Cookie Monster Meets the Fibonacci Numbers. Mmmmmm – Theorems!

Research and Results in REUs: Steven J. Miller

http://www.williams.edu/Mathematics/sjmiller/public_html

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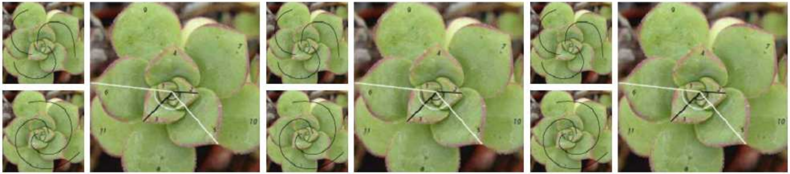
Introduction

Goals of the Talk

- Research: What questions to ask? How? With whom?
- Explore: Look for the right perspective.
- Utilize: What are your tools and how can they be used?
- succeed: Control what you can: reports, talks, ...



Joint with many students and junior faculty over the years.

Research: What questions to ask? How? With whom?

- Build on what you know and can learn.
- What will be interesting?
- How will you work?
- Where are the questions? Classes, arXiv, conferences, ...

Explore: Look for the right perspective.

- Ask interesting questions.
- Look for connections.
- Be a bit of a jack-of-all trades.

Leads naturally into....

Utilize: What are your tools and how can they be used?

Law of the Hammer:

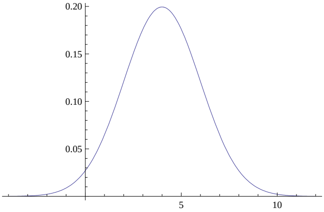
- Abraham Kaplan: I call it the law of the instrument, and it may be formulated as follows: Give a small boy a hammer, and he will find that everything he encounters needs pounding.
- Abraham Maslow: I suppose it is tempting, if the only tool you have is a hammer, to treat everything as if it were a nail.
- Bernard Baruch: If all you have is a hammer, everything looks like a nail.

Succeed: Control what you can: reports, talks

- Write up your work: post on the arXiv, submit.
- Go to conferences: present and mingle (no spam and P&J).
- Turn things around fast: show progress, no more than 24 hours on mundane.
- Service: refereeing, MathSciNet, ...

Pre-requisites

Pre-requisites: Probability Review



- Let X be random variable with density $p(x)$:
  - $\diamond$ $p(x) \geq 0$; $\int_{-\infty}^{\infty} p(x)dx = 1$;
  - $\diamond$ $\text{Prob}\,(a \leq X \leq b) = \int_a^b p(x)dx$.
- Mean: $\mu = \int_{-\infty}^{\infty} xp(x)dx$.
- Variance: $\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 p(x)dx$.
- Gaussian: Density $(2\pi\sigma^2)^{-1/2} \exp(-(x-\mu)^2/2\sigma^2)$.

Pre-requisites: Combinatorics Review

- $n!$: number of ways to order n people, order matters.
- $\frac{n!}{k!(n-k)!} = n\mathrm{C}k = \binom{n}{k}$: number of ways to choose k from n, order doesn't matter.
- Stirling's Formula: $n! \approx n^n e^{-n}\sqrt{2\pi n}$.

Previous Results

Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$;
First few: $1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \dots$.

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Example: $51 = ?$

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Example: $51 = 34 + 17 = F_8 + 17$.

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Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers.

Example: $51 = 34 + 13 + 4 = F_8 + F_6 + 4$.

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Zeckendorf's Theorem

Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers.

Example: $51 = 34 + 13 + 3 + 1 = F_8 + F_6 + F_3 + 1$.

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Example: $51 = 34 + 13 + 3 + 1 = F_8 + F_6 + F_3 + F_1$.
Example: $83 = 55 + 21 + 5 + 2 = F_9 + F_7 + F_4 + F_2$.
Observe: 51 miles $\approx$ 82.1 kilometers.

Old Results

Central Limit Type Theorem

As $n \to \infty$ distribution of number of summands in Zeckendorf decomposition for $m \in [F_n, F_{n+1})$ is Gaussian (normal).

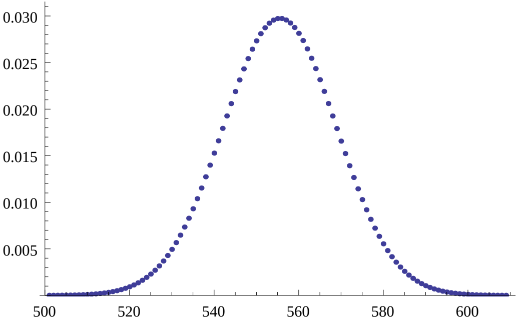

Figure: Number of summands in $[F_{2010}, F_{2011})$; $F_{2010} \approx 10^{420}$.

New Results: Bulk Gaps: $m \in [F_n, F_{n+1})$ and $\phi = \frac{1+\sqrt{5}}{2}$

$$m \;=\; \sum_{j=1}^{k(m)=n} F_{i_j}, \quad \nu_{m;n}(x) \;=\; \frac{1}{k(m)-1} \sum_{j=2}^{k(m)} \delta\left(x - (i_j - i_{j-1})\right).$$

Theorem (Zeckendorf Gap Distribution)

Gap measures $\nu_{m;n}$ converge almost surely to average gap measure where $P(k) = 1/\phi^k$ for $k \geq 2$.

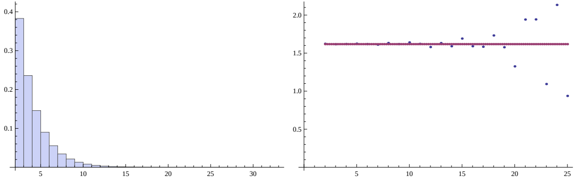

Figure: Distribution of gaps in $[F_{1000}, F_{1001})$; $F_{2010} \approx 10^{208}$.

New Results: Longest Gap

Theorem (Longest Gap)

As $n \to \infty$, the probability that $m \in [F_n, F_{n+1})$ has longest gap less than or equal to $f(n)$ converges to

$$\mathrm{Prob}\left(L_n(m) \le f(n)\right) \ \approx\ e^{-e^{\log n - f(n)/\log \phi}}.$$

Immediate Corollary: If $f(n)$ grows ***slower*** or ***faster*** than $\log n/\log \phi$, then $\mathrm{Prob}(L_n(m) \le f(n))$ goes to **0** or ***1***, respectively.

Preliminaries: The Cookie Problem

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Reinterpreting the Cookie Problem

The number of solutions to $x_1 + \cdots + x_P = C$ with $x_i \geq 0$ is $\binom{C+P-1}{P-1}$.

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Let $p_{n,k} = \# \{N \in [F_n, F_{n+1})$: the Zeckendorf decomposition of N has exactly k summands$\}$.

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$$N = F_{i_1} + F_{i_2} + \cdots + F_{i_{k-1}} + F_n,$$
$$1 \leq i_1 < i_2 < \cdots < i_{k-1} < i_k = n,\ i_j - i_{j-1} \geq 2.$$

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$$d_1 := i_1 - 1,\ d_j := i_j - i_{j-1} - 2\ (j > 1).$$

$$d_1 + d_2 + \cdots + d_k = n - 2k + 1,\ d_j \geq 0.$$

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$$d_1 + d_2 + \cdots + d_k = n - 2k + 1,\ d_j \geq 0.$$

Cookie counting $\Rightarrow p_{n,k} = \binom{n-2k+1\ +\ k-1}{k-1} = \binom{n-k}{k-1}$.

Gaussian Behavior

Generalizing Lekkerkerker: Erdos-Kac type result

Theorem (KKMW 2010)

As $n \to \infty$, the distribution of the number of summands in Zeckendorf's Theorem is a Gaussian.

Sketch of proof: Use Stirling's formula,

$$n! \approx n^n e^{-n} \sqrt{2\pi n}$$

to approximates binomial coefficients, after a few pages of algebra find the probabilities are approximately Gaussian.

(Sketch of the) Proof of Gaussianity

The probability density for the number of Fibonacci numbers that add up to an integer in $[F_n, F_{n+1})$ is $f_n(k) = \binom{n-1-k}{k}/F_{n-1}$. Consider the density for the $n+1$ case. Then we have, by Stirling

$$\begin{aligned} f_{n+1}(k) &= \binom{n-k}{k}\frac{1}{F_n} \\ &= \frac{(n-k)!}{(n-2k)!k!}\frac{1}{F_n} = \frac{1}{\sqrt{2\pi}}\frac{(n-k)^{n-k+\frac{1}{2}}}{k^{(k+\frac{1}{2})}(n-2k)^{n-2k+\frac{1}{2}}}\frac{1}{F_n} \end{aligned}$$

plus a lower order correction term.

Also we can write $F_n = \frac{1}{\sqrt{5}}\phi^{n+1} = \frac{\phi}{\sqrt{5}}\phi^n$ for large n, where ϕ is the golden ratio (we are using relabeled Fibonacci numbers where $1 = F_1$ occurs once to help dealing with uniqueness and $F_2 = 2$). We can now split the terms that exponentially depend on n.

$$f_{n+1}(k) = \left(\frac{1}{\sqrt{2\pi}}\sqrt{\frac{(n-k)}{k(n-2k)}}\frac{\sqrt{5}}{\phi}\right)\left(\phi^{-n}\frac{(n-k)^{n-k}}{k^k(n-2k)^{n-2k}}\right).$$

Define

$$N_n = \frac{1}{\sqrt{2\pi}}\sqrt{\frac{(n-k)}{k(n-2k)}}\frac{\sqrt{5}}{\phi}, \quad S_n = \phi^{-n}\frac{(n-k)^{n-k}}{k^k(n-2k)^{n-2k}}.$$

Thus, write the density function as

$$f_{n+1}(k) = N_n S_n$$

where N_n is the first term that is of order $n^{-1/2}$ and S_n is the second term with exponential dependence on n.

(Sketch of the) Proof of Gaussianity

Model the distribution as centered around the mean by the change of variable $k = \mu + x\sigma$ where μ and σ are the mean and the standard deviation, and depend on n. The discrete weights of $f_n(k)$ will become continuous. This requires us to use the change of variable formula to compensate for the change of scales:

$$f_n(k)dk = f_n(\mu + \sigma x)\sigma dx.$$

Using the change of variable, we can write N_n as

$$\begin{aligned} N_n &= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{n-k}{k(n-2k)}}\frac{\phi}{\sqrt{5}} \\ &= \frac{1}{\sqrt{2\pi n}}\sqrt{\frac{1-k/n}{(k/n)(1-2k/n)}}\frac{\sqrt{5}}{\phi} \\ &= \frac{1}{\sqrt{2\pi n}}\sqrt{\frac{1-(\mu+\sigma x)/n}{((\mu+\sigma x)/n)(1-2(\mu+\sigma x)/n)}}\frac{\sqrt{5}}{\phi} \\ &= \frac{1}{\sqrt{2\pi n}}\sqrt{\frac{1-C-y}{(C+y)(1-2C-2y)}}\frac{\sqrt{5}}{\phi} \end{aligned}$$

where $C = \mu/n \approx 1/(\phi+2)$ (note that $\phi^2 = \phi + 1$) and $y = \sigma x/n$. But for large n, the y term vanishes since $\sigma \sim \sqrt{n}$ and thus $y \sim n^{-1/2}$. Thus

$$N_n \approx \frac{1}{\sqrt{2\pi n}}\sqrt{\frac{1-C}{C(1-2C)}}\frac{\sqrt{5}}{\phi} = \frac{1}{\sqrt{2\pi n}}\sqrt{\frac{(\phi+1)(\phi+2)}{\phi}}\frac{\sqrt{5}}{\phi} = \frac{1}{\sqrt{2\pi n}}\sqrt{\frac{5(\phi+2)}{\phi}} = \frac{1}{\sqrt{2\pi\sigma^2}}$$

since $\sigma^2 = n\frac{\phi}{5(\phi+2)}$.

(Sketch of the) Proof of Gaussianity

For the second term S_n, take the logarithm and once again change variables by $k = \mu + x\sigma$,

$$\begin{aligned}
\log(S_n) &= \log\left(\phi^{-n}\frac{(n-k)^{(n-k)}}{k^k(n-2k)^{(n-2k)}}\right)\\
&= -n\log(\phi) + (n-k)\log(n-k) - (k)\log(k)\\
&\quad - (n-2k)\log(n-2k)\\
&= -n\log(\phi) + (n-(\mu+x\sigma))\log(n-(\mu+x\sigma))\\
&\quad - (\mu+x\sigma)\log(\mu+x\sigma)\\
&\quad - (n-2(\mu+x\sigma))\log(n-2(\mu+x\sigma))\\
&= -n\log(\phi)\\
&\quad + (n-(\mu+x\sigma))\left(\log(n-\mu)+\log\left(1-\frac{x\sigma}{n-\mu}\right)\right)\\
&\quad - (\mu+x\sigma)\left(\log(\mu)+\log\left(1+\frac{x\sigma}{\mu}\right)\right)\\
&\quad - (n-2(\mu+x\sigma))\left(\log(n-2\mu)+\log\left(1-\frac{x\sigma}{n-2\mu}\right)\right)\\
&= -n\log(\phi)\\
&\quad + (n-(\mu+x\sigma))\left(\log\left(\frac{n}{\mu}-1\right)+\log\left(1-\frac{x\sigma}{n-\mu}\right)\right)\\
&\quad - (\mu+x\sigma)\log\left(1+\frac{x\sigma}{\mu}\right)\\
&\quad - (n-2(\mu+x\sigma))\left(\log\left(\frac{n}{\mu}-2\right)+\log\left(1-\frac{x\sigma}{n-2\mu}\right)\right).
\end{aligned}$$

(Sketch of the) Proof of Gaussianity

Note that, since $n/\mu = \phi + 2$ for large n, the constant terms vanish. We have $\log(S_n)$

$$\begin{aligned}
= \quad & -n\log(\phi) + (n-k)\log\left(\frac{n}{\mu} - 1\right) - (n-2k)\log\left(\frac{n}{\mu} - 2\right) + (n-(\mu+x\sigma))\log\left(1 - \frac{x\sigma}{n-\mu}\right) \\
& - (\mu + x\sigma)\log\left(1 + \frac{x\sigma}{\mu}\right) - (n - 2(\mu+x\sigma))\log\left(1 - \frac{x\sigma}{n-2\mu}\right) \\
= \quad & -n\log(\phi) + (n-k)\log(\phi+1) - (n-2k)\log(\phi) + (n-(\mu+x\sigma))\log\left(1 - \frac{x\sigma}{n-\mu}\right) \\
& - (\mu + x\sigma)\log\left(1 + \frac{x\sigma}{\mu}\right) - (n - 2(\mu+x\sigma))\log\left(1 - \frac{x\sigma}{n-2\mu}\right) \\
= \quad & n(-\log(\phi) + \log\left(\phi^2\right) - \log(\phi)) + k(\log(\phi^2) + 2\log(\phi)) + (n-(\mu+x\sigma))\log\left(1 - \frac{x\sigma}{n-\mu}\right) \\
& - (\mu + x\sigma)\log\left(1 + \frac{x\sigma}{\mu}\right) - (n - 2(\mu+x\sigma))\log\left(1 - 2\frac{x\sigma}{n-2\mu}\right) \\
= \quad & (n-(\mu+x\sigma))\log\left(1 - \frac{x\sigma}{n-\mu}\right) - (\mu + x\sigma)\log\left(1 + \frac{x\sigma}{\mu}\right) \\
& - (n - 2(\mu+x\sigma))\log\left(1 - 2\frac{x\sigma}{n-2\mu}\right).
\end{aligned}$$

(Sketch of the) Proof of Gaussianity

Finally, we expand the logarithms and collect powers of $x\sigma/n$.

$$\begin{aligned}
\log(S_n) &= (n-(\mu+x\sigma))\left(-\frac{x\sigma}{n-\mu}-\frac{1}{2}\left(\frac{x\sigma}{n-\mu}\right)^2+\dots\right) \\
&\quad -(\mu+x\sigma)\left(\frac{x\sigma}{\mu}-\frac{1}{2}\left(\frac{x\sigma}{\mu}\right)^2+\dots\right) \\
&\quad -(n-2(\mu+x\sigma))\left(-2\frac{x\sigma}{n-2\mu}-\frac{1}{2}\left(2\frac{x\sigma}{n-2\mu}\right)^2+\dots\right) \\
&= (n-(\mu+x\sigma))\left(-\frac{x\sigma}{n\frac{(\phi+1)}{(\phi+2)}}-\frac{1}{2}\left(\frac{x\sigma}{n\frac{(\phi+1)}{(\phi+2)}}\right)^2+\dots\right) \\
&\quad -(\mu+x\sigma)\left(\frac{x\sigma}{\frac{n}{\phi+2}}-\frac{1}{2}\left(\frac{x\sigma}{\frac{n}{\phi+2}}\right)^2+\dots\right) \\
&\quad -(n-2(\mu+x\sigma))\left(-\frac{2x\sigma}{n\frac{\phi}{\phi+2}}-\frac{1}{2}\left(\frac{2x\sigma}{n\frac{\phi}{\phi+2}}\right)^2+\dots\right) \\
&= \frac{x\sigma}{n}n\left(-\left(1-\frac{1}{\phi+2}\right)\frac{(\phi+2)}{(\phi+1)}-1+2\left(1-\frac{2}{\phi+2}\right)\frac{\phi+2}{\phi}\right) \\
&\quad -\frac{1}{2}\left(\frac{x\sigma}{n}\right)^2 n\left(-2\frac{\phi+2}{\phi+1}+\frac{\phi+2}{\phi+1}+2(\phi+2)-(\phi+2)+4\frac{\phi+2}{\phi}\right) \\
&\quad +O\left(n\,(x\sigma/n)^3\right)
\end{aligned}$$

(Sketch of the) Proof of Gaussianity

$$\begin{aligned}
\log(S_n) &= \frac{x\sigma}{n} n \left(-\frac{\phi+1}{\phi+2}\frac{\phi+2}{\phi+1} - 1 + 2\frac{\phi}{\phi+2}\frac{\phi+2}{\phi} \right) \\
&\quad -\frac{1}{2}\left(\frac{x\sigma}{n}\right)^2 n(\phi+2)\left(-\frac{1}{\phi+1} + 1 + \frac{4}{\phi}\right) \\
&\quad +O\left(n\left(\frac{x\sigma}{n}\right)^3\right) \\
&= -\frac{1}{2}\frac{(x\sigma)^2}{n}(\phi+2)\left(\frac{3\phi+4}{\phi(\phi+1)} + 1\right) + O\left(n\left(\frac{x\sigma}{n}\right)^3\right) \\
&= -\frac{1}{2}\frac{(x\sigma)^2}{n}(\phi+2)\left(\frac{3\phi+4+2\phi+1}{\phi(\phi+1)}\right) + O\left(n\left(\frac{x\sigma}{n}\right)^3\right) \\
&= -\frac{1}{2}x^2\sigma^2\left(\frac{5(\phi+2)}{\phi n}\right) + O\left(n\,(x\sigma/n)^3\right).
\end{aligned}$$

(Sketch of the) Proof of Gaussianity

But recall that

$$\sigma^2 = \frac{\phi n}{5(\phi+2)}.$$

Also, since $\sigma \sim n^{-1/2}$, $n\left(\frac{x\sigma}{n}\right)^3 \sim n^{-1/2}$. So for large n, the $O\left(n\left(\frac{x\sigma}{n}\right)^3\right)$ term vanishes. Thus we are left with

$$\begin{aligned} \log S_n &= -\frac{1}{2}x^2 \\ S_n &= e^{-\frac{1}{2}x^2}. \end{aligned}$$

Hence, as n gets large, the density converges to the normal distribution:

$$\begin{aligned} f_n(k)dk &= N_n S_n dk \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}x^2} \sigma dx \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx. \end{aligned}$$

$\square$

Generalizations

Generalizing from Fibonacci numbers to linearly recursive sequences with arbitrary nonnegative coefficients.

$$H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_L H_{n-L+1}, \ n \geq L$$

with $H_1 = 1$, $H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_n H_1 + 1, \ n < L$, coefficients $c_i \geq 0$; $c_1, c_L > 0$ if $L \geq 2$; $c_1 > 1$ if $L = 1$.

- Zeckendorf: Every positive integer can be written uniquely as $\sum a_i H_i$ with natural constraints on the a_i's (e.g. cannot use the recurrence relation to remove any summand).
- Lekkerkerker
- Central Limit Type Theorem

Generalizing Lekkerkerker

Generalized Lekkerkerker's Theorem

The average number of summands in the generalized Zeckendorf decomposition for integers in $[H_n, H_{n+1})$ tends to $Cn + d$ as $n \to \infty$, where $C > 0$ and d are computable constants determined by the c_i's.

$$C = -\frac{y'(1)}{y(1)} = \frac{\sum_{m=0}^{L-1}(s_m + s_{m+1} - 1)(s_{m+1} - s_m)y^m(1)}{2\sum_{m=0}^{L-1}(m+1)(s_{m+1} - s_m)y^m(1)}.$$

$$s_0 = 0, s_m = c_1 + c_2 + \cdots + c_m.$$

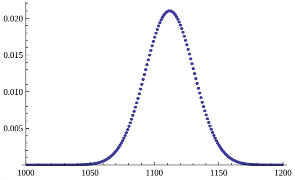
$y(x)$ is the root of $1 - \sum_{m=0}^{L-1}\sum_{j=s_m}^{s_{m+1}-1} x^j y^{m+1}$.

$y(1)$ is the root of $1 - c_1 y - c_2 y^2 - \cdots - c_L y^L$.

Central Limit Type Theorem

Central Limit Type Theorem

As $n \to \infty$, the distribution of the number of summands, i.e., $a_1 + a_2 + \cdots + a_m$ in the generalized Zeckendorf decomposition $\sum_{i=1}^{m} a_i H_i$ for integers in $[H_n, H_{n+1})$ is Gaussian.



Example: the Special Case of $L = 1$, $c_1 = 10$

$H_{n+1} = 10H_n$, $H_1 = 1$, $H_n = 10^{n-1}$.

- Legal decomposition is decimal expansion: $\sum_{i=1}^{m} a_i H_i$:
  $a_i \in \{0, 1, \dots, 9\}$ $(1 \le i < m)$, $a_m \in \{1, \dots, 9\}$.
- For $N \in [H_n, H_{n+1})$, $m = n$, i.e., first term is $a_n H_n = a_n 10^{n-1}$.
- A_i: the corresponding random variable of a_i.
  The A_i's are independent.
- For large n, the contribution of A_n is immaterial.
  A_i $(1 \le i < n)$ are identically distributed random variables
  with mean 4.5 and variance 8.25.
- Central Limit Theorem: $A_2 + A_3 + \cdots + A_n \to$ Gaussian
  with mean $4.5n + O(1)$
  and variance $8.25n + O(1)$.

Generating Function (Example: Binet's Formula)

Binet's Formula

$$F_1 = F_2 = 1;\ F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{-1+\sqrt{5}}{2}\right)^n\right].$$

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&\Rightarrow g(x) = x/(1 - x - x^2).
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Partial Fraction Expansion (Example: Binet's Formula)

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Coefficient of x^n (power series expansion):

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{-1+\sqrt{5}}{2}\right)^n\right]$$ - Binet's Formula!

(using geometric series: $\frac{1}{1-r} = 1 + r + r^2 + r^3 + \cdots$).

Differentiating Identities and Method of Moments

- Differentiating identities

  Example: Given a random variable X such that

  $\Pr(X=1)=\frac{1}{2}$, $\Pr(X=2)=\frac{1}{4}$, $\Pr(X=3)=\frac{1}{8}$, ...

  then what's the mean of X (i.e., $E[X]$)?

  Solution: Let $f(x)=\frac{1}{2}x+\frac{1}{4}x^2+\frac{1}{8}x^3+\cdots=\frac{1}{1-x/2}-1.$

  $$f'(x)=1\cdot\frac{1}{2}+2\cdot\frac{1}{4}x+3\cdot\frac{1}{8}x^2+\cdots.$$

  $$f'(1)=1\cdot\frac{1}{2}+2\cdot\frac{1}{4}+3\cdot\frac{1}{8}+\cdots=E[X].$$

- Method of moments: Random variables $X_1, X_2, \dots$.

  If ℓ^{th} moments $E[X_n^\ell]$ converges those of standard normal then X_n converges to a Gaussian.

  Standard normal distribution:

  $2m^{\text{th}}$ moment: $(2m-1)!!=(2m-1)(2m-3)\cdots 1$,

  $(2m-1)^{\text{th}}$ moment: 0.

New Approach: Case of Fibonacci Numbers

$p_{n,k} = \#\{N \in [F_n, F_{n+1})$: the Zeckendorf decomposition of N has exactly k summands$\}$.

- Recurrence relation:
  $N \in [F_{n+1}, F_{n+2})$: $N = F_{n+1} + F_t + \cdots$, $t \leq n - 1$.

$$p_{n+1,k+1} \quad = \quad p_{n-1,k} + p_{n-2,k} + \cdots$$

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- Generating function: $\sum_{n,k>0} p_{n,k} x^k y^n = \frac{y}{1-y-xy^2}$.
- Partial fraction expansion:

$$\frac{y}{1-y-xy^2} = -\frac{y}{y_1(x) - y_2(x)}\left(\frac{1}{y-y_1(x)} - \frac{1}{y-y_2(x)}\right)$$

where $y_1(x)$ and $y_2(x)$ are the roots of $1 - y - xy^2 = 0$.

Coefficient of y^n: $g(x) = \sum_{k>0} p_{n,k} x^k$.

New Approach: Case of Fibonacci Numbers (Continued)

K_n: the corresponding random variable associated with k.
$g(x) = \sum_{k>0} p_{n,k} x^k$.

- Differentiating identities:

  $g(1) = \sum_{k>0} p_{n,k} = F_{n+1} - F_n$,
  $g'(x) = \sum_{k>0} k p_{n,k} x^{k-1}$, $g'(1) = g(1) E[K_n]$,
  $(xg'(x))' = \sum_{k>0} k^2 p_{n,k} x^{k-1}$,
  $(xg'(x))' \,|_{x=1} = g(1) E[K_n^2]$,
  $\left(x\,(xg'(x))'\right)' \,|_{x=1} = g(1) E[K_n^3]$, ...

  Similar results hold for the centralized K_n:
  $K_n' = K_n - E[K_n]$.

- Method of moments (for normalized K_n'):

  $E[(K_n')^{2m}]/(SD(K_n'))^{2m} \to (2m-1)!!$,

  $E[(K_n')^{2m-1}]/(SD(K_n'))^{2m-1} \to 0$. $\Rightarrow K_n \to$ Gaussian.

New Approach: General Case

Let $p_{n,k} = \#\{N \in [H_n, H_{n+1})$: the generalized Zeckendorf decomposition of N has exactly k summands$\}$.

- Recurrence relation:

  Fibonacci: $p_{n+1,k+1} = p_{n,k+1} + p_{n,k}$.

  General: $p_{n+1,k} = \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} p_{n-m,k-j}$.

  where $s_0 = 0, s_m = c_1 + c_2 + \cdots + c_m$.

- Generating function:

  Fibonacci: $\frac{y}{1-y-xy^2}$.

  General:

$$\frac{\sum_{n\leq L} p_{n,k} x^k y^n - \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} x^j y^{m+1} \sum_{n<L-m} p_{n,k} x^k y^n}{1 - \sum_{m=0}^{L-1} \sum_{j=s_m}^{s_{m+1}-1} x^j y^{m+1}}.$$

New Approach: General Case (Continued)

- Partial fraction expansion:

  Fibonacci: $-\frac{y}{y_1(x)-y_2(x)}\left(\frac{1}{y-y_1(x)}-\frac{1}{y-y_2(x)}\right)$.

  General:

$$-\frac{1}{\sum_{j=s_{L-1}}^{s_L-1} x^j}\sum_{i=1}^{L}\frac{B(x,y)}{(y-y_i(x))\prod_{j\neq i}(y_j(x)-y_i(x))}.$$

$$B(x,y)=\sum_{n\leq L} p_{n,k}x^k y^n-\sum_{m=0}^{L-1}\sum_{j=s_m}^{s_{m+1}-1} x^j y^{m+1}\sum_{n<L-m} p_{n,k}x^k y^n,$$

$$y_i(x)\text{: root of } 1-\sum_{m=0}^{L-1}\sum_{j=s_m}^{s_{m+1}-1} x^j y^{m+1}=0.$$

  Coefficient of y^n: $g(x)=\sum_{n,k>0} p_{n,k}x^k$.

- Differentiating identities
- Method of moments: implies $K_n \to$ Gaussian.

Gaps in the Bulk

Distribution of Gaps

For $F_{r_1} + F_{r_2} + \cdots + F_{r_n}$, the gaps are the differences $r_n - r_{n-1}, r_{n-1} - r_{n-2}, \dots, r_2 - r_1$.

Example: For $F_1 + F_8 + F_{18}$, the gaps are 7 and 10.

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Can ask similar questions about binary or other expansions: $2012 = 2^{10} + 2^9 + 2^8 + 2^7 + 2^6 + 2^4 + 2^3 + 2^2$.

Main Result

Theorem (Distribution of Bulk Gaps (SMALL 2012))

Let $H_{n+1} = c_1 H_n + c_2 H_{n-1} + \cdots + c_L H_{n+1-L}$ be a positive linear recurrence of length L where $c_i \geq 1$ for all $1 \leq i \leq L$. Then

$$P(j) = \begin{cases} 1 - \left(\frac{a_1}{C_{Lek}}\right)(2\lambda_1^{-1} + a_1^{-1} - 3) & : j = 0 \\ \lambda_1^{-1}\left(\frac{1}{C_{Lek}}\right)(\lambda_1(1 - 2a_1) + a_1) & : j = 1 \\ (\lambda_1 - 1)^2 \left(\frac{a_1}{C_{Lek}}\right) \lambda_1^{-j} & : j \geq 2. \end{cases}$$

Special Cases

Theorem (Base B Gap Distribution (SMALL 2011))

For base B decompositions, $P(0) = \frac{(B-1)(B-2)}{B^2}$, and for $k \geq 1$, $P(k) = c_B B^{-k}$, with $c_B = \frac{(B-1)(3B-2)}{B^2}$.

Theorem (Zeckendorf Gap Distribution (SMALL 2011))

For Zeckendorf decompositions, $P(k) = 1/\phi^k$ for $k \geq 2$, with $\phi = \frac{1+\sqrt{5}}{2}$ the golden mean.

Proof of Bulk Gaps for Fibonacci Sequence

Lekkerkerker $\Rightarrow$ total number of gaps $\sim F_{n-1}\frac{n}{\phi^2+1}$.

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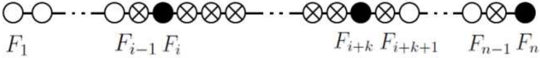
$$P(k) = \lim_{n\to\infty} \frac{\sum_{i=1}^{n-k} X_{i,i+k}}{F_{n-1}\frac{n}{\phi^2+1}}.$$

Calculating $X_{i,i+k}$

How many decompositions contain a gap from F_i to F_{i+k}?



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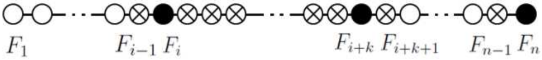

For the indices less than i: F_{i-1} choices. Why? Have F_i as largest summand and follows by Zeckendorf: $\#[F_i, F_{i+1}) = F_{i+1} - F_i = F_{i-1}$.

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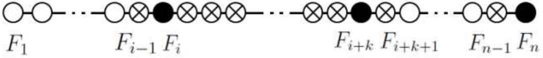

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For the indices greater than $i + k$: $F_{n-k-i-2}$ choices. Why? Shift. Choose summands from $\{F_1, \dots, F_{n-k-i+1}\}$ with $F_1, F_{n-k-i+1}$ chosen. Decompositions with largest summand $F_{n-k-i+1}$ minus decompositions with largest summand F_{n-k-i}.

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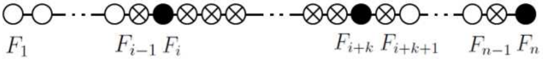

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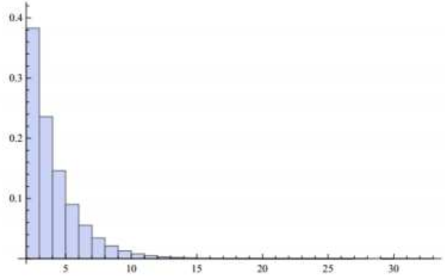
So total number of choices is $F_{n-k-2-i}F_{i-1}$.

Determining $P(k)$

Recall

$$P(k) = \lim_{n\to\infty} \frac{\sum_{i=1}^{n-k} X_{i,i+k}}{F_{n-1}\frac{n}{\phi^2+1}} = \lim_{n\to\infty} \frac{\sum_{i=1}^{n-k} F_{n-k-2-i}F_{i-1}}{F_{n-1}\frac{n}{\phi^2+1}}.$$

Use Binet's formula. Sums of geometric series: $P(k) = 1/\phi^k$.



Kentucky Sequence and Quilts
with Minerva Catral, Pari Ford, Pamela Harris & Dawn Nelson

Kentucky Sequence

Rule: (s, b)-Sequence: Bins of length b, and:

- cannot take two elements from the same bin, and
- if have an element from a bin, cannot take anything from the first s bins to the left or the first s to the right.

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Kentucky: These are $(s, b) = (1, 2)$.

$[1,\ \ 2],\ \ [3,\ \ 4],\ \ [5,\ \ 8],\ \ [11,$

Kentucky Sequence

Rule: (s, b)-Sequence: Bins of length b, and:

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- $a_{2n} = 2^n$ and $a_{2n+1} = \frac{1}{3}(2^{2+n} - (-1)^n)$:
  $a_{n+1} = a_{n-1} + 2a_{n-3}, a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 4.$

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- $a_{n+1} = a_{n-1} + 2a_{n-3}$: New as leading term 0.

What’s in a name?

www.11points.com/Dating-Sex/11_State_Laws_About_Marrying_Your_Cousins,_From_Strictest_to_Loosest

arxiv PeopleSoft AMSGradBlog Glow 317 331 389 wiki Sched FR HealthHub Survey LISTSERV Directory SMALL PMEJ2 Pr

fall under the "well, we *have* to invite your Uncle Bernie" umbrella!

1. ***A ban on marriages between first cousins and first cousins once removed:*** **Indiana, Kentucky, Nevada, Ohio, Washington and Wisconsin.** These states have the strictest laws (especially Kentucky, Nevada and Ohio, as you'll see the others below all make exceptions). In these six states, you can't marry your first cousin OR first cousin once removed (your first cousin once removed is the child of your first cousin).

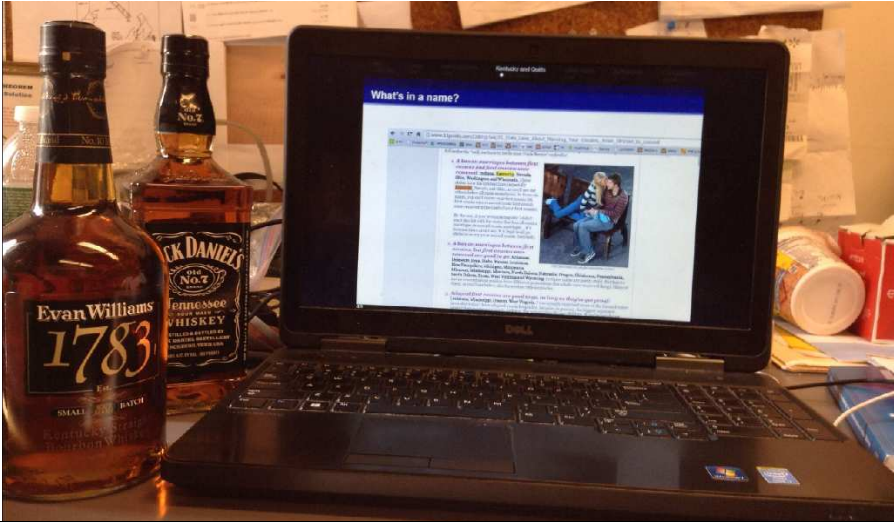
   By the way, if you're wondering why I didn't start this list with the states that ban all cousin marriages or second cousin marriages... it's because there aren't any. It is legal in all 50 states to marry your second cousin. Seriously.

I felt like these two people looked like cousins.

2. ***A ban on marriages between first cousins, but first cousins once removed are good to go:*** **Arkansas, Delaware, Iowa, Idaho, Kansas, Louisiana, New Hampshire, Michigan, Minnesota, Missouri, Mississippi, Montana, North Dakota, Nebraska, Oregon, Oklahoma, Pennsylvania, South Dakota, Texas, West Virginia and Wyoming.** So these states are pretty strict. But they're not as worried about cousins from different generations (the whole once removed thing). Many of them, as you'll see below, also have other little loopholes.

3. ***Adopted first cousins are good to go, as long as they've got proof:*** **Louisiana, Mississippi, Oregon, West Virginia.** I was actually surprised more of the banned states from above don't have adopted cousin loopholes. Because, in general, the biggest argument against first cousin marriage is, ya know, the potential for flipper children. If you're legislating

What's in a name?



Gaussian Behavior

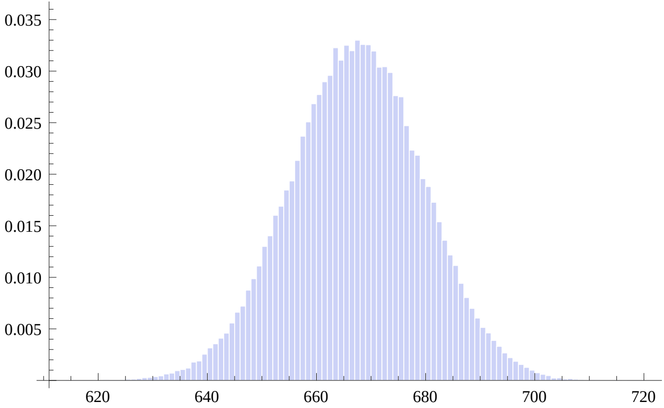


Figure: Plot of the distribution of the number of summands for 100,000 randomly chosen $m \in [1, a_{4000}) = [1, 2^{2000})$ (so m has on the order of 602 digits).

Gaps

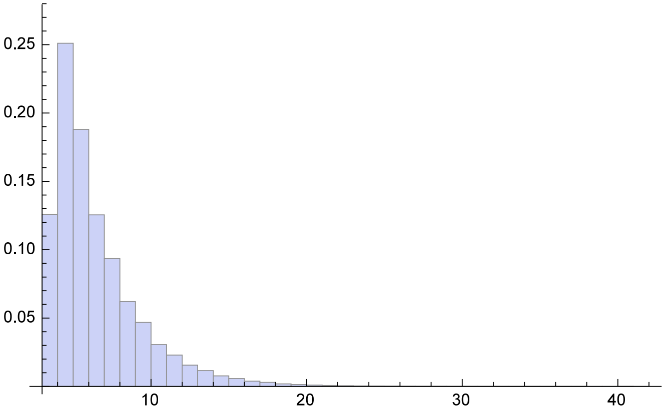


Figure: Plot of the distribution of gaps for 10,000 randomly chosen $m \in [1, a_{400}) = [1, 2^{200})$ (so m has on the order of 60 digits).

Gaps

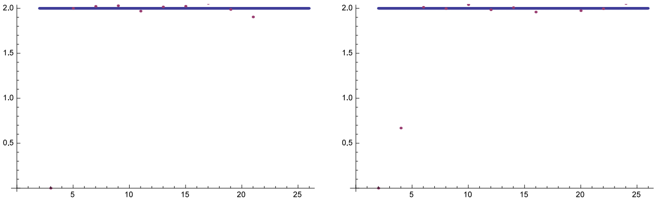
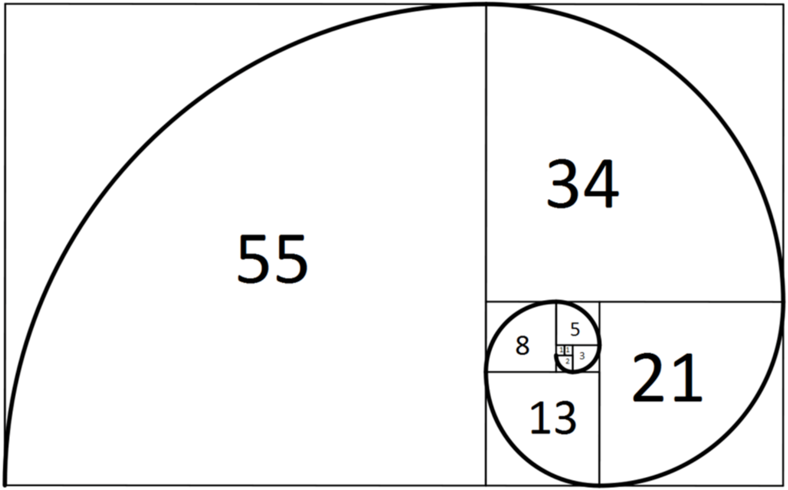


Figure: Plot of the distribution of gaps for 10,000 randomly chosen $m \in [1, a_{400}) = [1, 2^{200})$ (so m has on the order of 60 digits). Left (resp. right): ratio of adjacent even (resp odd) gap probabilities.

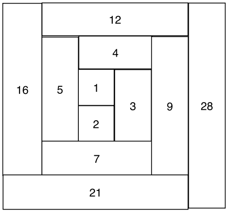
Again find geometric decay, but parity issues so break into even and odd gaps.

Fibonacci Spiral

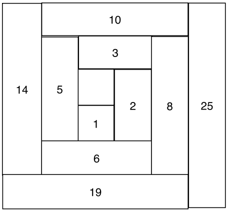
Fibonacci Spiral



The Fibonacci (or Log Cabin) Quilt: Work in Progress



1, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28, ...



1, 2, 3, 5, 6, 8, 10, 14, 19, 25, 33, ...

- $a_{n+1} = a_{n-1} + a_{n-2}$, non-uniqueness (average number of decompositions grows exponentially).
- In process of investigating Gaussianity, Gaps, $K_{\rm min}$, $K_{\rm ave}$, $K_{\rm max}$, $K_{\rm greedy}$.

Average Number of Representations

- d_n: the number of FQ-legal decompositions using only elements of $\{a_1, a_2, \dots, a_n\}$.
- c_n requires a_n to be used, b_n requires a_n *and* a_{n-2} to be used.

n	d_n	c_n	b_n	a_n
1	2	1	0	1
2	3	1	0	2
3	4	1	0	3
4	6	2	1	4
5	8	2	1	5
6	11	3	1	7
7	15	4	1	9
8	21	6	2	12
9	30	9	3	16

Table: First few terms. Find $d_n = d_{n-1} + d_{n-2} - d_{n-3} + d_{n-5} - d_{n-9}$, implying $d_{\mathrm{FQ;ave}}(n) \approx C \cdot 1.05459^n$.

Greedy Algorithm

h_n: number of integers from 1 to $a_{n+1} - 1$ where the greedy algorithm successfully terminates in a legal decomposition.

n	a_n	h_n	ρ_n
1	1	1	100.0000
2	2	2	100.0000
3	3	3	100.0000
4	4	4	100.0000
5	5	5	83.3333
6	7	7	87.5000
10	21	25	92.5926
11	28	33	91.6667
17	151	184	92.4623

Table: First few terms, yields $h_n = h_{n-1} + h_{n-5} + 1$ and percentage converges to about 0.92627.

Future Work and References

Future Research

Future Research

- Generalizing results beyond PLRS, signed decompositions, higher dimensions....
- Other systems such as f-Decompositions of Demontigny, Do, Miller and Varma.

References

References

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  `http://arxiv.org/pdf/1107.2718.pdf`.