$$
$$\cdot \left(\int_{\mathbb{R}} F_\ell(x)\,dx - \frac{2}{M\log X} \sum_{m=1}^{\infty} \frac{\Lambda(m)a_\pi(m)}{m^{1/2}} \chi_d(m) \right.$$
$$\left. \cdot \hat{F}_\ell\left(\frac{\log m}{M\log X}\right) - F_\ell(0) + O(1/\log X)\right). \tag{3.68}$$

But, by (3.66),

$$-F_\ell(0) = \lim_{X\to\infty} \frac{4}{M\log X} \sum_p \frac{\Lambda(p^2)a_\pi(p^2)}{p} \hat{F}_\ell\left(\frac{2\log p}{M\log X}\right),$$

and this has the effect, in (3.68), of changing the sign of the contribution from the squares of primes. □

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Department of Mathematics, University of Texas at Austin, Austin, Texas 78705, USA; miker@math.utexas.edu; current: American Institute of Mathematics, 360 Portage Avenue, Palo Alto, California 94306, USA.