From the Manhattan Project to Elliptic Curves: Introduction to Random Matrix Theory

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Introduction

Goals

- Determine correct scale and statistics to study eigenvalues and zeros of L-functions.
- See similar behavior in different systems.
- Discuss the tools and techniques needed to prove the results.

Fundamental Problem: Spacing Between Events

General Formulation: Studying system, observe values at $t_1, t_2, t_3, \dots$.

Question: What rules govern the spacings between the t_i?

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Examples: Spacings between

- Energy Levels of Nuclei.
- Eigenvalues of Matrices.
- Zeros of L-functions.
- Summands in Zeckendorf Decompositions.
- Primes.
- $n^k \alpha \bmod 1$.

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Sketch of proofs

In studying many statistics, often three key steps:

1. Determine correct scale for events.
2. Develop an explicit formula relating what we want to study to something we understand.
3. Use an averaging formula to analyze the quantities above.

It is not always trivial to figure out what is the correct statistic to study!

Eigenvalue Review: I

$z \in \mathbb{C} : z = x + iy$ with $i = \sqrt{-1}$; $\overline{z} = z^H = x - iy$.

Eigenvalue/Eigenvector: $\lambda \in \mathbb{C}, \overrightarrow{v} \neq \overrightarrow{0}$:

$$A\overrightarrow{v} \;=\; \lambda\overrightarrow{v}.$$

Can find by $\det(A - \lambda I) = 0$ but computational nightmare!
Real Symmetric: $A = A^T$; Hermitian: $A = A^H$ (complex conjugate transpose).

Length of $\overrightarrow{v}$ is $\sqrt{\overrightarrow{v}^H\overrightarrow{v}}$; $||\overline{v}||^2 = \overrightarrow{v}^H\overrightarrow{v}$; $\overrightarrow{v} \cdot \overrightarrow{w} = \overrightarrow{v}^H\overrightarrow{w}$.

Eigenvalue Review: II

A real implies eigenvalues real: If $A\vec{v} = \lambda\vec{v}$ then

$$\begin{aligned}
\vec{v}^{\,H} A^H \vec{v} &= \vec{v}^{\,H} A \vec{v} \\
(A\vec{v})^H \vec{v} &= \vec{v}^{\,H} (A\vec{v}) \\
(\lambda\vec{v})^H \vec{v} &= \vec{v}^{\,H} (\lambda\vec{v}) \\
\overline{\lambda}\vec{v}^{\,H}\vec{v} &= \lambda\vec{v}^{\,H}\vec{v} \\
\overline{\lambda}||\vec{v}||^2 &= \lambda||\vec{v}||^2,
\end{aligned}$$

and thus as length is non-zero have $\lambda = \overline{\lambda}$ and is real, and then get coefficients of $\vec{v}$ real.

A complex Hermitian: similar proof shows eigenvalues real (coefficients can be complex).

Eigenvalue Review: III

Orthogonal: $Q^T Q = QQ^T = I$; Unitary: $U^H U = UU^H = I$.

Spectral Theorem: If A is real symmetric or complex Hermitian than can diagonalize (real symmetric: $A = Q^T \Lambda Q$, complex Hermitian $A = U^H \Lambda U$).

Proof: 'Trivial' if distinct eigenvalues as each has an eigenvector, mutually orthogonal, choose unit length and let these be columns of Q:

$$\begin{aligned} \overrightarrow{v_1}^T A^T \overrightarrow{v_2} &= \overrightarrow{v_1}^T A \overrightarrow{v_2} \\ \left(A\overrightarrow{v_1}\right)^T \overrightarrow{v_2} &= \overrightarrow{v_1}^T \left(A\overrightarrow{v_2}\right) \\ \lambda_1 \overrightarrow{v_1}^T \overrightarrow{v_2} &= \lambda_2 \overrightarrow{v_1}^T \overrightarrow{v_2}. \end{aligned}$$

Classical Random Matrix Theory

With Olivia Beckwith, Leo Goldmakher, Chris Hammond, Steven Jackson, Cap Khoury, Murat Koloğlu, Gene Kopp, Victor Luo, Adam Massey, Eve Ninsuwan, Vincent Pham, Karen Shen, Jon Sinsheimer, Fred Strauch, Nicholas Triantafillou, Wentao Xiong

Origins of Random Matrix Theory

Classical Mechanics: 3 Body Problem intractable.

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Classical Mechanics: 3 Body Problem intractable.

Heavy nuclei (Uranium: 200$+$ protons / neutrons) worse!

Get some info by shooting high-energy neutrons into nucleus, see what comes out.

Fundamental Equation:

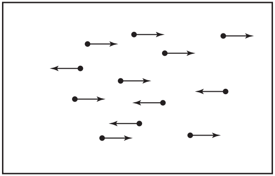
$$H\psi_n = E_n\psi_n$$

H : matrix, entries depend on system

E_n : energy levels

ψ_n : energy eigenfunctions

Origins of Random Matrix Theory



- Statistical Mechanics: for each configuration, calculate quantity (say pressure).
- Average over all configurations – most configurations close to system average.
- Nuclear physics: choose matrix at random, calculate eigenvalues, average over matrices (real Symmetric $A = A^T$, complex Hermitian $\overline{A}^T = A$).

Random Matrix Ensembles

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1N} & a_{2N} & a_{3N} & \cdots & a_{NN} \end{pmatrix} = A^T, \quad a_{ij} = a_{ji}$$

Fix p, define

$$\mathrm{Prob}(A) = \prod_{1 \le i \le j \le N} p(a_{ij}).$$

This means

$$\mathrm{Prob}\left(A : a_{ij} \in [\alpha_{ij}, \beta_{ij}]\right) = \prod_{1 \le i \le j \le N} \int_{x_{ij}=\alpha_{ij}}^{\beta_{ij}} p(x_{ij}) dx_{ij}.$$

Want to understand eigenvalues of A.

Eigenvalue Distribution

$\delta(x - x_0)$ is a unit point mass at x_0:
$\int f(x)\delta(x - x_0)dx = f(x_0)$.

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$$\mu_{A,N}(x) \quad = \quad \frac{1}{N}\sum_{i=1}^{N}\delta\left(x - \frac{\lambda_i(A)}{2\sqrt{N}}\right)$$

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$$\text{k}^{\text{th}} \text{ moment} = \frac{\sum_{i=1}^{N}\lambda_i(A)^k}{2^k N^{\frac{k}{2}+1}} = \frac{\mathrm{Trace}(A^k)}{2^k N^{\frac{k}{2}+1}}.$$

Wigner's Semi-Circle Law

Wigner's Semi-Circle Law

$N \times N$ real symmetric matrices, entries i.i.d.r.v. from a fixed $p(x)$ with mean 0, variance 1, and other moments finite. Then for almost all A, as $N \to \infty$

$$\mu_{A,N}(x) \longrightarrow \begin{cases} \frac{2}{\pi}\sqrt{1-x^2} & \text{if } |x| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

SKETCH OF PROOF: Eigenvalue Trace Lemma

Want to understand the eigenvalues of A, but choose the matrix elements randomly and independently.

Eigenvalue Trace Lemma

Let A be an $N \times N$ matrix with eigenvalues $\lambda_i(A)$. Then

$$\mathrm{Trace}(A^k) \;=\; \sum_{n=1}^{N} \lambda_i(A)^k,$$

where

$$\mathrm{Trace}(A^k) \;=\; \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_N i_1}.$$

SKETCH OF PROOF: Correct Scale

$$\mathrm{Trace}(A^2) \ = \ \sum_{i=1}^{N} \lambda_i(A)^2.$$

By the Central Limit Theorem:

$$\begin{aligned} \mathrm{Trace}(A^2) \ &= \ \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}a_{ji} \ = \ \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^2 \ \sim \ N^2 \\ \sum_{i=1}^{N} \lambda_i(A)^2 \ &\sim \ N^2 \end{aligned}$$

Gives $N\mathrm{Ave}(\lambda_i(A)^2) \ \sim \ N^2$ or $\mathrm{Ave}(\lambda_i(A)) \sim \sqrt{N}$.

SKETCH OF PROOF: Averaging Formula

Recall k-th moment of $\mu_{A,N}(x)$ is $\mathrm{Trace}(A^k)/2^k N^{k/2+1}$.

Average k-th moment is

$$\int \cdots \int \frac{\mathrm{Trace}(A^k)}{2^k N^{k/2+1}} \prod_{i \leq j} p(a_{ij}) da_{ij}.$$

Proof by method of moments: Two steps

- Show average of k-th moments converge to moments of semi-circle as $N \to \infty$;
- Control variance (show it tends to zero as $N \to \infty$).

SKETCH OF PROOF: Averaging Formula for Second Moment

Substituting into expansion gives

$$\frac{1}{2^2 N^2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ji}^2 \cdot p(a_{11}) da_{11} \cdots p(a_{NN}) da_{NN}$$

Integration factors as

$$\int_{a_{ij}=-\infty}^{\infty} a_{ij}^2 p(a_{ij}) da_{ij} \quad \cdot \prod_{\substack{(k,l)\neq(i,j)\\ k<l}} \int_{a_{kl}=-\infty}^{\infty} p(a_{kl}) da_{kl} \; = \; 1.$$

Higher moments involve more advanced combinatorics (Catalan numbers).

SKETCH OF PROOF: Averaging Formula for Higher Moments

Higher moments involve more advanced combinatorics (Catalan numbers).

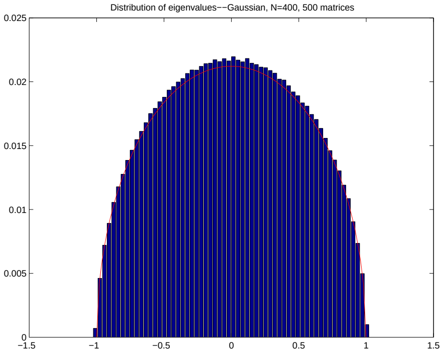
$$\frac{1}{2^k N^{k/2+1}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} a_{i_1 i_2} \cdots a_{i_k i_1} \cdot \prod_{i \leq j} p(a_{ij}) da_{ij}.$$

Main contribution when the $a_{i_\ell i_{\ell+1}}$'s matched in pairs, not all matchings contribute equally (if did would get a Gaussian and not a semi-circle; this is seen in Real Symmetric Palindromic Toeplitz matrices).

Distribution of eigenvalues of real symmetric palindromic Toeplitz matrices and circulant matrices (with Adam Massey and John Sinsheimer), Journal of Theoretical Probability **20** (2007), no. 3, 637–662.

http://arxiv.org/abs/math/0512146

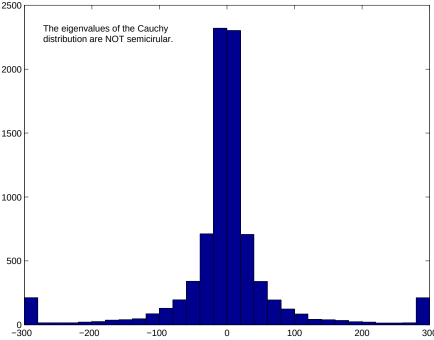
Numerical examples



500 Matrices: Gaussian 400×400

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Numerical examples



Cauchy Distribution: $p(x) = \frac{1}{\pi(1+x^2)}$

I. Zakharevich, *A generalization of Wigner's law*, Comm. Math. Phys. **268** (2006), no. 2, 403–414.

http://web.williams.edu/Mathematics/sjmiller/public_html/book/papers/innaz.pdf

GOE Conjecture

GOE Conjecture:

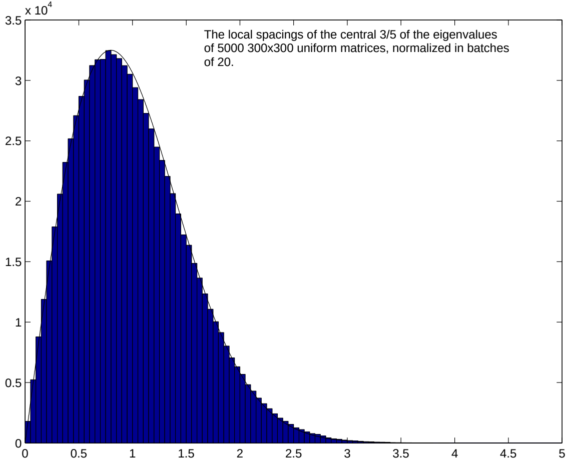
As $N \to \infty$, the probability density of the spacing b/w consecutive normalized eigenvalues approaches a limit independent of p.

Until recently only known if p is a Gaussian.

$$\mathrm{GOE}(x) \ \approx \ \tfrac{\pi}{2} x e^{-\pi x^2/4}.$$

Numerical Experiment: Uniform Distribution

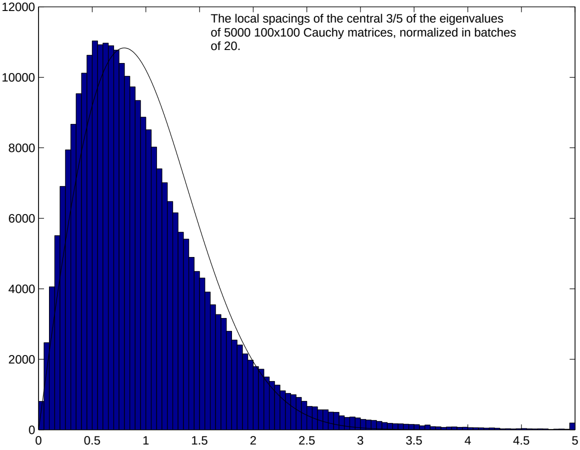
Let $p(x) = \frac{1}{2}$ for $|x| \leq 1$.



5000: 300×300 uniform on $[-1, 1]$

Cauchy Distribution

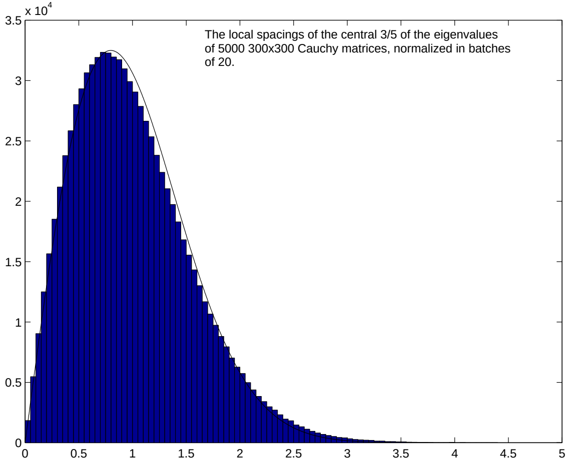
Let $p(x) = \frac{1}{\pi(1+x^2)}$.



5000: 100×100 Cauchy

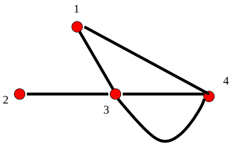
Cauchy Distribution

Let $p(x) = \frac{1}{\pi(1+x^2)}$.



5000: 300×300 Cauchy

Random Graphs



Degree of a vertex $=$ number of edges leaving the vertex.
Adjacency matrix: a_{ij} $=$ number edges b/w Vertex i and Vertex j.

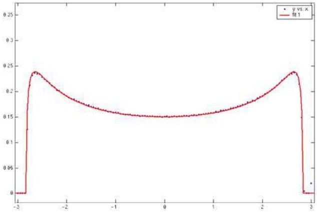
$$A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \end{pmatrix}$$

These are Real Symmetric Matrices.

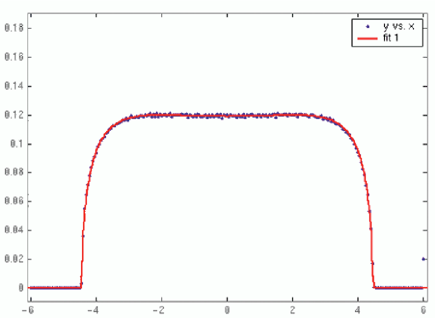
McKay's Law (Kesten Measure) with $d = 3$

Density of Eigenvalues for d-regular graphs

$$f(x) = \begin{cases} \frac{d}{2\pi(d^2 - x^2)}\sqrt{4(d-1) - x^2} & |x| \le 2\sqrt{d-1} \\ 0 & \text{otherwise.} \end{cases}$$



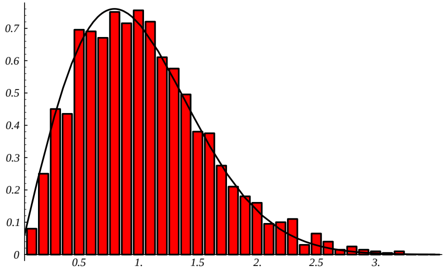
McKay's Law (Kesten Measure) with $d = 6$



Fat Thin: fat enough to average, thin enough to get something different than semi-circle (though as $d \to \infty$ recover semi-circle).

3-Regular Graph with 2000 Vertices: Comparison with the GOE

Spacings between eigenvalues of 3-regular graphs and the GOE:



Real Symmetric Toeplitz Matrices

Chris Hammond and Steven J. Miller

Toeplitz Ensembles

Toeplitz matrix is of the form

$$\begin{pmatrix} b_0 & b_1 & b_2 & \cdots & b_{N-1} \\ b_{-1} & b_0 & b_1 & \cdots & b_{N-2} \\ b_{-2} & b_{-1} & b_0 & \cdots & b_{N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{1-N} & b_{2-N} & b_{3-N} & \cdots & b_0 \end{pmatrix}$$

- Will consider Real Symmetric Toeplitz matrices.
- Main diagonal zero, $N-1$ independent parameters.
- Normalize Eigenvalues by $\sqrt{N}$.

Eigenvalue Density Measure

$$\mu_{A,N}(x)dx \ = \ \frac{1}{N}\sum_{i=1}^{N}\delta\left(x-\frac{\lambda_i(A)}{\sqrt{N}}\right)dx.$$

The k^{th} moment of $\mu_{A,N}(x)$ is

$$M_k(A,N) \ = \ \frac{1}{N^{\frac{k}{2}+1}}\sum_{i=1}^{N}\lambda_i^k(A) \ = \ \frac{\mathrm{Trace}(A^k)}{N^{\frac{k}{2}+1}}.$$

Let

$$M_k \ = \ \lim_{N\to\infty}\mathbb{E}_A\left[M_k(A,N)\right];$$

have $M_2 = 1$ and $M_{2k+1} = 0$.

Even Moments

$$M_{2k}(N) \;=\; \frac{1}{N^{k+1}} \sum_{1 \le i_1, \cdots, i_{2k} \le N} \mathbb{E}(b_{|i_1 - i_2|} b_{|i_2 - i_3|} \cdots b_{|i_{2k} - i_1|}).$$

Main Term: b_j's matched in pairs, say

$$b_{|i_m - i_{m+1}|} \;=\; b_{|i_n - i_{n+1}|}, \quad x_m \;=\; |i_m - i_{m+1}| \;=\; |i_n - i_{n+1}|.$$

Two possibilities:

$$i_m - i_{m+1} \;=\; i_n - i_{n+1} \quad \text{or} \quad i_m - i_{m+1} \;=\; -(i_n - i_{n+1}).$$

$(2k-1)!!$ ways to pair, 2^k choices of sign.

Main Term: All Signs Negative (else lower order contribution)

$$M_{2k}(N) = \frac{1}{N^{k+1}} \sum_{1\le i_1,\cdots,i_{2k}\le N} \mathbb{E}(b_{|i_1-i_2|} b_{|i_2-i_3|} \cdots b_{|i_{2k}-i_1|}).$$

Let $x_1, \dots, x_k$ be the values of the $|i_j - i_{j+1}|$'s, $\epsilon_1, \dots, \epsilon_k$ the choices of sign. Define $\widetilde{x}_1 = i_1 - i_2$, $\widetilde{x}_2 = i_2 - i_3, \dots$.

$$\begin{aligned}
i_2 &= i_1 - \widetilde{x}_1 \\
i_3 &= i_1 - \widetilde{x}_1 - \widetilde{x}_2 \\
&\vdots \\
i_1 &= i_1 - \widetilde{x}_1 - \cdots - \widetilde{x}_{2k} \\
\widetilde{x}_1 + \cdots + \widetilde{x}_{2k} &= \sum_{j=1}^{k} (1+\epsilon_j)\eta_j x_j \ = \ 0, \quad \eta_j \ = \ \pm 1.
\end{aligned}$$

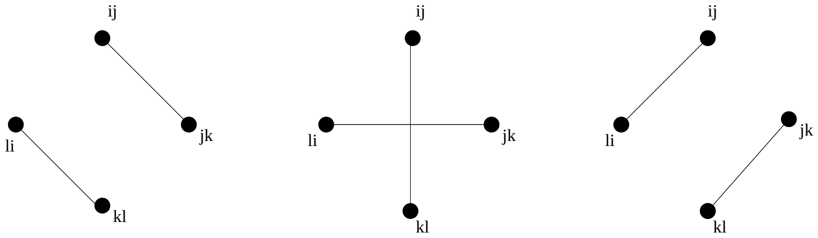
Even Moments: Summary

Main Term: paired, all signs negative.

$$M_{2k}(N) \ \leq \ (2k-1)!! + O_k\left(\frac{1}{N}\right).$$

Bounded by Gaussian.

The Fourth Moment



$$M_4(N) = \frac{1}{N^3} \sum_{1 \le i_1, i_2, i_3, i_4 \le N} \mathbb{E}(b_{|i_1-i_2|} b_{|i_2-i_3|} b_{|i_3-i_4|} b_{|i_4-i_1|})$$

Let $x_j = |i_j - i_{j+1}|$.

The Fourth Moment

Case One: $x_1 = x_2, x_3 = x_4$:

$$i_1 - i_2 \;=\; -(i_2 - i_3) \quad \text{and} \quad i_3 - i_4 \;=\; -(i_4 - i_1).$$

Implies

$$i_1 \;=\; i_3, \quad i_2 \text{ and } i_4 \text{ arbitrary.}$$

Left with $\mathbb{E}[b_{x_1}^2 b_{x_3}^2]$:

$$N^3 - N \text{ times get } 1, \; N \text{ times get } p_4 \;=\; \mathbb{E}[b_{x_1}^4].$$

Contributes 1 in the limit.

The Fourth Moment

$$M_4(N) = \frac{1}{N^3} \sum_{1 \le i_1, i_2, i_3, i_4 \le N} \mathbb{E}(b_{|i_1 - i_2|} b_{|i_2 - i_3|} b_{|i_3 - i_4|} b_{|i_4 - i_1|})$$

Case Two: Diophantine Obstruction: $x_1 = x_3$ and $x_2 = x_4$.

$$i_1 - i_2 \;=\; -(i_3 - i_4) \quad \text{and} \quad i_2 - i_3 \;=\; -(i_4 - i_1).$$

This yields

$$i_1 \;=\; i_2 + i_4 - i_3, \quad i_1, i_2, i_3, i_4 \in \{1, \dots, N\}.$$

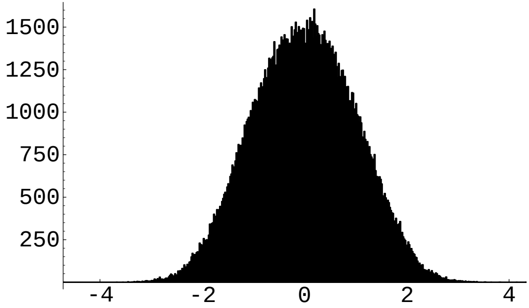
If $i_2, i_4 \geq \frac{2N}{3}$ and $i_3 < \frac{N}{3}$, $i_1 > N$: at most $(1 - \frac{1}{27})N^3$ valid choices.

The Fourth Moment

Theorem: Fourth Moment: Let p_4 be the fourth moment of p. Then

$$M_4(N) = 2\frac{2}{3} + O_{p_4}\left(\frac{1}{N}\right).$$

500 Toeplitz Matrices, 400×400.

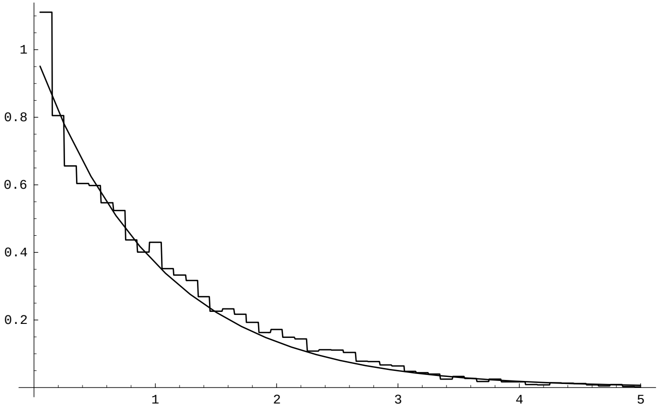


Main Result

Theorem: HM ’05

For real symmetric Toeplitz matrices, the limiting spectral measure converges in probability to a unique measure of unbounded support which is not the Gaussian. If p is even have strong convergence).

Poissonian Behavior?



Not rescaled. Looking at middle 11 spacings, 1000 Toeplitz matrices (1000 $\times$ 1000), entries iidrv from the standard normal.

Real Symmetric Palindromic Toeplitz Matrices

Adam Massey, Steven J. Miller, Jon Sinsheimer

Real Symmetric Palindromic Toeplitz matrices

$$\begin{pmatrix} b_0 & b_1 & b_2 & b_3 & \cdots & b_3 & b_2 & b_1 & b_0 \\ b_1 & b_0 & b_1 & b_2 & \cdots & b_4 & b_3 & b_2 & b_1 \\ b_2 & b_1 & b_0 & b_1 & \cdots & b_5 & b_4 & b_3 & b_2 \\ b_3 & b_2 & b_1 & b_0 & \cdots & b_6 & b_5 & b_4 & b_3 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ b_3 & b_4 & b_5 & b_6 & \cdots & b_0 & b_1 & b_2 & b_3 \\ b_2 & b_3 & b_4 & b_5 & \cdots & b_1 & b_0 & b_1 & b_2 \\ b_1 & b_2 & b_3 & b_4 & \cdots & b_2 & b_1 & b_0 & b_1 \\ b_0 & b_1 & b_2 & b_3 & \cdots & b_3 & b_2 & b_1 & b_0 \end{pmatrix}$$

- Extra symmetry fixes Diophantine Obstructions.
- Always have eigenvalue at 0.

Results

Theorem: MMS '07

For real symmetric palindromic matrices, converge in probability to the Gaussian (if p is even have strong convergence).

Results

Theorem: MMS '07

Let $X_0, \dots, X_{N-1}$ be iidrv (with $X_j = X_{N-j}$) from a distribution p with mean 0, variance 1, and finite higher moments. For $\omega = (x_0, x_1, \dots)$ set $X_\ell(\omega) = x_\ell$, and

$$S_N^{(k)}(\omega) \;=\; \frac{1}{\sqrt{N}} \sum_{\ell=0}^{N-1} X_\ell(\omega) \cos(2\pi k \ell / N).$$

Then as $n \to \infty$

$$\mathrm{Prob}\left(\left\{\omega \in \Omega : \sup_{x \in \mathbb{R}} \left| \frac{1}{N} \sum_{k=0}^{N-1} I_{S_N^{(k)}(\omega) \le x} - \Phi(x) \right| \to 0 \right\}\right) \;=\; 1;$$

I the indicator fn, Φ CDF of standard normal.

Real Symmetric Highly Palindromic Toeplitz Matrices

Steven Jackson, Victor Luo, Steven J. Miller, Vincent Pham, Nicholas George Triantafillou

Notation: Real Symmetric Highly Palindromic Toeplitz matrices

For fixed n, we consider $N \times N$ real symmetric Toeplitz matrices in which the first row is 2^n copies of a palindrome, entries are iidrv from a p with mean 0, variance 1 and finite higher moments.

For instance, a doubly palindromic Toeplitz matrix is of the form:

$$A_N = \begin{pmatrix} b_0 & b_1 & \cdots & b_1 & b_0 & b_0 & b_1 & \cdots & b_1 & b_0 \\ b_1 & b_0 & \cdots & b_2 & b_1 & b_0 & b_0 & \cdots & b_2 & b_1 \\ b_2 & b_1 & \cdots & b_3 & b_2 & b_1 & b_0 & \cdots & b_3 & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_2 & b_3 & \cdots & b_0 & b_1 & b_2 & b_3 & \cdots & b_1 & b_2 \\ b_1 & b_2 & \cdots & b_0 & b_0 & b_1 & b_2 & \cdots & b_0 & b_1 \\ b_0 & b_1 & \cdots & b_1 & b_0 & b_0 & b_1 & \cdots & b_1 & b_0 \end{pmatrix}.$$

Main Results

Theorem: JMP '12

Let n be a fixed positive integer, N a multiple of 2^n, consider the ensemble of real symmetric $N \times N$ palindromic Toeplitz matrices whose first row is 2^n copies of a fixed palindrome (independent entries iidrv from p with mean 0, variance 1 and finite higher moments).

1. As $N \to \infty$ the measures μ_{n,A_N} converge in probability to a limiting spectral measure which is even and has unbounded support.
2. If p is even, then converges almost surely.
3. The limiting measure has fatter tails than the Gaussian (or any previously seen distribution).

Work in Progress (with Victor Luo and Nicholas Triantafillou)

- Highly Palindromic Real Symmetric: all matchings contribute equally for fourth moment, conjectured equally in general.

- Highly Palindromic Hermitian: matchings do not contribute equally: fourth moment non-adjacent case is $\frac{1}{3}(2^n + 2^{-n})$, while the adjacent case is $\frac{1}{2}(2^n + 2^{-n})$.

Block Circulant Ensemble

With Murat Koloğlu, Gene Kopp, Fred Strauch and Wentao Xiong.

The Ensemble of m-Block Circulant Matrices

Symmetric matrices periodic with period m on wrapped diagonals, i.e., symmetric block circulant matrices.

8-by-8 real symmetric 2-block circulant matrix:

$$\left(\begin{array}{cc|cc|cc|cc} c_0 & c_1 & c_2 & c_3 & c_4 & d_3 & c_2 & d_1 \\ c_1 & d_0 & d_1 & d_2 & d_3 & d_4 & c_3 & d_2 \\ \hline c_2 & d_1 & c_0 & c_1 & c_2 & c_3 & c_4 & d_3 \\ c_3 & d_2 & c_1 & d_0 & d_1 & d_2 & d_3 & d_4 \\ \hline c_4 & d_3 & c_2 & d_1 & c_0 & c_1 & c_2 & c_3 \\ d_3 & d_4 & c_3 & d_2 & c_1 & d_0 & d_1 & d_2 \\ \hline c_2 & c_3 & c_4 & d_3 & c_2 & d_1 & c_0 & c_1 \\ d_1 & d_2 & d_3 & d_4 & c_3 & d_2 & c_1 & d_0 \end{array}\right).$$

Choose distinct entries i.i.d.r.v.

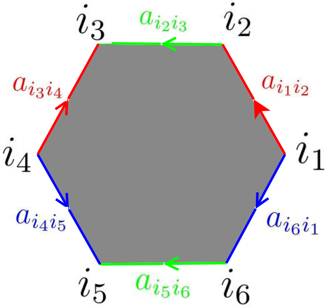
Oriented Matchings and Dualization

Compute moments of eigenvalue distribution (as m stays fixed and $N \to \infty$) using the combinatorics of pairings. Rewrite:

$$\begin{aligned} M_n(N) &= \frac{1}{N^{\frac{n}{2}+1}} \sum_{1 \le i_1, \dots, i_n \le N} \mathbb{E}(a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_n i_1}) \\ &= \frac{1}{N^{\frac{n}{2}+1}} \sum_{\sim} \eta(\sim) m_{d_1(\sim)} \cdots m_{d_l(\sim)}. \end{aligned}$$

where the sum is over oriented matchings on the edges $\{(1,2),(2,3),\dots,(n,1)\}$ of a regular n-gon.

Oriented Matchings and Dualization

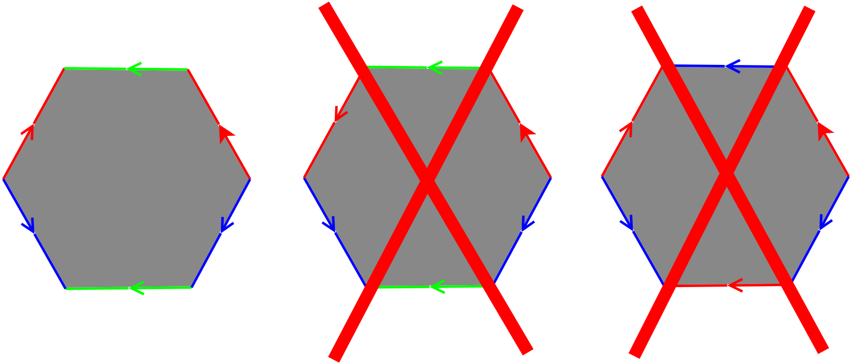


$$\left(\begin{array}{cc|cc|cc|cc}
c_0 & {\color{red}c_1} & c_2 & c_3 & c_4 & d_3 & c_2 & d_1 \\
c_1 & d_0 & d_1 & {\color{green}d_2} & d_3 & d_4 & c_3 & d_2 \\
\hline
c_2 & d_1 & c_0 & c_1 & c_2 & c_3 & c_4 & {\color{blue}d_3} \\
c_3 & d_2 & {\color{red}c_1} & d_0 & d_1 & d_2 & d_3 & d_4 \\
\hline
c_4 & d_3 & c_2 & d_1 & c_0 & c_1 & c_2 & c_3 \\
{\color{blue}d_3} & d_4 & c_3 & d_2 & c_1 & d_0 & d_1 & d_2 \\
\hline
c_2 & c_3 & c_4 & d_3 & c_2 & d_1 & c_0 & c_1 \\
d_1 & d_2 & d_3 & d_4 & c_3 & {\color{green}d_2} & c_1 & d_0
\end{array}\right)$$

Figure: An oriented matching in the expansion for $M_n(N) = M_6(8)$.

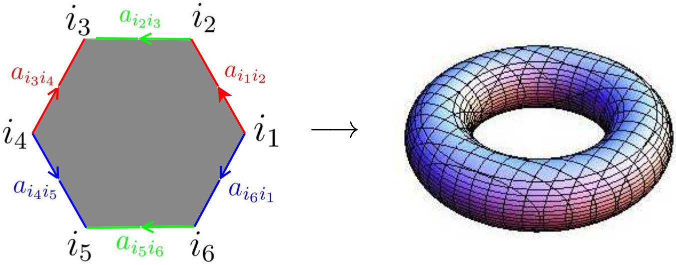
Contributing Terms

As $N \to \infty$, the only terms that contribute to this sum are those in which the entries are matched in pairs and with opposite orientation.



Only Topology Matters

Think of pairings as topological identifications; the contributing ones give rise to orientable surfaces.



Contribution from such a pairing is m^{-2g}, where g is the genus (number of holes) of the surface. Proof: combinatorial argument involving Euler characteristic.

Computing the Even Moments

Theorem: Even Moment Formula

$$M_{2k} = \sum_{g=0}^{\lfloor k/2 \rfloor} \varepsilon_g(k) m^{-2g} + O_k\left(\frac{1}{N}\right),$$

with $\varepsilon_g(k)$ the number of pairings of the edges of a $(2k)$-gon giving rise to a genus g surface.

J. Harer and D. Zagier (1986) gave generating functions for the $\varepsilon_g(k)$.

Harer and Zagier

$$\sum_{g=0}^{\lfloor k/2 \rfloor} \varepsilon_g(k) r^{k+1-2g} \;=\; (2k-1)!!\, c(k,r)$$

where

$$1 + 2\sum_{k=0}^{\infty} c(k,r) x^{k+1} \;=\; \left(\frac{1+x}{1-x}\right)^r .$$

Thus, we write

$$M_{2k} \;=\; m^{-(k+1)} (2k-1)!!\, c(k,m).$$

A multiplicative convolution and Cauchy's residue formula yield the characteristic function of the distribution.

$$\begin{aligned}
\phi(t) &= \sum_{k=0}^{\infty} \frac{(it)^{2k} M_{2k}}{(2k)!} = \frac{1}{m} \sum_{k=0}^{\infty} \frac{(-t^2/2m)^k}{k!} c(k,m) \\
&= \frac{1}{2\pi i m} \oint_{|z|=2} \frac{1}{2z^{-1}} \left(\left(\frac{1+z^{-1}}{1-z^{-1}} \right)^m - 1 \right) e^{-t^2 z/2m} \frac{dz}{z} \\
&= \frac{1}{m} e^{\frac{-t^2}{2m}} \sum_{\ell=1}^{m} \binom{m}{\ell} \frac{1}{(\ell-1)!} \left(\frac{-t^2}{m} \right)^{\ell-1}.
\end{aligned}$$

Results

Fourier transform and algebra yields

Theorem: Koloğlu, Kopp and Miller

The limiting spectral density function $f_m(x)$ of the real symmetric m-block circulant ensemble is given by the formula

$$f_m(x) = \frac{e^{-\frac{mx^2}{2}}}{\sqrt{2\pi m}} \sum_{r=0}^{m} \frac{1}{(2r)!} \sum_{s=0}^{m-r} \binom{m}{r+s+1} \frac{(2r+2s)!}{(r+s)!s!} \left(-\frac{1}{2}\right)^s (mx^2)^r.$$

As $m \to \infty$, the limiting spectral densities approach the semicircle distribution.

Results (continued)

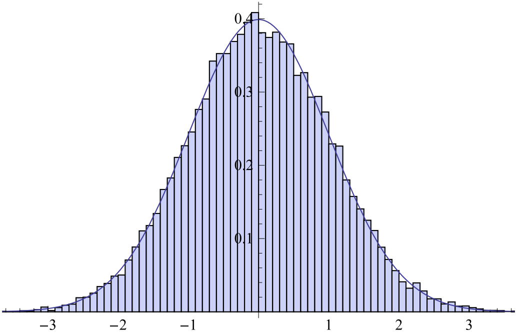


Figure: Plot for f_1 and histogram of eigenvalues of 100 circulant matrices of size 400×400.

Results (continued)

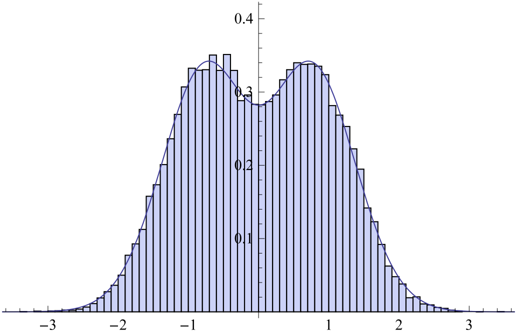


Figure: Plot for f_2 and histogram of eigenvalues of 100 2-block circulant matrices of size 400×400.

Results (continued)

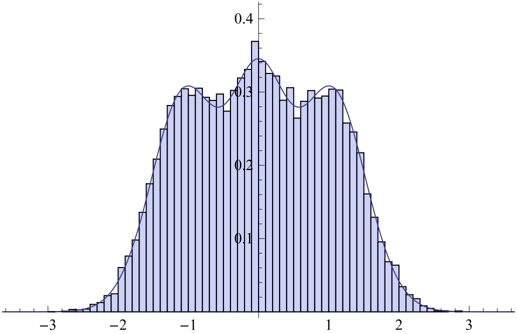


Figure: Plot for f_3 and histogram of eigenvalues of 100 3-block circulant matrices of size 402×402.

Results (continued)

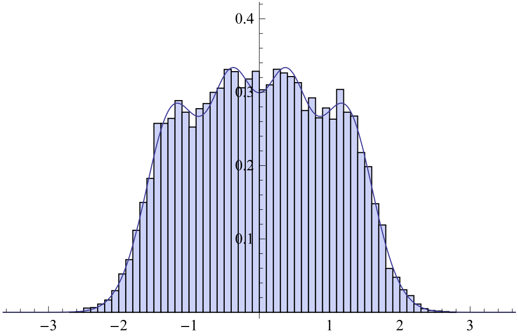


Figure: Plot for f_4 and histogram of eigenvalues of 100 4-block circulant matrices of size 400×400.

Results (continued)

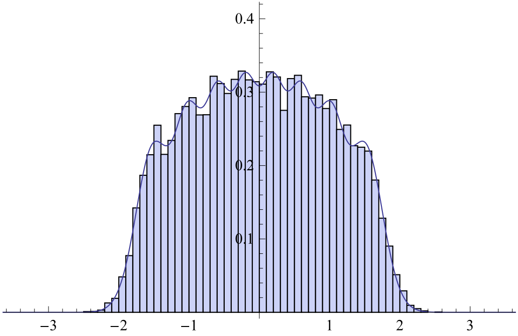


Figure: Plot for f_8 and histogram of eigenvalues of 100 8-block circulant matrices of size 400×400.

Results (continued)

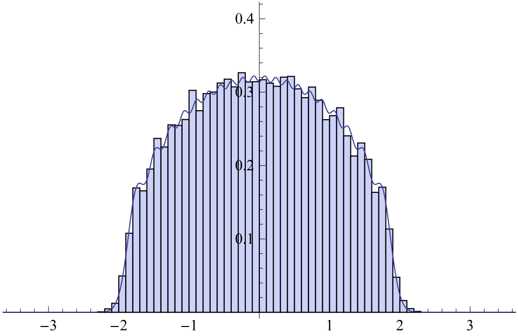


Figure: Plot for f_{20} and histogram of eigenvalues of 100 20-block circulant matrices of size 400×400.

Results (continued)

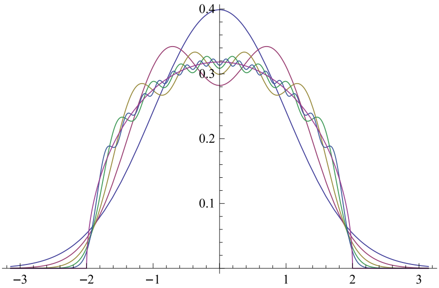


Figure: Plot of convergence to the semi-circle.

The Limiting Spectral Measure for Ensembles of Symmetric Block Circulant Matrices (with Murat Kologlu, Gene S. Kopp, Frederick W. Strauch and Wentao Xiong), Journal of Theoretical Probability **26** (2013), no. 4, 1020–1060. http://arxiv.org/abs/1008.4812

Weighted Real Symmetric Toeplitz Matrices

Olivia Beckwith, Steven J. Miller and Karen Shen

New Ensemble: Signed Toeplitz and Palindromic Toeplitz Matrices

For each entry, multiply by a randomly chosen $\epsilon_{ij} = \{1, -1\}$ with $p = \mathbb{P}(\epsilon_{ij} = 1)$ such that $\epsilon_{ij} = \epsilon_{ji}$.

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Varying p allows us to *continuously* interpolate between:

- Real Symmetric at $p = \frac{1}{2}$ (less structured)
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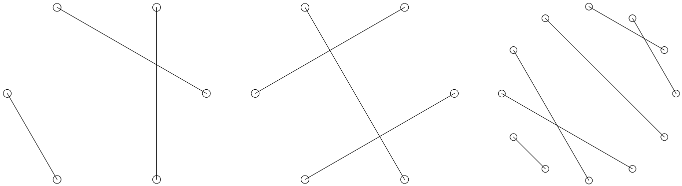
What is the eigenvalue distribution of these signed ensembles?

Weighted Contributions

Theorem:

Each configuration weighted by $(2p-1)^{2m}$, where $2m$ is the number of points on the circle whose edge crosses another edge.

Example:



2m = 4 2m=6 2m=8

Proof of Weighted Contributions Theorem

We compute the average k^{th} moment to be:

$$\frac{1}{N^{\frac{k}{2}+1}} \sum_{1 \le i_1, \dots, i_k \le N} \mathbb{E}\left(\epsilon_{i_1 i_2} b_{|i_1 - i_2|} \epsilon_{i_2 i_3} b_{|i_2 - i_3|} \cdots \epsilon_{i_k i_1} b_{|i_k - i_1|}\right)$$

where the b's are matched in pairs.

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Can show two ϵ's are matched if and only if their b's are not in a crossing.

Counting Crossing Configurations

Problem: Out of the $(2k-1)!!$ ways to pair $2k$ vertices, how many will have $2m$ vertices crossing ($Cross_{2k,2m}$)?

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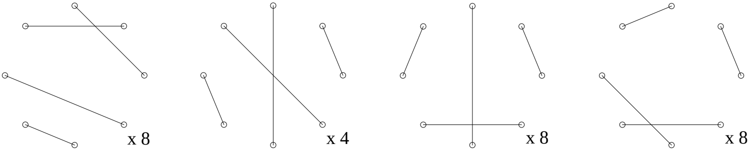
Example: $Cross_{8,4} = 28$



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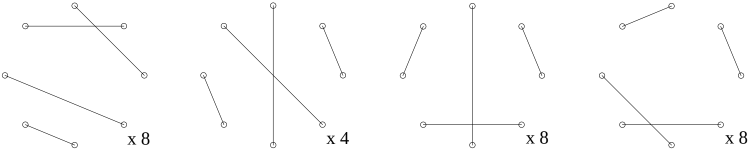

Fact:

$Cross_{2k,0} = C_k$, the k^{th} Catalan number.

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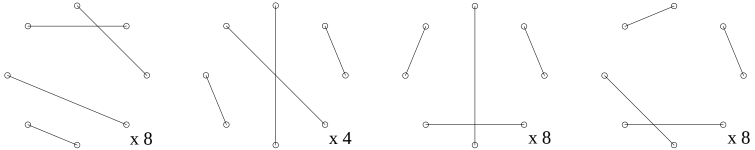

Fact:

$Cross_{2k,0} = C_k$, the k^{th} Catalan number.

What about for higher m?

Counting Crossing Configurations

To calculate $Cross_{2k,2m}$, we write it as the following sum:

$$Cross_{2k,2m} = \sum_{p=1}^{\lfloor \frac{m}{4} \rfloor} P_{2k,2m,p}.$$

where $P_{2k,2m,p}$ is the number of configurations of $2k$ vertices with $2m$ vertices crossing in p partitions.

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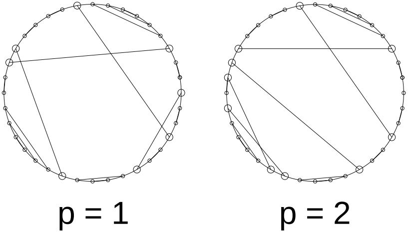
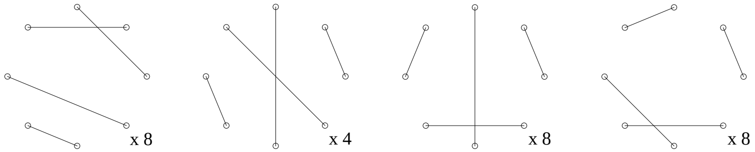


Non-Crossing Regions

Theorem:

If $2m$ vertices are already paired, the number of ways to pair and place the remaining $2k - 2m$ vertices as non-crossing non-partitioning edges is $\binom{2k}{k-m}$.

Example: $\binom{8}{2} = 28$ pairings with 4 crossing vertices.



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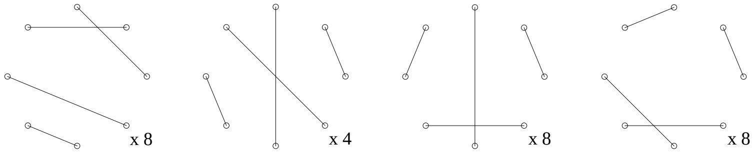


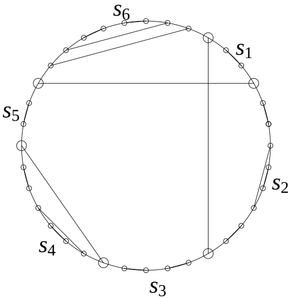
Lemma:

$P_{2k,2m,1} = Cross_{2m,2m}\binom{2k}{k-m}$.

Proof of Non-Crossing Regions Theorem

We showed the following equivalence:

$$\sum_{s_1+s_2+\cdots+s_{2m}=2k-2m} C_{s_1} C_{s_2} \cdots C_{s_{2m}} = \binom{2k}{k-m}.$$



Summary of Results

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- $p = \frac{1}{2}$: Semicircle Distribution (Bounded Support)
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  - Weight of each configuration as a function of p and the number of vertices in a crossing ($2m$): $(2p-1)^{2m}$
  - A way to count the number of configurations with $2m$ vertices crossing for small m
- Tight bounds on the moments in the limit
  - The expected number of vertices involved in a crossing is

$$\frac{2k}{2k-1}\left(2k-2-\frac{{}_2F_1(1,\frac{3}{2},\frac{5}{2}-k;-1)}{2k-3}-(2k-1)\,{}_2F_1\left(1,\frac{1}{2}+k,\frac{3}{2};-1\right)\right),$$

    which is $2k-2-\frac{2}{k}+O\left(\frac{1}{k^2}\right)$ as $k\to\infty$.
  - The variance tends to 4 as $k\to\infty$.

Introduction to L-Functions

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

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Unique Factorization: $n = p_1^{r_1} \cdots p_m^{r_m}$.

$$\prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \;=\; \left[1 + \frac{1}{2^s} + \left(\frac{1}{2^s}\right)^2 + \cdots\right] \left[1 + \frac{1}{3^s} + \left(\frac{1}{3^s}\right)^2 + \cdots\right] \cdots$$

$$= \sum_n \frac{1}{n^s}.$$

Riemann Zeta Function (cont)

$$\begin{aligned}\zeta(s) &= \sum_n \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1 \\ \pi(x) &= \#\{p : p \text{ is prime}, p \leq x\}\end{aligned}$$

Properties of $\zeta(s)$ and Primes:

Riemann Zeta Function (cont)

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Properties of $\zeta(s)$ and Primes:

- $\lim_{s \to 1^+} \zeta(s) = \infty$, $\pi(x) \to \infty$.

Riemann Zeta Function (cont)

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Properties of $\zeta(s)$ and Primes:

- $\lim_{s \to 1^+} \zeta(s) = \infty$, $\pi(x) \to \infty$.
- $\zeta(2) = \frac{\pi^2}{6}$, $\pi(x) \to \infty$.

Riemann Zeta Function

$$\zeta(s) \;=\; \sum_{n=1}^{\infty} \frac{1}{n^s} \;=\; \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Functional Equation:

$$\xi(s) \;=\; \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) \;=\; \xi(1-s).$$

Riemann Hypothesis (RH):

All non-trivial zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2}+i\gamma$.

Observation: Spacings b/w zeros appear same as b/w eigenvalues of Complex Hermitian matrices $\overline{A}^T = A$.

General L-functions

$$L(s,f) \;=\; \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} \;=\; \prod_{p \text{ prime}} L_p(s,f)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Functional Equation:

$$\Lambda(s,f) \;=\; \Lambda_\infty(s,f) L(s,f) \;=\; \Lambda(1-s,f).$$

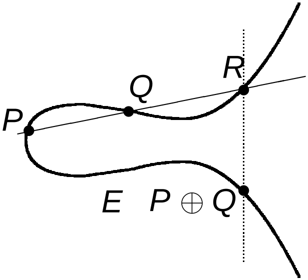
Generalized Riemann Hypothesis (RH):

All non-trivial zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

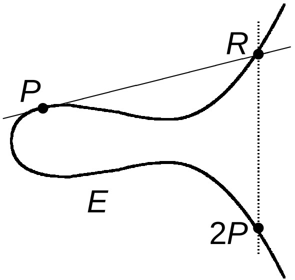
Observation: Spacings b/w zeros appear same as b/w eigenvalues of Complex Hermitian matrices $\overline{A}^T = A$.

Elliptic Curves: Mordell-Weil Group

Elliptic curve $y^2 = x^3 + ax + b$ with rational solutions $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ and connecting line $y = mx + b$.



Addition of distinct points P and Q



Adding a point P to itself

$$E(\mathbb{Q}) \approx E(\mathbb{Q})_{\mathrm{tors}} \oplus \mathbb{Z}^r$$

Elliptic curve L-function

$E : y^2 = x^3 + ax + b$, associate L-function

$$L(s,E) = \sum_{n=1}^{\infty} \frac{a_E(n)}{n^s} = \prod_{p \text{ prime}} L_E(p^{-s}),$$

where

$$a_E(p) = p - \#\{(x,y) \in (\mathbb{Z}/p\mathbb{Z})^2 : y^2 \equiv x^3 + ax + b \bmod p\}.$$

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Birch and Swinnerton-Dyer Conjecture

Rank of group of rational solutions equals order of vanishing of $L(s,E)$ at $s = 1/2$.

Properties of zeros of *L*-functions

- infinitude of primes, primes in arithmetic progression.
- Chebyshev's bias: $\pi_{3,4}(x) \geq \pi_{1,4}(x)$ 'most' of the time.
- Birch and Swinnerton-Dyer conjecture.
- Goldfeld, Gross-Zagier: bound for $h(D)$ from L-functions with many central point zeros.
- Even better estimates for $h(D)$ if a positive percentage of zeros of $\zeta(s)$ are at most $1/2 - \epsilon$ of the average spacing to the next zero.

Distribution of zeros

- $\zeta(s) \neq 0$ for $\mathfrak{Re}(s) = 1$: $\pi(x)$, $\pi_{a,q}(x)$.
- GRH: error terms.
- GSH: Chebyshev's bias.
- Analytic rank, adjacent spacings: $h(D)$.

Explicit Formula (Contour Integration)

$$-\frac{\zeta'(s)}{\zeta(s)} \;=\; -\frac{d}{ds}\log\zeta(s) \;=\; -\frac{d}{ds}\log\prod_p\left(1-p^{-s}\right)^{-1}$$

Explicit Formula (Contour Integration)

$$
\begin{aligned}
-\frac{\zeta'(s)}{\zeta(s)} &= -\frac{\mathrm{d}}{\mathrm{d}s}\log\zeta(s) \;=\; -\frac{\mathrm{d}}{\mathrm{d}s}\log\prod_p\left(1-p^{-s}\right)^{-1}\\
&= \frac{\mathrm{d}}{\mathrm{d}s}\sum_p\log\left(1-p^{-s}\right)\\
&= \sum_p\frac{\log p\cdot p^{-s}}{1-p^{-s}} \;=\; \sum_p\frac{\log p}{p^s} + \mathrm{Good}(s).
\end{aligned}
$$

Explicit Formula (Contour Integration)

$$-\frac{\zeta'(s)}{\zeta(s)} = -\frac{\mathrm{d}}{\mathrm{d}s}\log\zeta(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\log\prod_p\left(1-p^{-s}\right)^{-1}$$

$$= \frac{\mathrm{d}}{\mathrm{d}s}\sum_p \log\left(1-p^{-s}\right)$$

$$= \sum_p \frac{\log p \cdot p^{-s}}{1-p^{-s}} = \sum_p \frac{\log p}{p^s} + \mathrm{Good}(s).$$

Contour Integration:

$$\int -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\,ds \quad \text{vs} \quad \sum_p \log p \int \left(\frac{x}{p}\right)^s \frac{ds}{s}.$$

Explicit Formula (Contour Integration)

$$-\frac{\zeta'(s)}{\zeta(s)} = -\frac{d}{ds}\log\zeta(s) = -\frac{d}{ds}\log\prod_p\left(1-p^{-s}\right)^{-1}$$

$$= \frac{d}{ds}\sum_p \log\left(1-p^{-s}\right)$$

$$= \sum_p \frac{\log p \cdot p^{-s}}{1-p^{-s}} = \sum_p \frac{\log p}{p^s} + \mathrm{Good}(s).$$

Contour Integration:

$$\int -\frac{\zeta'(s)}{\zeta(s)}\phi(s)ds \quad \text{vs} \quad \sum_p \log p \int \phi(s)p^{-s}ds.$$

Explicit Formula (Contour Integration)

$$
\begin{aligned}
-\frac{\zeta'(s)}{\zeta(s)} &= -\frac{\mathrm{d}}{\mathrm{d}s}\log\zeta(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\log\prod_p\left(1-p^{-s}\right)^{-1} \\
&= \frac{\mathrm{d}}{\mathrm{d}s}\sum_p \log\left(1-p^{-s}\right) \\
&= \sum_p \frac{\log p \cdot p^{-s}}{1-p^{-s}} = \sum_p \frac{\log p}{p^s} + \mathrm{Good}(s).
\end{aligned}
$$

Contour Integration (see Fourier Transform arising):

$$
\int -\frac{\zeta'(s)}{\zeta(s)}\,\phi(s)ds \quad \text{vs} \quad \sum_p \log p \int \phi(s) e^{-\sigma\log p} e^{-it\log p} ds.
$$

Knowledge of zeros gives info on coefficients.

Explicit Formula: Example

Dirichlet L-functions: Let ϕ be an even Schwartz function and $L(s,\chi) = \sum_n \chi(n)/n^s$ a Dirichlet L-function from a non-trivial character χ with conductor m and zeros $\rho = \frac{1}{2} + i\gamma_\chi$. Then

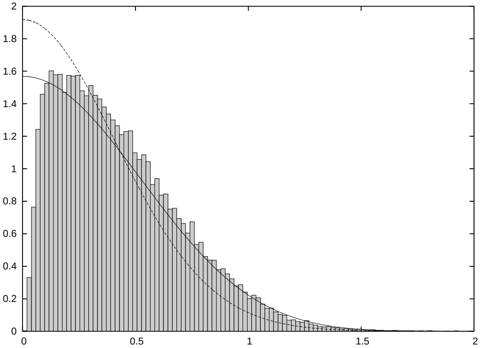
$$\begin{aligned}\sum_\rho \phi\left(\gamma_\chi \frac{\log(m/\pi)}{2\pi}\right) &= \int_{-\infty}^{\infty} \phi(y)dy \\ &\quad -2\sum_p \frac{\log p}{\log(m/\pi)} \widehat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right) \frac{\chi(p)}{p^{1/2}} \\ &\quad -2\sum_p \frac{\log p}{\log(m/\pi)} \widehat{\phi}\left(2\frac{\log p}{\log(m/\pi)}\right) \frac{\chi^2(p)}{p} + O\left(\frac{1}{\log m}\right).\end{aligned}$$

Takeaways

Very similar to Central Limit Theorem.

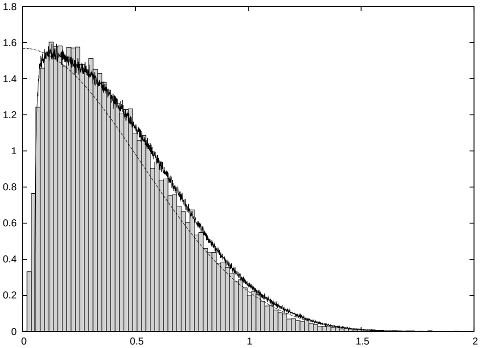
- Universal behavior: main term controlled by first two moments of Satake parameters, agrees with RMT.
- First moment zero save for families of elliptic curves.
- Higher moments control convergence and can depend on arithmetic of family.

Modeling lowest zero of $L_{E_{11}}(s, \chi_d)$ with $0 < d < 400,000$



Lowest zero for $L_{E_{11}}(s, \chi_d)$ (bar chart), lowest eigenvalue of SO(2N) with N_{eff} (solid), standard N_0 (dashed).

Modeling lowest zero of $L_{E_{11}}(s, \chi_d)$ with $0 < d < 400,000$



Lowest zero for $L_{E_{11}}(s, \chi_d)$ (bar chart); lowest eigenvalue of SO(2N): $N_{\rm eff}$ = 2 (solid) with discretisation, and $N_{\rm eff} = 2.32$ (dashed) without discretisation.

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Open Questions and References

Open Questions: Low-lying zeros of *L*-functions

- Generalize excised ensembles for higher weight GL_2 families where expect different discretizations.

- Obtain better estimates on vanishing at the central point by finding optimal test functions for the second and higher moment expansions.

- Further explore *L*-function Ratios Conjecture to predict lower order terms in families, compute these terms on number theory side.

See Dueñez-Huynh-Keating-Miller-Snaith, Miller, and the Ratios papers.

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2. *Variation in the number of points on elliptic curves and applications to excess rank*, C. R. Math. Rep. Acad. Sci. Canada **27** (2005), no. 4, 111–120. http://arxiv.org/abs/math/0506461

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7. *Models for zeros at the central point in families of elliptic curves* (with Eduardo Dueñez, Duc Khiem Huynh, Jon Keating and Nina Snaith), J. Phys. A: Math. Theor. **45** (2012) 115207 (32pp). http://arxiv.org/pdf/1107.4426

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2. *An orthogonal test of the L-Functions Ratios Conjecture*, Proceedings of the London Mathematical Society 2009, doi:10.1112/plms/pdp009. http://arxiv.org/abs/0805.4208
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5. *An elliptic curve family test of the Ratios Conjecture* (with Duc Khiem Huynh and Ralph Morrison), Journal of Number Theory **131** (2011), 1117–1147. http://arxiv.org/abs/1011.3298
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