Low-lying zeros of GL(2) L-functions

Steven J Miller
Williams College

Steven.J.Miller@williams.edu
http://www.williams.edu/Mathematics/sjmiller/

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Compound Families
Dueñez-Miller

Identifying the Symmetry Groups

- Often an analysis of the monodromy group in the function field case suggests the answer.
- All simple families studied to date are built from GL_1 or GL_2 L-functions.
- Tools: Explicit Formula, Orthogonality of Characters / Petersson Formula.
- How to identify symmetry group in general? One possibility is by the signs of the functional equation:
- **Folklore Conjecture:** If all signs are even and no corresponding family with odd signs, Symplectic symmetry; otherwise SO(even). (False!)

Explicit Formula

- π: cuspidal automorphic representation on GL_n.
- $Q_\pi > 0$: analytic conductor of $L(s,\pi) = \sum \lambda_\pi(n)/n^s$.
- By GRH the non-trivial zeros are $\frac{1}{2} + i\gamma_{\pi,j}$.
- Satake parameters $\{\alpha_{\pi,i}(p)\}_{i=1}^n$;
  $\lambda_\pi(p^\nu) = \sum_{i=1}^n \alpha_{\pi,i}(p)^\nu$.
- $L(s,\pi) = \sum_n \frac{\lambda_\pi(n)}{n^s} = \prod_p \prod_{i=1}^n (1 - \alpha_{\pi,i}(p)p^{-s})^{-1}$.

$$\sum_j g\left(\gamma_{\pi,j}\frac{\log Q_\pi}{2\pi}\right) = \widehat{g}(0) - 2\sum_{p,\nu}\widehat{g}\left(\frac{\nu \log p}{\log Q_\pi}\right)\frac{\lambda_\pi(p^\nu)\log p}{p^{\nu/2}\log Q_\pi}$$

Some Results: Rankin-Selberg Convolution of Families

Symmetry constant: $c_{\mathcal{L}} = 0$ (resp, 1 or -1) if family $\mathcal{L}$ has unitary (resp, symplectic or orthogonal) symmetry.

Rankin-Selberg convolution: Satake parameters for $\pi_{1,p} \times \pi_{2,p}$ are

$$\{\alpha_{\pi_1 \times \pi_2}(k)\}_{k=1}^{nm} \; = \; \{\alpha_{\pi_1}(i) \cdot \alpha_{\pi_2}(j)\}_{\substack{1 \le i \le n \\ 1 \le j \le m}}.$$

Theorem (Dueñez-Miller)

If $\mathcal{F}$ and $\mathcal{G}$ are *nice* families of L-functions, then $c_{\mathcal{F} \times \mathcal{G}} = c_{\mathcal{F}} \cdot c_{\mathcal{G}}$.

1-Level Density

Assuming conductors constant in family $\mathcal{F}$, have to study

$$\begin{aligned}
\lambda_f(p^\nu) &= \alpha_{f,1}(p)^\nu + \cdots + \alpha_{f,n}(p)^\nu \\
S_1(\mathcal{F}) &= -2\sum_p \hat{g}\left(\frac{\log p}{\log R}\right)\frac{\log p}{\sqrt{p}\log R}\left[\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\lambda_f(p)\right] \\
S_2(\mathcal{F}) &= -2\sum_p \hat{g}\left(2\frac{\log p}{\log R}\right)\frac{\log p}{p\log R}\left[\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\lambda_f(p^2)\right]
\end{aligned}$$

The corresponding classical compact group is determined by

$$\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\lambda_f(p^2) = c_{\mathcal{F}} = \begin{cases} 0 & \text{Unitary} \\ 1 & \text{Symplectic} \\ -1 & \text{Orthogonal.} \end{cases}$$

1-Level Density for Rankin-Selberg Convolution of Families

Families $\mathcal{F}$ and $\mathcal{G}$.
Satake parameters $\{\alpha_{f,i}(p)\}_{i=1}^{n}$ and $\{\beta_{g,j}(p)\}_{j=1}^{m}$.
Family $\mathcal{F} \times \mathcal{G}$, $L(s, f \times g)$ has parameters $\{\alpha_{f,i}(p)\beta_{g,j}(p)\}_{i=1\dots n, j=1\dots m}$.

$$\begin{aligned} a_{f\times g}(p^\nu) &= \sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_{f,i}(p)^\nu \beta_{g,j}(p)^\nu \\ &= \sum_{i=1}^{n} \alpha_{f,i}(p)^\nu \sum_{j=1}^{m} \beta_{g,j}(p)^\nu \\ &= \lambda_f(p^\nu) \cdot \lambda_g(p^\nu). \end{aligned}$$

Technical restriction: need f and g unrelated (i.e., g is not the contragredient of f) for our applications.

1-Level Density for Rankin-Selberg Convolution of Families (cont)

To analyze $S_\nu(\mathcal{F} \times \mathcal{G})$ we must study

$$\frac{1}{|\mathcal{F} \times \mathcal{G}|} \sum_{f \times g \in \mathcal{F} \times \mathcal{G}} \lambda_f(p^\nu) \cdot \lambda_g(p^\nu) = \left[\frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} \lambda_f(p^\nu)\right] \cdot \left[\frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} \lambda_g(p^\nu)\right]$$

- $\nu = 1$: If one of the families is rank zero, so is $\mathcal{F} \times \mathcal{G}$; $S_1(\mathcal{F} \times \mathcal{G})$ will not contribute.
- $\nu = 2$: $c_{\mathcal{F} \times \mathcal{G}} \ = \ c_{\mathcal{F}} \cdot c_{\mathcal{G}}$.

Proves if each family is of rank 0, the symmetry type of the convolution is the product of the symmetry types. □

Cuspidal Newforms
Hughes-Miller

Results from Iwaniec-Luo-Sarnak

- Orthogonal: Iwaniec-Luo-Sarnak: 1-level density for holomorphic even weight k cuspidal newforms of square-free level N (SO(even) and SO(odd) if split by sign) in $(-2, 2)$.

- Symplectic: Iwaniec-Luo-Sarnak: 1-level density for $\text{sym}^2(f)$, f holomorphic cuspidal newform.

Will review Orthogonal case and talk about extensions (joint with Chris Hughes).

Modular Form Preliminaries

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{array}{c} ad - bc = 1 \\ c \equiv 0(N) \end{array} \right\}$$

f is a weight k holomorphic cuspform of level N if

$$\forall \gamma \in \Gamma_0(N), \ \ f(\gamma z) = (cz+d)^k f(z).$$

- Fourier Expansion: $f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i z}$, $L(s,f) = \sum_{n=1}^{\infty} a_n n^{-s}$.
- Petersson Norm: $\langle f, g \rangle = \int_{\Gamma_0(N)\backslash \mathbf{H}} f(z)\overline{g(z)} y^{k-2} dxdy$.
- Normalized coefficients:

$$\psi_f(n) = \sqrt{\frac{\Gamma(k-1)}{(4\pi n)^{k-1}}} \frac{1}{||f||} a_f(n).$$

Modular Form Preliminaries: Petersson Formula

$B_k(N)$ an orthonormal basis for weight k level N. Define

$$\Delta_{k,N}(m,n) \;=\; \sum_{f \in B_k(N)} \psi_f(m)\overline{\psi}_f(n).$$

Petersson Formula

$$\Delta_{k,N}(m,n) \;=\; 2\pi i^k \sum_{c \equiv 0(N)} \frac{S(m,n,c)}{c} J_{k-1}\left(4\pi \frac{\sqrt{mn}}{c}\right) + \delta(m,n).$$

Modular Form Preliminaries: Explicit Formula

Let $\mathcal{F}$ be a family of cupsidal newforms (say weight k, prime level N and possibly split by sign) $L(s,f) = \sum_n \lambda_f(n)/n^s$. Then

$$\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\sum_{\gamma_f}\phi\left(\frac{\log R}{2\pi}\gamma_f\right) = \widehat{\phi}(0)+\frac{1}{2}\phi(0)-\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}P(f;\phi) + O\left(\frac{\log\log R}{\log R}\right)$$

$$P(f;\phi) = \sum_{p\nmid N}\lambda_f(p)\widehat{\phi}\left(\frac{\log p}{\log R}\right)\frac{2\log p}{\sqrt{p}\log R}.$$

Modular Form Preliminaries:Fourier Coefficient Review

$$\begin{aligned}\lambda_f(n) &= a_f(n)n^{\frac{k-1}{2}}\\ \lambda_f(m)\lambda_f(n) &= \sum_{\substack{d|(m,n)\\(d,M)=1}} \lambda_f\left(\frac{mn}{d}\right).\end{aligned}$$

For a newform of level N, $\lambda_f(N)$ is trivially related to the sign of the form:

$$\epsilon_f = i^k \mu(N)\lambda_f(N)\sqrt{N}.$$

The above will allow us to split into even and odd families: $1 \pm \epsilon_f$.

Key Kloosterman-Bessel integral from ILS

Ramanujan sum:

$$R(n,q) = \sum_{a \bmod q}^{*} e(an/q) = \sum_{d|(n,q)} \mu(q/d)d,$$

where $*$ restricts the summation to be over all a relatively prime to q.

Theorem (ILS)

Let Ψ be an even Schwartz function with $\mathrm{supp}(\widehat{\Psi}) \subset (-2,2)$. Then

$$\sum_{m \le N^{\epsilon}} \frac{1}{m^2} \sum_{(b,N)=1} \frac{R(m^2,b)R(1,b)}{\varphi(b)} \int_{y=0}^{\infty} J_{k-1}(y) \widehat{\Psi}\left(\frac{2\log(by\sqrt{N}/4\pi m)}{\log R}\right) \frac{\mathrm{d}y}{\log R}$$
$$= -\frac{1}{2}\left[\int_{-\infty}^{\infty} \Psi(x)\frac{\sin 2\pi x}{2\pi x}\mathrm{d}x - \frac{1}{2}\Psi(0)\right] + O\left(\frac{k\log\log kN}{\log kN}\right),$$

where $R = k^2 N$ and φ is Euler's totient function.

Limited Support ($\sigma < 1$): Sketch of proof

- Estimate Kloosterman-Bessel terms trivially.
  $\diamond$ Kloosterman sum: $d\overline{d} \equiv 1 \bmod q$, $\tau(q)$ is the number of divisors of q,

$$S(m,n;q) \;=\; \sideset{}{^*}\sum_{d \bmod q} e\left(\frac{md}{q} + \frac{n\overline{d}}{q}\right)$$

$$|S(m,n;q)| \;\le\; (m,n,q)\sqrt{\min\left\{\frac{q}{(m,q)}, \frac{q}{(n,q)}\right\}}\;\tau(q).$$

  $\diamond$ Bessel function: integer $k \geq 2$,
  $J_{k-1}(x) \ll \min\left(x, x^{k-1}, x^{-1/2}\right)$.

- Use Fourier Coefficients to split by sign: N fixed:
  $\pm \sum_f \lambda_f(N) * (\cdots)$.

Increasing Support ($\sigma < 2$): Sketch of the proof

- Using Dirichlet Characters, handle Kloosterman terms.

- Have terms like

$$\int_0^\infty J_{k-1}\left(4\pi\frac{\sqrt{m^2yN}}{c}\right)\widehat{\phi}\left(\frac{\log y}{\log R}\right)\frac{dy}{\sqrt{y}}$$

  with arithmetic factors to sum outside.

- Works for support up to $(-2, 2)$.

Increasing Support ($\sigma < 2$): Kloosterman-Bessel details

Stating in greater generality for later use.

Gauss sum: χ a character modulo q: $|G_\chi(n)| \le \sqrt{q}$ with

$$G_\chi(n) \;=\; \sum_{a \bmod q} \chi(a) \exp(2\pi i a n/q).$$

Increasing Support ($\sigma < 2$): Kloosterman-Bessel details

Kloosterman expansion:

$$S(m^2, p_1 \cdots p_n N; Nb) = \frac{-1}{\varphi(b)} \sum_{\chi (\bmod b)} \chi(N) G_\chi(m^2) G_\chi(1) \overline{\chi}(p_1 \cdots p_n).$$

Lemma: Assuming GRH for Dirichlet L-functions, $\operatorname{supp}(\widehat{\phi}) \subset \left(-\frac{2}{n}, \frac{2}{n}\right)$, non-principal characters negligible.
Proof: use $J_{k-1}(x) \ll x$ and see

$$\ll \frac{1}{\sqrt{N}} \sum_{m \le N^\epsilon} \frac{1}{m} \sum_{\substack{(b,N)=1 \\ b < N^{2006}}} \frac{1}{b} \frac{1}{\varphi(b)} \sum_{\substack{\chi (\bmod b) \\ \chi \neq \chi_0}} \left| G_\chi(m^2) G_\chi(1) \right|$$

$$\times \frac{m}{b\sqrt{N}} \prod_{j=1}^{n} \left| \sum_{p_j \neq N} \overline{\chi}(p_j) \log p_j \cdot \frac{1}{\log R} \widehat{\phi}\left(\frac{\log p_j}{\log R}\right) \right|.$$

2-Level Density

$$\int_{x_1=2}^{R^\sigma}\int_{x_2=2}^{R^\sigma}\widehat{\phi}\left(\frac{\log x_1}{\log R}\right)\widehat{\phi}\left(\frac{\log x_2}{\log R}\right)J_{k-1}\left(4\pi\frac{\sqrt{m^2x_1x_2N}}{c}\right)\frac{dx_1dx_2}{\sqrt{x_1x_2}}$$

Change of variables and Jacobian:

$$\begin{array}{rclcrcl} u_2 & = & x_1x_2 & \qquad & x_2 & = & \frac{u_2}{u_1} \\ u_1 & = & x_1 & & x_1 & = & u_1 \end{array}$$

$$\left|\frac{\partial x}{\partial u}\right| = \left|\begin{array}{cc} 1 & 0 \\ -\frac{u_2}{u_1^2} & \frac{1}{u_1}\end{array}\right| = \frac{1}{u_1}.$$

Left with

$$\int\int\widehat{\phi}\left(\frac{\log u_1}{\log R}\right)\widehat{\phi}\left(\frac{\log\left(\frac{u_2}{u_1}\right)}{\log R}\right)\frac{1}{\sqrt{u_2}}J_{k-1}\left(4\pi\frac{\sqrt{m^2u_2N}}{c}\right)\frac{du_1du_2}{u_1}$$

2-Level Density

Changing variables, u_1-integral is

$$\int_{w_1=\frac{\log u_2}{\log R}-\sigma}^{\sigma} \widehat{\phi}(w_1)\,\widehat{\phi}\left(\frac{\log u_2}{\log R}-w_1\right)dw_1.$$

Support conditions imply

$$\Psi_2\left(\frac{\log u_2}{\log R}\right) = \int_{w_1=-\infty}^{\infty} \widehat{\phi}(w_1)\,\widehat{\phi}\left(\frac{\log u_2}{\log R}-w_1\right)dw_1.$$

Substituting gives

$$\int_{u_2=0}^{\infty} J_{k-1}\left(4\pi\frac{\sqrt{m^2u_2N}}{c}\right)\Psi_2\left(\frac{\log u_2}{\log R}\right)\frac{du_2}{\sqrt{u_2}}$$

3-Level Density

$$\int_{x_1=2}^{R^\sigma}\int_{x_2=2}^{R^\sigma}\int_{x_3=2}^{R^\sigma}\widehat{\phi}\left(\frac{\log x_1}{\log R}\right)\widehat{\phi}\left(\frac{\log x_2}{\log R}\right)\widehat{\phi}\left(\frac{\log x_3}{\log R}\right)$$

$$* \quad J_{k-1}\left(4\pi\frac{\sqrt{m^2x_1x_2x_3N}}{c}\right)\frac{dx_1dx_2dx_3}{\sqrt{x_1x_2x_3}}$$

Change variables as below and get Jacobian:

$$\begin{array}{rclrcl} u_3 & = & x_1x_2x_3 & x_3 & = & \frac{u_3}{u_2} \\ u_2 & = & x_1x_2 & x_2 & = & \frac{u_2}{u_1} \\ u_1 & = & x_1 & x_1 & = & u_1 \end{array}$$

$$\left|\frac{\partial x}{\partial u}\right| = \begin{vmatrix} 1 & 0 & 0 \\ -\frac{u_2}{u_1^2} & \frac{1}{u_1} & 0 \\ 0 & -\frac{u_3}{u_2^2} & \frac{1}{u_2} \end{vmatrix} = \frac{1}{u_1u_2}.$$

n-Level Density: Determinant Expansions from RMT

- U(N), $U_k(N)$: $\det\left(K_0(x_j,x_k)\right)_{1\le j,k\le n}$
- USp(N): $\det\left(K_{-1}(x_j,x_k)\right)_{1\le j,k\le n}$
- SO(even): $\det\left(K_1(x_j,x_k)\right)_{1\le j,k\le n}$
- SO(odd): $\det\left(K_{-1}(x_j,x_k)\right)_{1\le j,k\le n} + \sum_{\nu=1}^{n}\delta(x_\nu)\det\left(K_{-1}(x_j,x_k)\right)_{1\le j,k\neq\nu\le n}$

where

$$K_\epsilon(x,y) = \frac{\sin\left(\pi(x-y)\right)}{\pi(x-y)} + \epsilon\frac{\sin\left(\pi(x+y)\right)}{\pi(x+y)}.$$

n-Level Density: Sketch of proof

Expand Bessel-Kloosterman piece, use GRH to drop non-principal characters, change variables, main term is

$$\frac{b\sqrt{N}}{2\pi m}\int_0^\infty J_{k-1}(x)\widehat{\Phi_n}\left(\frac{2\log(bx\sqrt{N}/4\pi m)}{\log R}\right)\frac{\mathrm{d}x}{\log R}$$

with $\Phi_n(x) = \phi(x)^n$.

Main Idea

Difficulty in comparison with classical RMT is that instead of having an n-dimensional integral of $\phi_1(x_1)\cdots\phi_n(x_n)$ we have a 1-dimensional integral of a new test function. This leads to harder combinatorics but allows us to appeal to the result from ILS.

Support for *n*-Level Density

Careful book-keeping gives (originally just had $\frac{1}{n-1/2}$)

$$\sigma_n \ < \ \frac{1}{n-1}.$$

n-Level Density is trivial for $\sigma_n < \frac{1}{n}$, non-trivial up to $\frac{1}{n-1}$.

Expected $\frac{2}{n}$. Obstruction from partial summation on primes.

Support Problems: 2-Level Density

Partial Summation on p_1 first, looks like

$$\sum_{\substack{p_1 \\ p_1 \neq p_2}} S(m^2, p_1p_2N, c) \frac{2\log p_1}{\sqrt{p_1}\log R} \widehat{\phi}\left(\frac{\log p_1}{\log R}\right) J_{k-1}\left(4\pi \frac{\sqrt{m^2 p_1 p_2 N}}{c}\right)$$

Similar to ILS, obtain ($c = bN$):

$$\sum_{\substack{p_1 \leq x_1 \\ p_1 \nmid b}} S(m^2, p_1p_2N, c)\frac{\log p}{\sqrt{p}} = \frac{2\mu(N)}{\phi(b)}\widetilde{R}(m^2, b, p_2)x_1^{\frac{1}{2}} + O\left(b(bx_1N)^{\epsilon}\right)$$

$\sum_{p_1}$ to $\int_{x_1}$, error $\ll b(bN)^{\epsilon} m\sqrt{p_2N}N^{\sigma_2/2}/bN$, yields

$$\sqrt{N}\sum_{m \leq N^{\epsilon}} \frac{1}{m} \sum_{b \leq N^5} \frac{1}{bN} \sum_{p_2 \leq N_2^{\sigma}} \frac{1}{\sqrt{p_2}} \frac{b(bN)^{\epsilon} m\sqrt{p_2N}N^{\frac{\sigma_2}{2}}}{bN}$$

$$\ll \quad N^{\frac{1}{2}+\epsilon'+\sigma_2+\frac{1}{2}+\frac{\sigma_2}{2}-2} \ll N^{\frac{3}{2}\sigma_2 - 1 + \epsilon'}$$

Support Problems: n-Level Density: Why is $\sigma_2 < 1$?

- If no $\sum_{p_2}$, have above *without* the N^σ which arose from $\sum_{p_2}$, giving

$$\ll \ N^{\frac12+\epsilon'+\frac12+\frac{\sigma_1}{2}-2} \ = \ N^{\frac12\sigma_1-1+\epsilon'}.$$

- Fine for $\sigma_1 < 2$. For 3-Level, have two sums over primes giving N^{σ_3}, giving

$$\ll \ N^{\frac12+\epsilon'+2\sigma_3+\frac12+\frac{\sigma_3}{2}-2} \ = \ N^{\frac52\sigma_3-1+\epsilon'}$$

- n-Level, have an additional $(n-1)$ prime sums, each giving N^{σ_n}, yields

$$\ll \ N^{\frac12+\epsilon'+(n-1)\sigma_n+\frac12+\frac{\sigma_n}{2}-2} \ = \ N^{\frac{(2n-1)}{2}\sigma_n-1+\epsilon'}$$

Summary

- More support for RMT Conjectures.
- Control of Conductors.
- Averaging Formulas.

Theorem (Hughes-Miller 2007)

n-level densities of weight k cuspidal newforms of prime level N, $N \to \infty$, agree with orthogonal in non-trivial range (with or without splitting by sign).

New Cuspidal Results
Iyer, Miller and Triantafillou

Goal:

Prove n-level densities agree for $\text{supp}(\widehat{\phi}) \subset \left(-\frac{1}{n-2}, \frac{1}{n-2}\right)$.

Philosophy:

Number theory harder - adapt tools to get an answer.
Random matrix theory easier - manipulate known answer.

Theorem (ILS)

Let Ψ be an even Schwartz function with $\text{supp}(\widehat{\Psi}) \subset (-2, 2)$. Then

$$\sum_{m \le N^\epsilon} \frac{1}{m^2} \sum_{(b,N)=1} \frac{R(m^2,b)R(1,b)}{\varphi(b)} \int_{y=0}^{\infty} J_{k-1}(y)\widehat{\Psi}\left(\frac{2\log(by\sqrt{N}/4\pi m)}{\log R}\right)\frac{\mathrm{d}y}{\log R}$$
$$= -\frac{1}{2}\left[\int_{-\infty}^{\infty} \Psi(x)\frac{\sin 2\pi x}{2\pi x}\mathrm{d}x - \frac{1}{2}\Psi(0)\right] + O\left(\frac{k\log\log kN}{\log kN}\right),$$

where $R = k^2N$, φ is Euler's totient function, and $R(n,q)$ is a Ramanujan sum.

Number Theory Side: Iyer-Miller-Triantafillou: $\text{supp}(\widehat{\phi}) \subset \left(-\frac{1}{n-2}, \frac{1}{n-2}\right)$

Sequence of Lemmas - New Contributions Arise

1. Apply Petersson Formula
2. Restrict Certain Sums
3. Convert Kloosterman Sums to Gauss Sums
4. Remove Non-Principal Characters
5. Convert Sums to Integrals

New Results: Number Theory Side: Iyer-Miller-Triantafillou: $\mathrm{supp}(\widehat{\phi}) \subset \left(-\frac{1}{n-2}, \frac{1}{n-2}\right)$

Typical Argument

If any prime is ‘special’, bound error terms

$$\ll \frac{1}{\sqrt{N}} \sum_{p_1,\dots,p_n \le N^\sigma} \sum_{m \le N^\epsilon} \sum_{r=1}^{\infty} \frac{m^2 r}{r p_1} \frac{\sqrt{p_1 \cdots p_n}}{r p_1 \sqrt{N}} \frac{1}{\sqrt{p_1 \cdots p_n}}$$

$$\ll N^{-1+\epsilon'} \left(\sum_{p \le N^\sigma} 1 \right)^{n-1} \ll N^{-1+(n-1)\sigma+\epsilon'}$$

Bounds fail for large support - new terms arise.

Converting Sums to Integrals - Hughes-Miller: $\text{supp}(\widehat{\phi}) \subset (-\frac{1}{n-1}, \frac{1}{n-1})$

We want to evaluate

$$\sum_{n_1,\dots,n_n} \left[\prod_{i=1}^{n} \hat{\phi}\left(\frac{\log n_i}{\log R}\right) \frac{\chi_0(n_i)\Lambda(n_i)}{\sqrt{n_i}\log R}\right] J_{k-1}\left(\frac{4\pi m\sqrt{n_1\cdots n_n}}{b\sqrt{N}}\right)$$

$$= \frac{1}{2\pi i}\int_{\mathfrak{Re}(s)=1} \left[\prod_{i=1}^{n}\sum_{n_i} \hat{\phi}\left(\frac{\log n_i}{\log R}\right)\frac{\chi_0(n_i)\Lambda(n_i)}{\sqrt{n_i}\log R}\right]$$

$$\times \left(\frac{4\pi m\sqrt{n_1\cdots n_n}}{b\sqrt{N}}\right)^{-s} G_{k-1}(s)ds$$

Converting Sums to Integrals - Hughes-Miller: $\text{supp}(\widehat{\phi}) \subset (-\frac{1}{n-1}, \frac{1}{n-1})$

Important Observation

$$\sum_{r=1}^{\infty} \hat{\phi}\left(\frac{\log r}{\log R}\right) \frac{\chi_0(r)\Lambda(r)}{r^{(1+s)/2}\log R} = \phi\left(\frac{1-s}{4\pi i}\log R\right) + \mathcal{E}(s),$$

where

$$\mathcal{E}(s) = -\frac{1}{2\pi i}\int_{\mathfrak{Re}(z)=c} \phi\left(\frac{(2z-1-s)\log R}{4\pi i}\right)\frac{L'}{L}(z,\chi_0)dz.$$

For convenience, rename expressions, $X = Y + Z$.

Converting Sums to Integrals - Hughes-Miller: $\text{supp}(\widehat{\phi}) \subset (-\frac{1}{n-1}, \frac{1}{n-1})$

Important Observation

By the binomial theorem,

$$\frac{1}{2\pi i}\int_{\Re(s)=1} X^n \left(\frac{4\pi m}{b\sqrt{N}}\right)^{-s} G_{k-1}(s)ds$$

$$=\frac{1}{2\pi i}\int_{\Re(s)=1} (Y+Z)^n \left(\frac{4\pi m}{b\sqrt{N}}\right)^{-s} G_{k-1}(s)ds$$

$$=\sum_{j=0}^{n}\binom{n}{j}\frac{1}{2\pi i}\int_{\Re(s)=1} Y^{n-j}Z^j \left(\frac{4\pi m}{b\sqrt{N}}\right)^{-s} G_{k-1}(s)ds.$$

Converting Sums to Integrals - Hughes-Miller: $\text{supp}(\widehat{\phi}) \subset (-\frac{1}{n-1}, \frac{1}{n-1})$

$Z = \mathcal{E}(s)$ is easy to bound (shift coutours), get

$$\frac{1}{2\pi i}\int_{\mathfrak{Re}(s)=1} Y^{n-j}Z^{j}\left(\frac{4\pi m}{b\sqrt{N}}\right)^{-s} G_{k-1}(s)ds$$
$$\ll N^{(n-j)\sigma/2+\epsilon''}$$

Converting Sums to Integrals - Hughes-Miller: $\text{supp}(\widehat{\phi}) \subset (-\frac{1}{n-1}, \frac{1}{n-1})$

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For $\sigma < \frac{1}{n-1}$, only $j = 0$ term is non-negligible.

Converting Sums to Integrals - Hughes-Miller: $\text{supp}(\widehat{\phi}) \subset (-\frac{1}{n-1}, \frac{1}{n-1})$

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$$\frac{1}{2\pi i} \int_{\mathfrak{Re}(s)=1} Y^{n-j} Z^j \left(\frac{4\pi m}{b\sqrt{N}}\right)^{-s} G_{k-1}(s) ds$$
$$\ll N^{(n-j)\sigma/2+\epsilon''}$$

For $\sigma < \frac{1}{n-1}$, only $j = 0$ term is non-negligible.

For $\sigma < \frac{1}{n-2}$, $j = 1$ term also non-negligible.

Converting Sums to Integrals - Hughes-Miller: $\text{supp}(\widehat{\phi}) \subset (-\frac{1}{n-1}, \frac{1}{n-1})$

$Z = \mathcal{E}(s)$ is easy to bound (shift coutours), get

$$\frac{1}{2\pi i} \int_{\Re\mathfrak{e}(s)=1} Y^{n-j} Z^j \left(\frac{4\pi m}{b\sqrt{N}}\right)^{-s} G_{k-1}(s) ds$$
$$\ll N^{(n-j)\sigma/2+\epsilon''}$$

For $\sigma < \frac{1}{n-1}$, only $j = 0$ term is non-negligible.

For $\sigma < \frac{1}{n-2}$, $j = 1$ term also non-negligible.

Unfortunately, $Z = \mathcal{E}(s)$ is hard to compute with.

Converting Sums to Integrals - Iyer-Miller-Triantafillou: $\text{supp}(\widehat{\phi}) \subset (-\frac{1}{n-2}, \frac{1}{n-2})$

Key idea: Recall $X = Y + Z$, write $Z = X - Y$.

Converting Sums to Integrals - Iyer-Miller-Triantafillou: $\text{supp}(\widehat{\phi}) \subset (-\frac{1}{n-2}, \frac{1}{n-2})$

Key idea: Recall $X = Y + Z$, write $Z = X - Y$.

$$\frac{n}{2\pi i}\int_{\Re(s)=1} Y^{n-1}Z\left(\frac{4\pi m}{b\sqrt{N}}\right)^{-s} G_{k-1}(s)ds$$
$$= \frac{n}{2\pi i}\int_{\Re(s)=1} XY^{n-1}\left(\frac{4\pi m}{b\sqrt{N}}\right)^{-s} G_{k-1}(s)ds$$
$$-\frac{n}{2\pi i}\int_{\Re(s)=1} Y^{n}\left(\frac{4\pi m}{b\sqrt{N}}\right)^{-s} G_{k-1}(s)ds.$$

Hughes-Miller handle Y^n term, XY^{n-1} term is similar.

Converting Sums to Integrals - Iyer-Miller-Triantafillou: $\text{supp}(\widehat{\phi}) \subset (-\frac{1}{n-2}, \frac{1}{n-2})$

When the dust clears, we see

$$\sum_{n_1,\dots,n_n}\left[\prod_{i=1}^{n}\hat{\phi}\left(\frac{\log n_i}{\log R}\right)\frac{\chi_0(n_i)\Lambda(n_i)}{\sqrt{n_i}\log R}\right]J_{k-1}\left(\frac{4\pi m\sqrt{n_1\cdots n_n}}{b\sqrt{N}}\right)$$

$$=(1-n)\left(\int_{-\infty}^{\infty}\phi(x)^n\frac{\sin 2\pi x}{2\pi x}dx-\frac{1}{2}\phi(0)^n\right)$$

$$+n\left(\int_{-\infty}^{\infty}\hat{\phi}(x_2)\int_{-\infty}^{\infty}\phi^{n-1}(x_1)\frac{\sin(2\pi x_1(1-|x_2|))}{2\pi x_1}dx_1dx_2\right.$$

$$\left.-\frac{1}{2}\phi^n(0)\right)+o(1)$$

New Results: Number Theory Side: Iyer-Miller-Triantafillou: $\text{supp}(\widehat{\phi}) \subset (-\frac{1}{n-2}, \frac{1}{n-2})$

Theorem

Fix $n \geq 4$ and let ϕ be an even Schwartz function with $\text{supp}(\widehat{\phi}) \subset \left(-\frac{1}{n-2}, \frac{1}{n-2}\right)$. Then, the nth centered moment of the 1-level density for holomorphic cusp forms is

$$\frac{1+(-1)^n}{2}(-1)^n(n-1)!!\left(2\int_{-\infty}^{\infty}\hat{\phi}(y)^2|y|dy\right)^{n/2}$$
$$\pm(-2)^{n-1}\left(\int_{-\infty}^{\infty}\phi(x)^n\frac{\sin 2\pi x}{2\pi x}dx-\frac{1}{2}\phi(0)^n\right)$$
$$\mp(-2)^{n-1}n\left(\int_{-\infty}^{\infty}\hat{\phi}(x_2)\int_{-\infty}^{\infty}\phi^{n-1}(x_1)\frac{\sin(2\pi x_1(1+|x_2|))}{2\pi x_1}dx_1dx_2-\frac{1}{2}\phi^n(0)\right).$$

Compare to ...

Random Matrix Theory: New Combinatorial Vantage
Iyer-Miller-Triantafillou

n-Level Density: Katz-Sarnak Determinant Expansions

Example: $\mathrm{SO(even)}$

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \widehat{\phi}(x_1) \cdots \widehat{\phi}(x_n) \det \Big(K_1(x_j, x_k) \Big)_{1 \le j,k \le n} dx_1 \cdots dx_n,$$

where

$$K_1(x, y) = \frac{\sin\Big(\pi(x-y)\Big)}{\pi(x-y)} + \frac{\sin\Big(\pi(x+y)\Big)}{\pi(x+y)}.$$

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Problem: n-dimensional integral - looks very different.

Preliminaries

Easier to work with cumulants.

$$\sum_{n=1}^{\infty} C_n \frac{(it)^n}{n!} = \log \widehat{P}(t),$$

where P is the probability density function.

$$\mu_n' = C_n + \sum_{m=1}^{n-1} \binom{n-1}{m-1} C_m \mu_{n-m}',$$

where μ_n' is uncentered moment.

Preliminaries

Manipulating determinant expansions leads to analysis of

$$K(y_1,\dots,y_n) = \sum_{m=1}^{n} \sum_{\substack{\lambda_1+\dots+\lambda_m=n \\ \lambda_j \ge 1}} \frac{(-1)^{m+1}}{m} \frac{n!}{\lambda_1!\cdots\lambda_m!} \sum_{\epsilon_1,\dots,\epsilon_n=\pm1} \prod_{\ell=1}^{m} \chi_{\left\{\left|\sum_{j=1}^{n}\eta(\ell,j)\epsilon_j y_j\right|\le 1\right\}},$$

where

$$\eta(\ell,j) = \begin{cases} +1 & \text{if } j \le \sum_{k=1}^{\ell}\lambda_k \\ -1 & \text{if } j > \sum_{k=1}^{\ell}\lambda_k. \end{cases}$$

New Result: Iyer-Miller-Triantafillou: Large Support: $\mathrm{supp}(\widehat{\phi}) \subseteq \left(-\frac{1}{n-2}, \frac{1}{n-2}\right)$

Hughes-Miller solved for $\mathrm{supp}(\widehat{\phi}) \subseteq (-\frac{1}{n-1}, \frac{1}{n-1})$.

New Complications: If $\mathrm{supp}(\widehat{\phi}) \subseteq (-\frac{1}{n-2}, \frac{1}{n-2})$,

1. $\eta(\ell, j)\epsilon_j y_j$ need not have same sign (at most one can differ);
2. more than one term in product can be zero (for fixed m, λ_j, ϵ_j).

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Solution: Double count terms and subtract a correcting term ρ_j.

Generating Function Identity - $\lambda_i \geq 1$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n = e^{-x} = \frac{1}{1-(1-e^x)} = \sum_{m=0}^{\infty} (1-e^x)^m$$

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$$= \sum_{m=0}^{\infty} (-1)^m \left(\sum_{\lambda=1}^{\infty} \frac{1}{\lambda!} x^{\lambda} \right)^m = \sum_{n=0}^{\infty} \left(\sum_{m=1}^{\infty} \sum_{\lambda_1+\cdots+\lambda_m=n} \frac{(-1)^m}{\lambda_1! \cdots \lambda_m!} \right) x^n$$

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$$\Rightarrow (-1)^n = \sum_{m=1}^{\infty} \sum_{\lambda_1+\cdots+\lambda_m=n} \frac{(-1)^m n!}{\lambda_1! \cdots \lambda_m!}.$$

New Result: Dealing With 'ρ_j's

All $\lambda_i, \lambda_i' \geq 1$.

$$\rho_j = \sum_{m=1}^{n} \sum_{l=1}^{m} \sum_{\substack{\lambda_1+\cdots+\lambda_{\ell-1}=j-1 \\ \lambda_\ell=1 \\ \lambda_{\ell+1}+\cdots+\lambda_m=n-j}} \frac{(-1)^m}{m} \frac{n!}{\lambda_1!\cdots\lambda_m!}$$

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$$= n \sum_{m=1}^{n} \sum_{\ell=1}^{m} \sum_{\lambda_1'+\cdots+\lambda_{m-1}'=n-1} \frac{(-1)^m}{m} \frac{(n-1)!}{\lambda_1'! \cdots \lambda_{m-1}'!}$$

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$$= n \sum_{m=1}^{n} \sum_{\lambda_1'+\cdots+\lambda_{m-1}'=n-1} (-1) \frac{(-1)^{m-1}(n-1)!}{\lambda_1'!\cdots\lambda_{m-1}'!}$$

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$$= n(-1)^n.$$

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After Fourier transform identities:

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After Fourier transform identities:

$$\begin{aligned} C_n^{SO}(\phi) &= \frac{(-1)^{n-1}}{2}\left(\int_{-\infty}^{\infty} \phi(x)^n \frac{\sin 2\pi x}{2\pi x} dx - \frac{1}{2}\phi(0)^n\right) \\ &+ \frac{n(-1)^n}{2}\left(\int_{-\infty}^{\infty} \hat{\phi}(x_2) \int_{-\infty}^{\infty} \phi^{n-1}(x_1)\right. \\ &\left. \frac{\sin(2\pi x_1(1+|x_2|))}{2\pi x_1} dx_1 dx_2 - \frac{1}{2}\phi^n(0)\right). \end{aligned}$$

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After Fourier transform identities:

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Agrees with number theory!

Recap

1. Difficult to compare n-dimensional integral from RMT with NT in general. Harder combinatorics worthwhile to appeal to result from ILS.

2. Solve combinatorics by using cumulants; support restrictions translate to which terms can contribute.

3. Extend number theory results by bounding Bessel functions, Kloosterman sums, etc. New terms arise and match random matrix theory prediction.

4. Better bounds on percent of forms vanishing to large order at the center point.

Cuspidal Maass Forms
Alpoge-Amersi-Iyer-Lazarev-Miller-Zhang

Maass Forms

Definition: Maass Forms

A Maass form on a group $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ is a function $f : \mathcal{H} \to \mathbb{R}$ which satisfies:

1. $f(\gamma z) = f(z)$ for all $\gamma \in \Gamma$,
2. f vanishes at the cusps of Γ, and
3. $\Delta f = \lambda f$ for some $\lambda = s(1-s) > 0$, where

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

is the Laplace-Beltrami operator on $\mathcal{H}$.

- Coefficients contain information about partitions.
- For full modular group, $s = 1/2 + it_j$ with $t_j \in \mathbb{R}$.
- Test Katz-Sarnak conjecture.

L-function associated to Maass forms

Write Fourier expansion of Maass form u_j as

$$u_j(z) = \cosh(t_j) \sum_{n \neq 0} \sqrt{y}\,\lambda_j(n) K_{it_j}(2\pi|n|y) e^{2\pi inx}.$$

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Define L-function attached to u_j as

$$L(s, u_j) = \sum_{n \geq 1} \frac{\lambda_j(n)}{n^s} = \prod_p \left(1 - \frac{\alpha_j(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_j(p)}{p^s}\right)^{-1}$$

where $\alpha_j(p) + \beta_j(p) = \lambda_j(p), \quad \alpha_j(p)\beta_j(p) = 1, \quad \lambda_j(1) = 1.$

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where $\alpha_j(p) + \beta_j(p) = \lambda_j(p), \quad \alpha_j(p)\beta_j(p) = 1, \quad \lambda_j(1) = 1.$
Also,

$$\lambda_j(p) \ll p^{7/64}.$$

n-level over a family

- Recall for Katz-Sarnak Conjecture,

$$\frac{1}{|\mathcal{F}_N|}\sum_{f\in\mathcal{F}_N} D_{n,f}(\phi) = \frac{1}{|\mathcal{F}_N|}\sum_{f\in\mathcal{F}_N}\sum_{\substack{j_1,\dots,j_n \\ j_i\neq\pm j_k}}\prod_i \phi_i\left(L_f\gamma_E^{(j_i)}\right)$$
$$\to \int\cdots\int \phi(x)W_{n,\mathcal{G}(\mathcal{F})}(x)dx.$$

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$$\to \int\cdots\int \phi(x)W_{n,\mathcal{G}(\mathcal{F})}(x)dx.$$

- For Dirichlet/cuspidal newform L-functions, there are many with a given conductor.
- Problem: For Maass forms, expect at most one with a given conductor.

n-level over a family: Last Year (Amersi-Iyer-Lazarev-Miller-Zhang)

- Solution: Average over Laplace eigenvalues $\lambda_f = 1/4 + t_j^2$.

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- two choices for the weight function h_T:

$$h_{1,T}(t_j) = \exp\left(-t_j^2/T^2\right),$$

which picks out eigenvalues near the origin,

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$$h_{2,T}(t_j) = \exp\left(-(t_j - T)^2/L^2\right) + \exp\left(-(t_j + T)^2/L^2\right),$$

which picks out eigenvalues centered at $\pm T$.

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which picks out eigenvalues centered at $\pm T$.
- Weighted 1-level density becomes

$$\frac{1}{\sum_j \frac{h_T(t_j)}{\|u_j\|^2}} \sum_j \frac{h_T(t_j)}{\|u_j\|^2} D_{n,u_j}(\phi)$$

$$= \frac{1}{\sum_j \frac{h_T(t_j)}{\|u_j\|^2}} \sum_j \frac{h_T(t_j)}{\|u_j\|^2} \sum_{j_1,\dots,j_n} \prod_i \phi_i\left(\frac{\gamma}{2\pi}\log R\right)$$

1-level density: new test function

Joint with Levent Alpoge.

Work with a better test function.

Not as general, but can exploit holomorphicity.

$h(r) = rh(ir)/\sinh(\pi r)$.

Results
Amersi-Iyer-Lazarev-Miller-Zhang 2011

1-Level Density

1-level density for one function

$$D(u_j;\phi)$$
$$= \hat{\phi}(0)\frac{\log(1+t_j^2)}{\log R} + \frac{\phi(0)}{2} + O\left(\frac{\log\log R}{\log R}\right)$$
$$-\sum_p \frac{2\lambda_j(p)\log p}{p^{\frac{1}{2}}\log R}\hat{\phi}\left(\frac{\log p}{\log R}\right) - \sum_p \frac{2\lambda_j(p^2)\log p}{p\log R}\hat{\phi}\left(\frac{2\log p}{\log R}\right)$$

1. Explicit formula.
2. Gamma function identities
3. Prime Number Theorem

Average 1-level density

The weighted 1-level density becomes:

$$\frac{1}{\sum_j \frac{h_T(t_j)}{\|u_j\|^2}} \sum_j \frac{h_t(t_j)}{\|u_j\|^2} D(u_j;\phi)$$

$$= \frac{\phi(0)}{2} + O\left(\frac{\log\log R}{\log R}\right) + \frac{1}{\sum_j \frac{h_t(t_j)}{\|u_j\|^2}} \sum_j \frac{h_t(t_j)}{\|u_j\|^2} \widehat{\phi}(0) \frac{\log(1+t_j^2)}{\log R}$$

$$-\frac{1}{\sum_j \frac{h_T(t_j)}{\|u_j\|^2}} \sum_p \frac{2\log p}{p^{\frac{1}{2}}\log R} \widehat{\phi}\left(\frac{\log p}{\log R}\right) \sum_j \frac{h_T(t_j)}{\|u_j\|^2} \lambda_j(p)$$

$$-\frac{1}{\sum_j \frac{h_T(t_j)}{\|u_j\|^2}} \sum_p \frac{2\log p}{p\log R} \widehat{\phi}\left(\frac{2\log p}{\log R}\right) \sum_j \frac{h_T(t_j)}{\|u_j\|^2} \lambda_j(p^2)$$

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$$- \frac{1}{\sum_j \frac{h_T(t_j)}{\|u_j\|^2}} \sum_p \frac{2\log p}{p^{\frac{1}{2}}\log R} \widehat{\phi}\left(\frac{\log p}{\log R}\right) \sum_j \frac{h_T(t_j)}{\|u_j\|^2} \lambda_j(p)$$

$$- \frac{1}{\sum_j \frac{h_T(t_j)}{\|u_j\|^2}} \sum_p \frac{2\log p}{p\log R} \widehat{\phi}\left(\frac{2\log p}{\log R}\right) \sum_j \frac{h_T(t_j)}{\|u_j\|^2} \lambda_j(p^2)$$

Kuznetsov Trace Formula

To tackle terms with $\lambda_j(p)$ and $\lambda_j(p^2)$ we need the Kuznetsov Trace Formula:

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$$\sum_j \frac{h(t_j)}{\|u_j\|^2}\lambda_j(m)\overline{\lambda_j(n)}$$

$$= \text{ some function that depends just on h, m, and n}$$

Kuznetsov Trace Formula

$$\sum_j \frac{h(t_j)}{\|u_j\|^2}\lambda_j(m)\overline{\lambda_j(n)} + \frac{1}{4\pi}\int_{\mathbb{R}} \overline{\tau(m,r)}\tau(n,r)\frac{h(r)}{\cosh(\pi r)}dr =$$

$$\frac{\delta_{n,m}}{\pi^2}\int_{\mathbb{R}} r\tanh(r)h(r)dr + \frac{2i}{\pi}\sum_{c\geq 1}\frac{S(n,m;c)}{c}\int_{\mathbb{R}} J_{ir}\left(\frac{4\pi\sqrt{mn}}{c}\right)\frac{h(r)r}{\cosh(\pi r)}dr$$

where
$\tau(m,r) = \pi^{\frac{1}{2}+ir}\Gamma(1/2+ir)^{-1}\zeta(1+2ir)^{-1}n^{-\frac{1}{2}}\sum_{ab=|m|}\left(\frac{a}{b}\right)^{ir}$.

$$S(n,m;c) = \sum_{0\leq x\leq c-1, gcd(x,c)=1} e^{2\pi i(nx+mx^*)/c}$$

$$J_{ir}(x) = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+ir+1)}\left(\frac{1}{2}x\right)^{2m+ir}.$$

Kuznetsov Formula

- The only $\lambda(m)\overline{\lambda(n)}$ term that contributes is when $m = n = 1$.
- The $m = 1, n = p$ and $m = 1, n = p^2$ terms do not contribute because of the $\delta_{m,n}$ function.

Result: 1-level density

Theorem (AILMZ, 2011)

If $h_T = h_{1,T}$ or $h_{2,T}$, $T \to \infty$ and $L \ll T / \log T$, and $\sigma < 1/6$ then 1-level density is

$$\frac{1}{\sum_j \frac{h_T(t_j)}{\|u_j\|^2}} \sum_j \frac{h_T(t_j)}{\|u_j\|^2} D(u_j;\phi) = \frac{\phi(0)}{2} + \widehat{\phi}(0) + O\left(\frac{\log\log R}{\log R}\right)$$
$$+ O(T^{3\sigma/2-1/4+\epsilon} + T^{\sigma/2-1/4+\epsilon}).$$

- This matches with the orthogonal family density as predicted by Katz-Sarnak.

Support

Main reason support so small due to bounds on Bessel-Kloosterman piece.

Can distinguish unitary and symplectic from the 3 orthogonal groups, but 1-level density cannot distinguish the orthogonal groups from each other if support in $(-1, 1)$.

2-level density can distinguish orthogonal groups with arbitrarily small support; additional term depending on distribution of signs of functional equations.

2- Level Density

To differentiate between even and odd in orthogonal family, we calculated the 2-level density:

$$D_2^*(\phi) := \frac{1}{\sum_j \frac{h_T(t_j)}{\|u_j\|^2}} \sum_j \frac{h_T(t_j)}{\|u_j\|^2} \sum_{j_1, j_2} \phi_1(\gamma^{(j_1)}) \overline{\phi_2(\gamma^{(j_2)})}$$

$$= \frac{1}{\sum_j \frac{h_T(t_j)}{\|u_j\|^2}} \sum_j \frac{h_T(t_j)}{\|u_j\|^2} \prod_{i=1}^{2} \left| \frac{\phi_i(0)}{2} + \widehat{\phi}_i(0) \frac{\log(1+t_j^2)}{\log R} + O\left(\frac{\log\log R}{\log R}\right) \right.$$

$$\left. - \sum_p \frac{2\lambda_j(p)\log p}{p^{\frac{1}{2}}\log R} \widehat{\phi}_i\left(\frac{\log p}{\log R}\right) - \sum_p \frac{2\lambda_j(p^2)\log p}{p\log R} \widehat{\phi}_i\left(\frac{2\log p}{\log R}\right) \right| .$$

25 terms, handled by Cauchy-Schwarz or Kuznetsov.

Result: 2-level density

Theorem (AILMZ, 2011)

Same conditions as before, for $\sigma < 1/12$ have

$$\begin{aligned} D^*_{2,\mathcal{F}} &= \prod_{i=1}^{2}\left[\frac{\phi_i(0)}{2} + \widehat{\phi}_i(0)\right] + 2\int_{-\infty}^{\infty} |z|\widehat{\phi}_1(z)\widehat{\phi}_2(z)dz \\ &\quad -\phi_1(0)\phi_1(0) - 2\widehat{\phi_1\phi_2}(0) + (\phi_1\phi_2)(0)\mathcal{N}(-1) \\ &\quad + O\left(\frac{\log\log R}{\log R}\right). \end{aligned}$$

Note that $\mathcal{N}(-1)$ is the weighted percent that have odd sign in functional equation.

Results
Alpoge-Miller 2012

Sketch of proof

Theorem (Alpoge-Miller 2012)

For $h(r) = rh(ir)/\sinh(\pi r)$ the Katz-Sarnak conjecture holds for level 1 cuspidal Maass forms for test functions whose Fourier transform is supported in $(-4/3, 4/3)$.

- Express $\int_{-\infty}^{\infty} J_{2ir}(X)\frac{rh_T(r)}{\cosh(\pi r)}dr$ as a sum of $J_{2k+1}(X)$ and h_T at imaginary arguments and $J_{2kT}(X)$ and h at k.
- Bound contributions from sums. Apply Poisson summation, analyze, and Poisson summation again.
- In calculations key steps are Taylor expanding exponentials and invoking Fourier transform identities (especially relating differentiation and multiplication).

Conclusion

Recap

- Understand compound families in terms of simple ones.
- Choose combinatorics to simplify calculations.
- Extending support often related to deep arithmetic questions.