EXTENDING THE SUPPORT IN THE 1-LEVEL DENSITY FOR FAMILIES OF DIRICHLET CHARACTERS

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ABSTRACT. We study the distribution of the zeros near the central point for following families of primitive Dirichlet characters:

(1) all primitive characters of conductor m, m a fixed prime;
(2) all primitive characters of conductor m, m an odd square-free number with r factors (r fixed);
(3) all primitive characters whose conductor is a square-free odd integer $m \in [N, 2N]$.

For these families the 1-level densities agree with the Unitary Group for even Schwartz functions $\widehat{\phi}$ with supp$(\widehat{\phi}) \subset (-2, 2)$. We investigate the consequences of conjectures about the modulus dependence in the error terms in the distribution of primes in congruence classes. We show how some natural conjectures imply the 1-level densities agree with Unitary matrices for arbitrary support. Further, we show how some weaker conjectures still give an improvement over $(-2, 2)$, allowing support up to $(-4, 4)$. **These are very rough notes.**

1. INTRODUCTION

Assuming GRH, the non-trivial zeros of any nice L-function lie on its critical line, and therefore it is possible to investigate the statistics of its normalized zeros. Let $\mathcal{F}$ be a family of L-functions, and $\mathcal{F}_N$ the subset with analytic conductors N (or at most N). For example, we can study L-functions arising from Dirichlet characters [Ru, HR], families of elliptic curves [Mil, Yo], weight k level N cuspidal newforms [ILS, Ro, HM], L-functions attached to quadratic fields [FI] and symmetric powers of GL_2 automorphic representations [Gü] to name a few. Katz and Sarnak [KS1, KS2] have conjectured that the behavior of zeros near the central point $s = \frac{1}{2}$ in a family of L-functions (as the conductors tend to infinity) agrees with the behavior of eigenvalues near 1 of a classical compact group (unitary, symplectic, or some flavor of orthogonal).

Let ϕ be an even Schwartz test function on $\mathbb{R}$ whose Fourier transform

$$\widehat{\phi}(y) \ = \ \int_{-\infty}^{\infty} \phi(x) e^{-2\pi i x y} dx \tag{1.1}$$

has compact support. Let $\mathcal{F}_N$ be a (finite) family of L-functions satisfying GRH. The 1-level density associated to $\mathcal{F}_N$ is defined by

$$D_{1,\mathcal{F}_N}(\phi) \ = \ \frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} \sum_j \phi\left(\frac{\log c_f}{2\pi} \gamma_f^{(j)}\right), \tag{1.2}$$

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where $\frac{1}{2}+i\gamma_f^{(j)}$ runs through the non-trivial zeros of $L(s,f)$. Here c_f is the "analytic conductor" of f, and gives the natural scale for the low zeros. As ϕ is Schwartz, only low-lying zeros (i.e., zeros within a distance $\ll \frac{1}{\log c_f}$ of the central point $s=\frac{1}{2}$) contribute significantly. Thus the 1-level density can help identify the symmetry type of the family.

Based in part on the function-field analysis where $G(\mathcal{F})$ is the monodromy group associated to the family $\mathcal{F}$, it is conjectured that for each reasonable irreducible family of L-functions there is an associated symmetry group $G(\mathcal{F})$ (one of the following five: unitary U, symplectic USp, orthogonal O, SO(even), SO(odd)), and that the distribution of critical zeros near $\frac{1}{2}$ mirrors the distribution of eigenvalues near 1. The five groups have distinguishable 1-level densities.

To evaluate (1.2), one applies the explicit formula, converting sums over zeros to sums over primes. By [KS1], the 1-level densities for the classical compact groups are

$$\begin{aligned}
W_{1,\text{SO(even)}}(x) &= K_1(x,x)\\
W_{1,\text{SO(odd)}}(x) &= K_{-1}(x,x)+\delta(x)\\
W_{1,\text{O}}(x) &= \tfrac{1}{2}W_{1,\text{SO(even)}}(x)+\tfrac{1}{2}W_{1,\text{SO(odd)}}(x)\\
W_{1,\text{U}}(x) &= K_0(x,x)\\
W_{1,\text{USp}}(x) &= K_{-1}(x,x)
\end{aligned}$$

where $K(y)=\frac{\sin\pi y}{\pi y}$, $K_\epsilon(x,y)=K(x-y)+\epsilon K(x+y)$ for $\epsilon=0,\pm1$, and $\delta(x)$ is the Dirac delta functional. It is often more convenient to work with the Fourier transforms of the densities:

$$\begin{aligned}
\widehat{W}_{1,\text{SO(even)}}(u) &= \delta(u)+\tfrac{1}{2}I(u)\\
\widehat{W}_{1,\text{SO(odd)}}(u) &= \delta(u)-\tfrac{1}{2}I(u)+1\\
\widehat{W}_{1,\text{O}}(u) &= \delta(u)+\tfrac{1}{2}\\
\widehat{W}_{1,U}(u) &= \delta(u)\\
\widehat{W}_{1,\text{USp}}(u) &= \delta(u)-\tfrac{1}{2}I(u),
\end{aligned}$$

where $I(u)$ is the characteristic function of $[-1,1]$.

We study families of Dirichlet characters below. Hughes and Rudnick [HR] show that for the family of primitive characters with prime conductor, the 1-level density agrees with Unitary matrices for test functions ϕ with $\text{supp}(\widehat{\phi})\in[-2,2]$. Our goal is to show how reasonable conjectures allow us to increase the support. In this regard our work is similar to [ILS], where they show that if a classical exponential sum over primes has some cancellation, then the 1-level density of weight k level 1 cusp forms (split by sign) agrees with the corresponding orthogonal group for $\text{supp}(\widehat{\phi})\subset(-\frac{22}{9},\frac{22}{9})$. For us, the corresponding quantities involve the modulus dependence in the error terms in primes in residue classes, and relates how natural conjectures on the distributions of primes can be used to provide further support for the density conjectures. We sketch two cases below.

Let q either be prime or range over primes in $[N, 2N]$. Let

$$\begin{aligned}\psi(x) &= \sum_{n\le x}\Lambda(n)\\ \psi(x,q,a) &= \sum_{\substack{n\le x\\ n\equiv a \bmod q}}\Lambda(n)\\ E(x,q,a) &= \psi(x,q,a)-\frac{\psi(x)}{\phi(q)}.\end{aligned} \tag{1.3}$$

If we assume GRH, we have (we could replace ϵ with powers of $\log$ below) that

$$\begin{aligned}\psi(x) &= x+O(x^{\frac12+\epsilon})\\ \psi(x,q,a) &= \frac{\psi(x)}{\phi(q)}+O(x^{\frac12}\cdot(xq)^\epsilon)\\ E(x,q,a) &= O(x^{\frac12}\cdot(xq)^\epsilon).\end{aligned} \tag{1.4}$$

Probabilistic arguments suggest that $E(x,q,a)$ should be much smaller. Expecting square-root cancellation, we have $\phi(q)$ residue classes. Note $\sum_{a=1}^{q-1}E(x,q,a) = O(x^{\frac12+\epsilon})$. If the error of size $x^{\frac12+\epsilon}$ is spread among these $\phi(q)$ classes equally, we expect each $\psi(x,q,a)$ to be of size $\frac{\psi(x)}{\phi(q)}$ with errors of size $\sqrt{\frac{x}{\phi(q)}}\cdot(xq)^\epsilon$; see [Mon1]. It is by gaining some savings in q in the error $E(x,q,a)$ that we can increase the support for families of Dirichlet L-functions. Explicitly, consider the following weaker version of Montgomery's Conjecture:

Conjecture 1.1. *There is a $\theta\in[0,\frac12)$ such that for q prime*

$$E(x,q,1) \ \ll\ q^\theta\cdot\sqrt{\frac{x}{\phi(q)}}\cdot(xq)^\epsilon. \tag{1.5}$$

Conjecture 1.1 implies

Theorem 1.2. *Assume Conjecture 1.1 holds. Then the 1-level density agrees with Unitary matrices for any test function of finite support.*

Consider the total variance

$$V(x,q) \ =\ \sum_{\substack{a=1\\(a,q)=1}}^{q}\left|\psi(x,q,a)-\frac{x}{\phi(q)}\right|^2; \tag{1.6}$$

dividing by $\frac{1}{\phi(q)}$ would give the average variance.

Goldston and Vaughan [GV] have shown that under GRH,

$$\sum_{q\le Q}V(x,q) \ =\ Qx\log Q-cxQ+O\left(Q^2(x/Q)^{\frac14+\epsilon}+x^{\frac32}(\log 2x)^{\frac52}(\log\log 3x)^2\right). \tag{1.7}$$

We shall only use such results when q is prime. As each term is non-negative, this yields

Theorem 1.3 (Goldston-Vaughan)**.** *For q prime, assuming GRH we have*

$$\sum_{q\le Q} V(x,q) \ \ll\ Qx\log Q + Q^{\frac{7}{4}}x^{\frac{1}{4}+\epsilon} + x^{\frac{3}{2}+\epsilon} \tag{1.8}$$

Note we subtract $\frac{x}{\phi(q)}$ and not $\frac{\psi(x)}{\phi(q)}$ in the definition of $V(x,q)$, though assuming RH and $Q \gg x^{2\epsilon}$ either gives the same results in terms of increasing the support (the calculation is straightforward; see Lemma A.1 for details).

If each $E(x,q,a)$ were of size $\sqrt{\frac{x}{\phi(q)}}\cdot(xq)^\epsilon$, we would expect

$$V(x,q) \ \sim\ x\cdot(xq)^\epsilon \tag{1.9}$$

and

$$\sum_{q\le Q} V(x,q) \ \sim\ Qx\cdot(xQ)^\epsilon. \tag{1.10}$$

In fact, Hooley has conjectured that (1.9) holds for some unspecified range of q (replacing ϵ with logarithms), and we shall show later that such a result also leads to improving the support of the test function.

Instead of conjecturing bounds on individual $E(u,m,a)$ we consider the relation of $E(u,m,1)$ to the total variance.

Conjecture 1.4. *There exists a $\theta\in[0,1]$ such that for prime $m\in[N,2N]$ with $N^2 \ll u \ll N^{4-2\theta}$,*

$$\sum_{\substack{m=N\\ m\ \text{prime}}}^{2N} E(u,m,1)^2 \ \ll\ N^\theta\cdot\frac{1}{N}\sum_{\substack{m=N\\ m\ \text{prime}}}^{2N}\sum_{\substack{a=1\\ (a,m)=1}}^{m} E(u,m,a)^2. \tag{1.11}$$

Conjecture 1.4 is trivially true for $\theta = 1$, and while it is unlikely to be true for $\theta = 0$, it is reasonable to expect it to hold for $\theta = \epsilon$ (for any $\epsilon > 0$). What we need is some control over biases of primes congruent to $1 \bmod m$. For the residue class $a \bmod m$, $E(u,m,a)^2$ is the variance; the above conjecture can be interpreted as bounding $E(u,m,1)^2$ in terms of the average variance. Interestingly, $\theta = 1$ recovers the 1-level density result of support in $(-2,2)$. **ADD REMARKS THAT ENOUGH TO HAVE THE ERRORS OF QUADRATIC RESIDUES OF THE SAME MAGNITUDE. BY** [RubSa] **WE KNOW THE QUAD RES AND NON-RES BEHAVE DIFFERENTLY MOST OF THE TIME....** From Conjecture 1.4 and the results of Goldston and Vaughan we have

Theorem 1.5. *Let $\mathcal{F}_N$ be the family of primitive characters with prime conductor $m\in[N,2N]$. The 1-level density for $\mathcal{F}_N$ holds for test functions whose Fourier transforms are supported in $(-4+2\theta, 4-2\theta)$.*

In §2 we quickly review the proof that the 1-level density for primitive Dirichlet characters with prime conductor m agrees with Unitary matrices for test functions supported in $(-2,2)$; in Appendices B and C we show how to extend these results to odd square-free m and then all odd square-free numbers in $[N,2N]$. In §3 we show how the above (and other) conjectures on modulus dependence on the errors of primes in residue classes lead to increasing the support of the test functions, as well as commenting in what sense Conjecture 1.4 is weaker than Conjecture 1.1.

2. DIRICHLET CHARACTERS FROM A PRIME CONDUCTOR

2.1. **Review of Dirichlet Characters.** If m is prime, then $(\mathbb{Z}/m\mathbb{Z})^*$ is cyclic of order $m-1$ with generator g (so any element is of the form g^a for some a). Let $\zeta_{m-1} = e^{2\pi i/(m-1)}$. The principal character χ_0 is given by

$$\chi_0(k) \;=\; \begin{cases} 1 & \text{if } (k,m)=1 \\ 0 & \text{if } (k,m)>1. \end{cases} \tag{2.1}$$

Each of the $m-2$ primitive characters are determined (because they are multiplicative) once their action on a generator g is specified. As each $\chi : (\mathbb{Z}/m\mathbb{Z})^* \to \mathbb{C}^*$, for each χ there exists an l such that $\chi(g) = \zeta_{m-1}^l$. Hence for each $l, 1 \le l \le m-2$ we have

$$\chi_l(k) \;=\; \begin{cases} \zeta_{m-1}^{la} & \text{if } k \equiv g^a \bmod m \\ 0 & \text{if } (k,m)>0. \end{cases} \tag{2.2}$$

Let χ be a primitive character modulo m. Set

$$c(m,\chi) \;=\; \sum_{k=0}^{m-1} \chi(k) e^{2\pi i k/m}; \tag{2.3}$$

$c(m,\chi)$ is a Gauss sum of modulus $\sqrt{m}$. The associated L-function and its analytic continuation are given by

$$\begin{aligned} L(s,\chi) &= \prod_p (1-\chi(p)p^{-s})^{-1} \\ \Lambda(s,\chi) &= \pi^{-\frac{1}{2}(s+\epsilon)} \Gamma\left(\frac{s+\epsilon}{2}\right) m^{\frac{1}{2}(s+\epsilon)} L(s,\chi), \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} \epsilon &= \begin{cases} 0 & \text{if } \chi(-1) = \ \ 1 \\ 1 & \text{if } \chi(-1) = -1 \end{cases} \\ \Lambda(s,\chi) &= (-i)^{\epsilon} \frac{c(m,\chi)}{\sqrt{m}} \Lambda(1-s,\bar{\chi}). \end{aligned} \tag{2.5}$$

Let ϕ be an even Schwartz function with compact support, say contained in the interval $(-\sigma,\sigma)$, and let χ be a non-trivial primitive Dirichlet character of conductor m. The explicit formula (see [RS] for a proof) gives

$$\begin{aligned} \sum_\gamma \phi\left(\gamma \frac{\log(\frac{m}{\pi})}{2\pi}\right) &= \int_{-\infty}^{\infty} \phi(y)dy \\ &\quad - \sum_p \frac{\log p}{\log(m/\pi)} \widehat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right) [\chi(p)+\overline{\chi}(p)] p^{-\frac{1}{2}} \\ &\quad - \sum_p \frac{\log p}{\log(m/\pi)} \widehat{\phi}\left(2\frac{\log p}{\log(m/\pi)}\right) [\chi^2(p)+\overline{\chi}^2(p)] p^{-1} \\ &\quad + O\left(\frac{1}{\log m}\right), \end{aligned} \tag{2.6}$$

where we are assuming GRH to write the zeros as $\frac{1}{2}+i\gamma$, $\gamma\in\mathbb{R}$. Sometimes it is more convenient to normalize the zeros not by the logarithm of the analytic conductor but rather by something that is the same to first order. Explicitly, for $m\in[N,2N]$ we have

$$\begin{aligned}\sum_{\gamma}\phi\left(\gamma\frac{\log(\frac{N}{\pi})}{2\pi}\right) \quad = \quad & \frac{\log(m/\pi)}{\log(N/\pi)}\int_{-\infty}^{\infty}\phi(y)dy \\ & -\sum_p \frac{\log p}{\log(N/\pi)}\widehat{\phi}\left(\frac{\log p}{\log(N/\pi)}\right)[\chi(p)+\overline{\chi}(p)]p^{-\frac{1}{2}} \\ & -\sum_p \frac{\log p}{\log(N/\pi)}\widehat{\phi}\left(2\frac{\log p}{\log(N/\pi)}\right)[\chi^2(p)+\overline{\chi}^2(p)]p^{-1} \\ & +O\left(\frac{1}{\log N}\right); \qquad (2.7)\end{aligned}$$

for any subset $\mathcal{N}$ of $[N,2N]$ we have

$$\frac{1}{|\mathcal{N}|}\sum_{m\in\mathcal{N}}\frac{\log(m/\pi)}{\log(N/\pi)} \;=\; 1+O\left(\frac{1}{\log N}\right). \qquad (2.8)$$

Consider $\mathcal{F}_m$, the family of primitive characters modulo a prime m. There are $m-2$ elements in this family, given by $\{\chi_l\}_{1\le l\le m-2}$. As each χ_l is primitive, we may use the Explicit Formula. To determine the 1-level density we must evaluate

$$\begin{aligned}\int_{-\infty}^{\infty}\phi(y)dy \quad - \quad & \frac{1}{m-2}\sum_{\chi\neq\chi_0}\sum_p\frac{\log p}{\log(m/\pi)}\widehat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right)[\chi(p)+\overline{\chi}(p)]p^{-\frac{1}{2}} \\ - \quad & \frac{1}{m-2}\sum_{\chi\neq\chi_0}\sum_p\frac{\log p}{\log(m/\pi)}\widehat{\phi}\left(2\frac{\log p}{\log(m/\pi)}\right)[\chi^2(p)+\overline{\chi}^2(p)]p^{-1} \\ + \quad & O\left(\frac{1}{\log m}\right). \qquad (2.9)\end{aligned}$$

Definition 2.1 (First and Second Sums)**.** *We call the two sums above the First Sum and the Second Sum (respectively), denoting them by* $S_1(m)$ *and* $S_2(m)$*.*

The Density Conjecture states that the family average should converge to the Unitary Density:

$$\int_{-\infty}^{\infty}\phi(y)dy. \qquad (2.10)$$

We prove this for $\widehat{\phi}$ with suitable support, and show how various natural conjectures allow us to increase the support.

2.2. **The First Sum.** We must analyze (for m prime)

$$S_1(m) \;=\; \frac{1}{m-2}\sum_{\chi\neq\chi_0}\sum_p\frac{\log p}{\log(m/\pi)}\widehat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right)[\chi(p)+\overline{\chi}(p)]p^{-\frac{1}{2}}. \qquad (2.11)$$

Since

$$\sum_{\chi} \chi(k) \ = \ \begin{cases} m-1 & \text{if } k \equiv 1 \bmod m \\ 0 & \text{otherwise.} \end{cases} \tag{2.12}$$

we have for any prime $p \neq m$

$$\sum_{\chi \neq \chi_0} \chi(p) \ = \ \begin{cases} m-2 & \text{if } p \equiv 1 \bmod m \\ -1 & \text{otherwise.} \end{cases} \tag{2.13}$$

Let

$$\delta_m(p,1) \ = \ \begin{cases} 1 & \text{if } p \equiv 1 \bmod m \\ 0 & \text{otherwise.} \end{cases} \tag{2.14}$$

The contribution to the sum from $p = m$ is zero; if instead we substitute -1 for $\sum_{\chi \neq \chi_0} \chi(m)$, our error is $O\left(\frac{1}{\log m}\right)$ and hence negligible.

We now calculate $S_1(m)$, suppressing the errors of $O\left(\frac{1}{\log m}\right)$; $\widehat{\phi}$ will be an even Schwartz function with support in $(-\sigma, \sigma)$.

$$\begin{aligned} S_1(m) \ &= \ \frac{1}{m-2} \sum_{\chi \neq \chi_0} \sum_{p} \frac{\log p}{\log(m/\pi)} \widehat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right) [\chi(p) + \overline{\chi}(p)] p^{-\frac{1}{2}} \\ &= \ \frac{1}{m-2} \sum_{p} \frac{\log p}{\log(m/\pi)} \widehat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right) \sum_{\chi \neq \chi_0} [\chi(p) + \overline{\chi}(p)] p^{-\frac{1}{2}} \\ &= \ \frac{2}{m-2} \sum_{p} \frac{\log p}{\log(m/\pi)} \widehat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right) p^{-\frac{1}{2}} (-1 + (m-1)\delta_m(p,1)) \\ &= \ \frac{-2}{m-2} \sum_{p}^{m^\sigma} \frac{\log p}{\log(m/\pi)} \widehat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right) p^{-\frac{1}{2}} \\ &\quad + \ 2\frac{m-1}{m-2} \sum_{p \equiv 1(m)}^{m^\sigma} \frac{\log p}{\log(m/\pi)} \widehat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right) p^{-\frac{1}{2}} \\ &\ll \ \frac{1}{m} \sum_{p}^{m^\sigma} p^{-\frac{1}{2}} \ + \ \sum_{p \equiv 1(m)}^{m^\sigma} p^{-\frac{1}{2}} \\ &\ll \ \frac{1}{m} \sum_{k}^{m^\sigma} k^{-\frac{1}{2}} \ + \ \sum_{\substack{k \equiv 1(m) \\ k \geq m+1}}^{m^\sigma} k^{-\frac{1}{2}} \\ &\ll \ \frac{1}{m} \sum_{k}^{m^\sigma} k^{-\frac{1}{2}} \ + \ \frac{1}{m} \sum_{k}^{m^\sigma} k^{-\frac{1}{2}} \\ &\ll \ \frac{1}{m} m^{\sigma/2}. \end{aligned} \tag{2.15}$$

Note that we must be careful with the estimates of the second sum. Each residue class of k mod m has approximately the same sum, with the difference between two classes bounded by the first term of whichever class has the smallest element. Since we are dropping the first term $(k = 1)$, the class of $k \equiv 1(m)$ has the smallest sum of the m classes. Hence if we add all the classes and divide by m, we increase the sum, so the above arguments are valid.

Hence $S_1(m) = \frac{1}{m} m^{\sigma/2} + O\left(\frac{1}{\log m}\right)$, implying that there is no contribution from the first sum if $\sigma < 2$.

Remark 2.2. By a more careful analysis, in [HR] it is shown we may take $\sigma \leq 2$.

2.3. **The Second Sum.** We must analyze (for m prime)

$$S_2(m) = \frac{1}{m-2} \sum_{\chi \neq \chi_0} \sum_p \frac{\log p}{\log(m/\pi)} \widehat{\phi}\left(2 \frac{\log p}{\log(m/\pi)}\right) [\chi^2(p) + \overline{\chi}^2(p)] p^{-1}. \quad (2.16)$$

If $p \equiv \pm 1(m)$ then $\sum_{\chi \neq \chi_0} [\chi^2(p) + \overline{\chi}^2(p)] = 2(m-2)$. Otherwise, fix a generator g and write $p \equiv g^a(m)$. As $p \not\equiv \pm 1$, $a \not\equiv 0, \frac{m-1}{2}$ mod $(m-1)$, as $(\mathbb{Z}/m\mathbb{Z})^*$ is cyclic of order $m-1$. Hence $e^{4\pi i a/(m-1)} \neq 1$. Recall $\zeta_{m-1} = e^{2\pi i/(m-1)}$. Let $x = e^{4\pi i a/(m-1)} \neq 1$.

$$\begin{aligned} S &= \sum_{\chi \neq \chi_0} [\chi^2(p) + \overline{\chi}^2(p)] = \sum_{l=1}^{m-2} [\chi_l^2(p) + \overline{\chi}_l^2(p)] \\ &= \sum_{l=1}^{m-2} [\chi_l^2(g^a) + \overline{\chi}_l^2(g^a)] \\ &= \sum_{l=1}^{m-2} [(\chi_l(g))^{2a} + (\overline{\chi}_l(g))^{2a}] \\ &= \sum_{l=1}^{m-2} [(\zeta_{m-1}^l)^{2a} + (\zeta_{m-1}^l)^{-2a}] \\ &= \sum_{l=1}^{m-2} [(\zeta_{m-1}^{2a})^l + (\zeta_{m-1}^{-2a})^l] \\ &= \sum_{l=1}^{m-2} [x^l + (x^{-1})^l] \\ &= \frac{x-1}{1-x} + \frac{x^{-1}-1}{1-x^{-1}} = -2. \end{aligned} \quad (2.17)$$

The contribution to the sum from $p = m$ is zero; if instead we substitute -2 for $\sum_{\chi \neq \chi_0} \chi^2(m)$, our error is $O\left(\frac{1}{\log m}\right)$ and hence negligible.

Therefore

$$\sum_{\chi \neq \chi_0} [\chi^2(p) + \overline{\chi}^2(p)] = \begin{cases} 2(m-2) & p \equiv \pm 1(m) \\ -2 & p \not\equiv \pm 1(m). \end{cases} \quad (2.18)$$

Let

$$\delta_m(p, \pm) = \begin{cases} 1 & \text{if } p \equiv \pm 1 \bmod m \\ 0 & \text{otherwise} \end{cases} \quad (2.19)$$

Up to $O\left(\frac{1}{\log m}\right)$ we find that

$$\begin{aligned}
S_2(m) &= \frac{1}{m-2}\sum_{\chi\neq\chi_0}\sum_p \frac{\log p}{\log(m/\pi)}\widehat{\phi}\left(2\frac{\log p}{\log(m/\pi)}\right)[\chi^2(p)+\overline{\chi}^2(p)]p^{-1} \\
&= \frac{1}{m-2}\sum_p \frac{\log p}{\log(m/\pi)}\widehat{\phi}\left(2\frac{\log p}{\log(m/\pi)}\right)\sum_{\chi\neq\chi_0}[\chi^2(p)+\overline{\chi}^2(p)]p^{-1} \\
&= \frac{1}{m-2}\sum_p^{m^{\sigma/2}} \frac{\log p}{\log(m/\pi)}\widehat{\phi}\left(2\frac{\log p}{\log(m/\pi)}\right)p^{-1}[-2+(2m-2)\delta_m(p,\pm)] \\
&\ll \frac{1}{m-2}\sum_p^{m^{\sigma/2}} p^{-1} \ + \ \frac{2m-2}{m-2}\sum_{p\equiv\pm1(m)}^{m^{\sigma/2}} p^{-1} \\
&\ll \frac{1}{m-2}\sum_k^{m^{\sigma/2}} k^{-1} \ + \sum_{\substack{k\equiv1(m)\\k\geq m+1}}^{m^{\sigma/2}} k^{-1} \ + \sum_{\substack{k\equiv-1(m)\\k\geq m-1}}^{m^{\sigma/2}} k^{-1} \\
&\ll \frac{1}{m-2}\log(m^{\sigma/2}) \ + \ \frac{1}{m}\sum_k^{m^{\sigma/2}} k^{-1} \ + \ \frac{1}{m}\sum_k^{m^{\sigma/2}} k^{-1} \ + \ O\left(\frac{1}{m}\right) \\
&\ll \sigma\left(\frac{\log m}{m} + \frac{\log m}{m} + \frac{\log m}{m} + \frac{1}{m}\right). \qquad (2.20)
\end{aligned}$$

Therefore $S_2(m) = O(\sigma\frac{\log m}{m})$, so for all fixed, finite σ there is no contribution.

2.4. **Density Function from a Prime Conductor.**

Theorem 2.3 (Density Function from a Prime Conductor)**.** *Let $\widehat{\phi}$ be an even Schwartz function with supp($\widehat{\phi}$) $\subset (-2,2)$, m a prime, and $\mathcal{F}_m = \{\chi : \chi$ is primitive mod $m\}$. Then assuming GRH we have*

$$\frac{1}{\mathcal{F}_m}\sum_{\chi\in\mathcal{F}_m}\sum_{\gamma:L(\frac{1}{2}+i\gamma,\chi)=0}\phi\left(\gamma\frac{\log(m/\pi)}{2\pi}\right) = \int_{-\infty}^{\infty}\phi(y)dy + O\left(\frac{1}{\log m}\right). \qquad (2.21)$$

Note the above theorem can trivially be modified to handle the family

$$\mathcal{F}_N \ = \ \{\chi : \chi \text{ is primitive with conductor } m \text{ for some prime } m\in[N,2N]\}. \qquad (2.22)$$

It is possible to handle larger families (either m is square-free and tending to infinity, or m is square-free and in $[N,2N]$ with $N\to\infty$); see Appendices B and C for details. This theorem was also proved by Hughes and Rudnick [HR], where they show we may take $\sigma\leq 2$. We shall use this fact whenever needed below.

In the arguments below, where we try to extend the support, it is often useful to have $m\in[N,2N]$, as then we only need bounds to hold on average. There does not appear to be any gain by extending our family to include square-free m. The reason for this is that the cardinality of the two families, namely (1) primitive characters with prime conductor in $[N,2N]$ and (2) primitive characters with square-free conductor in $[N,2N]$, have approximately the same cardinality. The reason is that there are about

$\frac{N}{\log N}$ primes in $[N, 2N]$, so the families' cardinalities differ only by powers of $\log N$. Thus, in general if we are unable to obtain sufficient cancellation when m is restricted to prime values, we will not obtain the needed cancellation over square-free (as the prime m's are too large a subset).

3. Natural Conjectures to Extend the Support

Trivial estimation of prime sums yield the 1-level density for families of Dirichlet L-functions for supp$(\widehat{\phi}) \subset (-2, 2)$. We discuss some natural (we hope!) conjectures for the distribution of primes in residue classes, and how these would allow us to increase the support. Specifically, consider estimates of errors for the distribution of primes in residue classes. Assuming GRH (and other reasonable), how are the errors or excesses split among the various classes? Specifically, what is the modulus dependence on average?

3.1. **Sums to Investigate.**

We state the sums that we must bound. Again the families we shall investigate are either m prime and tending to infinity or

$$\mathcal{F}_N \ = \ \{\chi : \chi \text{ is primitive with conductor } m \text{ for some prime } m \in [N, 2N]\}. \qquad (3.1)$$

It is straightforward to extend many of our results to this second family. By (2.20) we see that the second sum (the sum over the squares of primes) is always negligible for any finite support. In fact, as long as $\sigma = o\left(\frac{m}{\log m}\right)$ these terms will not contribute.

The difficulty arises from the first sum (the sum over primes). By (2.15) we have

$$S_1(m) \quad = \quad \frac{2}{m-2} \sum_{p \le m^\sigma} \frac{\log p}{\log(m/\pi)} \widehat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right) p^{-\frac{1}{2}} (-1 + (m-1)\delta_m(p,1)), \qquad (3.2)$$

where

$$\delta_m(p,1) \ = \ \begin{cases} 1 & \text{if } p \equiv 1 \bmod m \\ 0 & \text{otherwise.} \end{cases} \qquad (3.3)$$

It is natural to analyze (3.2) by partial summation. For n prime let

$$\begin{aligned} a_n \quad &= \quad (-1 + (m-1)\delta_m(n,1)) \log n \\ h(n) \quad &= \quad \frac{1}{\log(m/\pi)} \widehat{\phi}\left(\frac{\log n}{\log(m/\pi)}\right) n^{-\frac{1}{2}}; \end{aligned} \qquad (3.4)$$

set $a_n = 0$ if n is not prime and $A_m(x) = \sum_{n \le x} a_n$. Then

$$S_1(m) \ = \ \frac{2}{m-2} \int_2^{m^\sigma} A_m(u) h'(u) du + O\left(\frac{1}{m}\right). \qquad (3.5)$$

By our previous arguments, there is no contribution from this term if $\sigma < 2$. Thus for $\sigma > 1$ it suffices to study

$$S_1(m) \ = \ \frac{2}{m-2} \int_m^{m^\sigma} A_m(u) h'(u) du + o(1). \qquad (3.6)$$

If needed, by [HR] we may replace the lower bound of m by m^2.

Note $h'(u) \ll u^{-\frac{3}{2}}$. What is the true size of $A_m(u)$? As m is prime, $\phi(m) = m-1$ and

$$\begin{aligned} A_m(u) &= \sum_{p \le u} [-1 + \phi(m)\delta_m(p,1)] \log p \\ &= \phi(m) \left[\sum_{\substack{p \le u \\ p \equiv 1 \bmod m}} \log p - \frac{1}{\phi(m)} \sum_{p \le u} \log p \right]. \end{aligned} \tag{3.7}$$

Let

$$B_m(u) \ = \ \phi(m) \left[\sum_{\substack{n \le u \\ n \equiv 1 \bmod m}} \Lambda(n) - \frac{1}{\phi(m)} \sum_{n \le u} \Lambda(n) \right]. \tag{3.8}$$

We want to show $|A_m(u) - B_m(u)|$ is small. Clearly the contributions to $B_m(u)$ from $n = p^\nu$ for $\nu \ge 3$ are bounded by $\phi(m) u^{\frac{1}{3}} \log u$. The $\nu = 2$ terms contribute

$$\phi(m) \left[\sum_{\substack{p \le \sqrt{u} \\ p \equiv 1 \bmod m}} \log p - \frac{1}{\phi(m)} \sum_{p \le \sqrt{u}} \log p \right] \ = \ A_m(\sqrt{u}). \tag{3.9}$$

Thus

$$B_m(u) - A_m(u) \ = \ A_m(\sqrt{u}) + O(m u^{\frac{1}{3}} \log u). \tag{3.10}$$

Repeating this argument gives

$$B_m(\sqrt{u}) - A_m(\sqrt{u}) \ = \ A_m(\sqrt[4]{u}) + O(m u^{\frac{1}{6}} \log u). \tag{3.11}$$

As trivially $A_m(x) \ll x$ we have

$$|B_m(u) - A_m(u)| \ \ll \ \left[B_m(\sqrt{u}) - A_m(\sqrt[4]{u}) \right] + m u^{\frac{1}{3}} \log u \ \ll \ B_m(\sqrt{u}) + m u^{\frac{1}{3}} \log u. \tag{3.12}$$

Assuming GRH, by (1.4) we have $B_m(x) = \phi(m)E(x,m,1) = O(m x^{\frac{1}{2}} \cdot (xm)^\epsilon)$. Thus the error from $B_m(\sqrt{u})$ may be absorbed by the other error, giving

$$A_m(u) \ = \ B_m(u) + O\left(m u^{\frac{1}{3}} \log u \right). \tag{3.13}$$

Thus (3.6) becomes

$$\begin{aligned} S_1(m) &= \frac{2}{m-2} \int_m^{m^\sigma} B_m(u) h'(u) du + O\left(\frac{1}{m} \int_m^{m^\sigma} m u^{\frac{1}{3}} \log u \cdot u^{-\frac{3}{2}} du \right) + o(1) \\ &= \frac{2}{m-2} \int_m^{m^\sigma} B_m(u) h'(u) du + O(m^{-\frac{1}{6}+\epsilon}) + o(1). \end{aligned} \tag{3.14}$$

Therefore to increase the support for the 1-level density for the family of Dirichlet characters with prime conductor m we need a good bound for $B_m(u)$.

3.2. **Bounds from Montgomery's Conjecture.** We now explore various bounds for (3.14). Note $B_m(u) = \phi(m)E(u,m,1)$. Probabilistic arguments (along the lines of the central limit theorem; see [Mon1]) lead to the conjecture $E(x,q,a) \ll \sqrt{\frac{x}{\phi(q)}} \cdot (xq)^\epsilon$. Assuming such a bound gives

$$\begin{aligned} S_1(m) &\ll \frac{1}{m}\int_m^{m^\sigma} \phi(m)\frac{u^{\frac{1}{2}+\epsilon}}{m^{\frac{1}{2}-\epsilon}} \cdot u^{-\frac{3}{2}} du + o(1) \\ &\ll m^{-\frac{1}{2}+\epsilon}\int_m^{m^\sigma} u^{\epsilon-1} du + o(1) \\ &\ll m^{\epsilon\sigma-\frac{1}{2}+\epsilon} + o(1), \end{aligned} \tag{3.15}$$

and this is $o(1)$ for fixed σ and ϵ sufficiently small.

Of course, we only need such a bound for $a = 1$. We consider weaker versions of Montgomery's conjecture. Explicitly, recall Conjecture 1.1:

Conjecture 1.1: *There is a* $\theta \in [0, \frac{1}{2})$ *such that for* q *prime*

$$E(x,q,1) \ \ll\ q^\theta \cdot \sqrt{\frac{x}{\phi(q)}} \cdot (xq)^\epsilon. \tag{3.16}$$

Conjecture 1.1 implies Theorem 1.2:

Proof of Theorem 1.2. As $B_m(u) = \phi(m)E(u,m,1)$, substituting the bound for $E(u,m,1)$ from Conjecture 1.1 in (3.14) yields

$$\begin{aligned} S_1(m) &\ll \frac{1}{m}\int_m^{m^\sigma} \phi(m) \cdot m^{\theta-\frac{1}{2}+\epsilon} u^{\frac{1}{2}+\epsilon} \cdot u^{-\frac{3}{2}} du + o(1) \\ &\ll m^{\theta-\frac{1}{2}+\epsilon}\int_m^{m^\sigma} u^{\epsilon-1} du + o(1) \\ &\ll m^{\epsilon\sigma-(\frac{1}{2}-\theta)} + o(1). \end{aligned} \tag{3.17}$$

As long as $\theta < \frac{1}{2}$, by choosing ϵ sufficiently small the above is $o(1)$ for any fixed σ. □

The above is amazing. Assuming GRH we have $E(u,m,1) \ll u^{\frac{1}{2}}(um)^\epsilon$. This bound just fails, giving $S_1(m) \ll m^{\epsilon(1+\sigma)}$. Any power savings in m ($\log^A m$ savings would suffice for A sufficiently large) in bounding $E(u,m,1)$ leads to arbitrarily large support!

Remark 3.1. The consequence of Montgomery's original conjecture extending the range of support has been independently observed by others as well. See for example [Sar]. Below we try and explore the consequences of weaker conjectures.

Let us consider the family $\mathcal{F}_N$, where we consider all primitive characters with prime conductor $m \in [N, 2N]$. The sum over primes (the first sum) is now just

$$S_1 \ =\ \frac{1}{|\mathcal{F}_N|} \sum_{\substack{m=N \\ m \text{ prime}}}^{2N} S_1(m). \tag{3.18}$$

There are $\pi(2N) - \pi(N) = \frac{N}{\log N} + o\left(\frac{N}{\log N}\right)$ primes in $[N, 2N]$, and for each prime m we have $\phi(m) = m - 2$ primitive characters. Thus

$$\frac{N^2}{\log N} \ll |\mathcal{F}_N| \ll \frac{N^2}{\log N}; \tag{3.19}$$

we will divide by $|\mathcal{F}_N|$ instead of $m-2$ as this is the cardinality of the family. Summing (3.14) over the family $\mathcal{F}_N$ now gives

$$S_1 = \frac{2}{|\mathcal{F}_N|} \sum_{\substack{m=N \\ m \text{ prime}}}^{2N} \int_m^{m^\sigma} B_m(u) h'(u) du + o(1); \tag{3.20}$$

Using $B_m(u) = \phi(m)E(u, m, 1)$ yields

$$S_1 = \frac{2}{|\mathcal{F}_N|} \int_N^{(2N)^\sigma} \left[\sum_{\substack{m=N \\ m \text{ prime}}}^{2N} \phi(m)E(u, m, 1) \right] h'(u) du + o(1). \tag{3.21}$$

ADD REMARK THAT CAN EXTEND INTEGRATION TO THESE BOUNDS.

Remark 3.2. We need to be a little careful as the above equation is slightly wrong. Technically $h'(u)$ has some m dependence. We have

$$\begin{aligned} h(u) &= \frac{1}{\log(m/\pi)} \widehat{\phi}\left(\frac{\log u}{\log(m/\pi)}\right) u^{-\frac{1}{2}} \\ h'(u) &= \frac{1}{\log(m/\pi)} \left[\widehat{\phi}'\left(\frac{\log u}{\log(m/\pi)}\right) \frac{1}{\log(m/\pi)} + \widehat{\phi}\left(\frac{\log u}{\log(m/\pi)}\right) \right] u^{-\frac{3}{2}}. \end{aligned} \tag{3.22}$$

As m varies from N to $2N$, there are oscillations of size $\frac{1}{\log N}$ in $h'(u)$. There are two solutions. If we are not interested in exploiting cancellation in sign in the $E(u, m, 1)$, then all is fine. If we do want to try and use the sign of $E(u, m, 1)$, instead of normalizing the zeros of $L(s, \chi)$ (where χ is a conductor with prime character m) by $\frac{\log(m/\pi)}{2\pi}$ we should instead use $\frac{\log(N/2\pi)}{2\pi}$. This leads to trivial modifications in the explicit formula (see (2.7)). The reason such a change in scaling is tractable is that the conductors are monotone increasing; see [Mil] for more on handling oscillation in conductors. In all arguments below we assume these corrections have been made (if needed).

The following conjecture is an average version of Montgomery's:

Conjecture 3.3. *There exists a $\theta \in [0, \frac{1}{2})$ such that for all ϵ and u with $N \ll u \ll N^\sigma$*

$$\sum_{\substack{m=N \\ m \text{ prime}}}^{2N} \phi(m)E(u, m, 1) \ll N^\theta \cdot N^2 \sqrt{\frac{u}{N}} (uN)^\epsilon \ll N^{2-(\frac{1}{2}-\theta-\epsilon)} u^{\frac{1}{2}+\epsilon}. \tag{3.23}$$

The above is weaker as we have the potential for some cancellation because we have signed errors; however, it is possible there may be no variation in sign (see Rubinstein and Sarnak's work on Chebyshev's Bias [RubSa]). We immediately obtain

Theorem 3.4. *Assuming Conjecture 3.3, there is no contribution from* S_1 *for* $\text{supp}(\widehat{\phi}) \subset (-\sigma, \sigma)$.

Proof. From (3.19), (3.21), and Conjecture 3.3 we have

$$\begin{aligned} S_1 &\ll \frac{\log N}{N^2} \int_N^{(2N)^\sigma} N^{2-(\frac{1}{2}-\theta-\epsilon)} u^{\frac{1}{2}+\epsilon} \cdot u^{-\frac{3}{2}} du + o(1) \\ &\ll N^{-(\frac{1}{2}-\theta-\epsilon)} \log N \int_N^{(2N)^\sigma} u^{\epsilon-1} du + o(1) \\ &\ll N^{-(\frac{1}{2}-\theta-\epsilon)} \cdot N^{\epsilon\sigma} \log N + o(1). \end{aligned} \tag{3.24}$$

As long as $\theta < \frac{1}{2}$ the above is $o(1)$. □

Conjecture 3.3 is very plausible. Here we are summing *signed* quantities, and only require slight cancellation in the modulus aspect. Consider the terms on the left hand side of (3.23). By GRH, each is at most $Nu^{\frac{1}{2}}(uN)^\epsilon$. Hence their sum is at most $N^{2+\epsilon}u^{\frac{1}{2}+\epsilon}$. The existence of a $\theta \in [0, \frac{1}{2})$ implies there is a small power savings in this signed sum, and this is enough to obtain unlimited support (if we assume the conjecture holds for all σ). It is also now possible that a small number of moduli have a large contribution; for instance, for any $\eta < 1$ there can be N^η choices of $m \in [N, 2N]$ such that

$$E(u, m, 1) \ll N^{\frac{1}{2}} \sqrt{\frac{u}{N}}\, (uN)^\epsilon \ll N^\epsilon u^{\frac{1}{2}+\epsilon} \tag{3.25}$$

Thus the sum of these terms (weighted by $\phi(m)$) is $\ll N^{1+\eta+\epsilon}u^{\frac{1}{2}+\epsilon}$. As long as $1+\eta+\epsilon < 2$, these terms will not contribute.

Is it reasonable to expect there to be oscillation in the signs of $E(u, m, 1)$, or do we expect these terms to have the same signs? Note 1 is always a quadratic residue, so perhaps by the arguments in [RubSa] we should expect that most of the time these are the same sign. Below we investigate conjectures where we do not try to exploit cancellation by sign.

3.3. **Conjectures for Distribution Among Residue Classes.** We now investigate some weaker conjectures. These will not yield unlimited support but have the advantage of incorporating known results as well as allowing some biases among the residue classes. We use the results of [HR] to replace $\int_m^{m^\sigma}$ with $\int_{m^2}^{m^\sigma}$. Recall Conjecture 1.4:

Conjecture 1.4: *There exists a* $\theta \in [0, 1]$ *such that for prime* $m \in [N, 2N]$ *with* $N^2 \ll u \ll N^{4-2\theta}$,

$$\sum_{\substack{m=N \\ m \text{ prime}}}^{2N} E(u, m, 1)^2 \ll N^\theta \cdot \frac{1}{N} \sum_{\substack{m=N \\ m \text{ prime}}}^{2N} \sum_{\substack{a=1 \\ (a,m)=1}}^{m} E(u, m, a)^2. \tag{3.26}$$

Note the right hand side of (3.26) is $N^{\theta-1}(V(u, 2N) - V(u, N))$. Conjecture 1.4 and Goldston-Vaughan's result imply Theorem 1.5:

Theorem 1.5: *Let $\mathcal{F}_N$ be the family of primitive characters with prime conductor $m \in [N, 2N]$. The 1-level density for $\mathcal{F}_N$ holds for test functions whose Fourier transforms are supported in $(-4+2\theta, 4-2\theta)$.*

Conjecture 1.4 is trivially true for $\theta = 1$, and while it is unlikely to be true for $\theta = 0$, it is reasonable to expect it to hold for $\theta = \epsilon$ (for any $\epsilon > 0$). What we need is some control over biases of primes to be congruent to $1 \bmod m$. For the residue class $a \bmod m$, $E(u,m,a)^2$ is the variance; the above conjecture can be interpreted as bounding $E(u,m,1)^2$ in terms of the average variance. Interestingly, $\theta = 1$ recovers the 1-level density result of support in $(-2, 2)$.

Bounds such as these are useful as, by using the Cauchy-Schwartz inequality, the variance $E(u,m,1)^2$ surfaces in investigating the 1-level density sums. If we can express the variance $E(u,m,1)^2$ in terms of the average variance, the bounds from Goldston-Vaughan are applicable. There is also the possibility of using higher moment bounds and Holder's Inequality instead of Cauchy-Schwartz (see [Va]); unfortunately, Vaughan's results only hold for m "close" to u. Explicitly, $u^{\frac{3}{4}+\epsilon} \ll m \ll u$. To obtain better support than $(-2, 2)$, we need $u \gg \sqrt{m}$.

The question is: for what θ is the above conjecture "reasonable"? Can we glean a reasonable value for θ from the arguments in say [RubSa], or from probabilistic arguments on random primes (where with probability one we know RH is true for a random sequence of primes – what is known there about error terms in congruence classes, and how that depends on the modulus)?

One could probably work with all square-free m and not just prime m in the Dirichlet L-function's densities; however, as the variances are positive, if bounds like this do not hold for m restricted to prime values, they will not hold for m square-free (because we are going for more than a logarithm savings).

Proof of Theorem 1.5. It suffices to show (3.21) is negligible for $\sigma < 4 - 2\theta$. We shall only do the case when the second part of Conjecture 1.4 holds. We must therefore study

$$S_1 \quad = \quad \frac{2}{|\mathcal{F}_N|} \int_N^{(2N)^\sigma} \left[\sum_{\substack{m=N \\ m \text{ prime}}}^{2N} \phi(m) E(u,m,1) h'(u) \right] du + o(1). \tag{3.27}$$

Using Cauchy-Schwartz and Conjecture 1.4 we have

$$\begin{aligned} \left| \sum_{\substack{m=N \\ m \text{ prime}}}^{2N} \phi(m) E(u,m,1) h'(u) \right| \quad &\leq \quad \sqrt{\sum_{\substack{m=N \\ m \text{ prime}}}^{2N} \phi(m)^2 h'(u)^2} \sqrt{\sum_{\substack{m=N \\ m \text{ prime}}}^{2N} E(u,m,1)^2} \\ &\ll \quad N^{\frac{3}{2}} |h'(u)| \cdot \sqrt{N^\theta \cdot \frac{1}{N} \sum_{\substack{m=N \\ m \text{ prime}}}^{2N} \sum_{\substack{a=1 \\ (a,m)=1}}^{m} E(u,m,a)^2} \\ &\ll \quad N^{\frac{3}{2}} |h'(u)| \cdot \sqrt{N^{\theta-1}(V(u,2N) - V(u,N))}. \end{aligned} \tag{3.28}$$

Using Goldston-Vaughan's bounds (Theorem 1.3) yields

$$\begin{aligned}\left|\sum_{\substack{m=N\\ m \text{ prime}}}^{2N} \phi(m)E(u,m,1)h'(u)\right| &\ll N^{1+\frac{\theta}{2}}|h'(u)| \cdot \sqrt{V(u,2N)-V(u,N)} \\ &\ll N^{1+\frac{\theta}{2}} \cdot \sqrt{Nu\log N + N^{\frac{7}{4}}u^{\frac{1}{4}+\epsilon} + u^{\frac{3}{2}+\epsilon}} \\ &\ll \left[N^{\frac{3}{2}+\epsilon+\frac{\theta}{2}}u^{\frac{1}{2}} + N^{\frac{15}{8}+\frac{\theta}{2}}u^{\frac{1}{8}+\epsilon} + N^{1+\frac{\theta}{2}}u^{\frac{3}{4}+\epsilon}\right]|h'(u)|.\end{aligned} \tag{3.29}$$

Recall $h'(u) \ll u^{-\frac{3}{2}}$ and $\log N \ll N^\epsilon$. We have (ϵ changes from line to line)

$$\begin{aligned} S_1 &\ll \frac{\log N}{N^2}\int_{N^2}^{(2N)^\sigma}\left[N^{\frac{3}{2}+\epsilon+\frac{\theta}{2}}u^{\frac{1}{2}} + N^{\frac{15}{8}+\frac{\theta}{2}}u^{\frac{1}{8}+\epsilon} + N^{1+\frac{\theta}{2}}u^{\frac{3}{4}+\epsilon}\right]u^{-\frac{3}{2}}du + o(1) \\ &\ll N^{-\frac{1}{2}+\frac{\theta}{2}+\epsilon}\int_{N^2}^{(2N)^\sigma} u^{-1}du + N^{-\frac{1}{16}+\frac{\theta}{2}+\epsilon}\int_{N^2}^{(2N)^\sigma} u^{-\frac{11}{8}+\epsilon}du \\ &\quad + N^{-1+\frac{\theta}{2}+\epsilon}\int_{N^2}^{(2N)^\sigma} u^{-\frac{3}{4}+\epsilon}du + o(1) \\ &\ll N^{-\frac{1}{2}+\frac{\theta}{2}+\epsilon} + N^{-\frac{1}{16}+\frac{\theta}{2}+\epsilon}(N^2)^{-\frac{3}{8}+\epsilon} + N^{-1+\frac{\theta}{2}+\epsilon}(N^\sigma)^{\frac{1}{4}+\epsilon} + o(1) \\ &\ll N^{-\frac{1-\theta+\epsilon}{2}} + N^{-\frac{13-8\theta+\epsilon}{16}} + N^{4(\sigma-(4-2\theta))} + o(1). \end{aligned} \tag{3.30}$$

The first term is negligible when $\theta < 1$, the second term when $\theta < \frac{13}{8}$ and the third when $\sigma < 4-2\theta$. Therefore, as long as $\theta < 1$ we may take any $\sigma < 4-2\theta$. □

Remark 3.5. The reason Conjecture 1.4 only allows us to go up to $\sigma < 4-2\theta$ is because of the $O(x^{\frac{3}{2}+\epsilon})$ error in Theorem 1.3. Assume we could replace that error with $O(x^{1+\eta+\epsilon})$ for some $\eta \in [0, \frac{1}{2}]$. Then we would replace $u^{\frac{3}{4}+\epsilon}$ with $u^{\frac{1}{2}+\frac{\eta}{2}+\epsilon}$. This piece would now give

$$\begin{aligned} N^{-1+\frac{\theta}{2}+\epsilon}\int_{N^2}^{(2N)^\sigma} u^{-1+\frac{\eta}{2}+\epsilon}du &\ll N^{-1+\frac{\theta}{2}+\epsilon}N^{\frac{\eta\sigma}{2}+\epsilon\sigma} \\ &\ll N^{(\frac{\eta}{2}+\epsilon)\sigma-(1-\frac{\theta}{2}+\epsilon)}. \end{aligned} \tag{3.31}$$

Thus there is no contribution for

$$\sigma \ < \ \frac{1-\frac{\theta}{2}+\epsilon}{\frac{\eta}{2}+\epsilon}, \tag{3.32}$$

or for

$$\sigma \ < \ \frac{2-\theta}{\eta}. \tag{3.33}$$

If we could take η arbitrarily close to 0 then we would have unlimited support. Note that $\theta = 1$ and $\eta = 1/2$ (both of which are valid choices) recovers $\sigma < 2$.

Remark 3.6. For $N^2 \ll u \ll N^\sigma$, we discuss in what sense Conjecture 1.4,

$$\sum_{\substack{m=N\\ m \text{ prime}}}^{2N} E(u,m,1)^2 \ \ll \ N^\eta \cdot \frac{1}{N}\sum_{\substack{m=N\\ m \text{ prime}}}^{2N}\ \sum_{\substack{a=1\\ (a,m)=1}}^{m} E(u,m,a)^2, \ \ \eta \in [0,1) \tag{3.34}$$

is weaker than Conjecture 1.1,

$$E(x,q,1) \ \ll\ q^{\theta} \cdot \sqrt{\frac{x}{\phi(q)}} \cdot (xq)^{\epsilon}, \ \ \theta \in [0, \frac{1}{2}); \tag{3.35}$$

we could use Conjecture 3.3 instead of Conjecture 1.1, but we would obtain similar results. The two conjectures are essentially equivalent for u such that $N^2 \ll u \ll N^{4-2\eta}$, with $\theta + \frac{1}{2}$ playing the role of η.

If each $E(u,m,1) \ll m^{\theta-\frac{1}{2}} u^{\frac{1}{2}} (um)^{\epsilon}$, then

$$\sum_{\substack{m=N \\ m \text{ prime}}}^{2N} E(u,m,1) \quad \ll \quad N^{\theta+\frac{1}{2}+\epsilon} u^{\frac{1}{2}}. \tag{3.36}$$

Using Theorem 1.3, Conjecture 1.4 and Cauchy-Schwartz gives

$$\begin{aligned}
\sum_{\substack{m=N \\ m \text{ prime}}}^{2N} E(u,m,1) \quad &\leq \quad \left(\sum_{\substack{m=N \\ m \text{ prime}}}^{2N} E(u,m,1)^2 \right)^{\frac{1}{2}} \left(\sum_{\substack{m=N \\ m \text{ prime}}}^{2N} 1 \right)^{\frac{1}{2}} \\
&\ll \quad \left(N^{\eta-1} \sum_{\substack{m=N \\ m \text{ prime}}}^{2N} \sum_{\substack{a=1 \\ (a,m)=1}}^{m} E(u,m,a)^2 \right)^{\frac{1}{2}} \cdot N^{\frac{1}{2}} \\
&\ll \quad \left(N^{\eta-1} \left(N^{1+\epsilon} u + N^{\frac{7}{8}+\epsilon} u^{\frac{1}{4}} + u^{\frac{3}{2}+\epsilon} \right) \right)^{\frac{1}{2}} \cdot N^{\frac{1}{2}} \\
&\ll \quad N^{\frac{\eta}{2}+\epsilon} \left(N^{\frac{1}{2}} + N^{\frac{7}{8}} u^{-\frac{3}{8}} + u^{\frac{1}{4}} \right) u^{\frac{1}{2}} \\
&\ll \quad N^{\frac{\eta}{2}+\epsilon} u^{\frac{1}{4}} \cdot u^{\frac{1}{2}},
\end{aligned} \tag{3.37}$$

because as $u \gg N^2$, the dominant term is the last. The two conjectures provide identical bounds for the sum of interest, $\sum_m E(u,m,1)$, when $N^2 \ll u \ll N^{4-2\eta-4\eta'}$ (with $\theta + \frac{1}{2}$ equivalent to $1-\eta'$).

While we expect the variance sum to be of size $Nu \cdot N^{\epsilon}$, the best bounds (Theorem 1.3) have an error of size $u^{\frac{3}{2}+\epsilon}$. This bound is larger than what we expect the truth to be for $u^{\frac{1}{2}} \gg N$; however, it is exactly such a range that we need to investigate, and when u is so much larger than N, results are harder to obtain. If the error were such that $Nu \cdot N^{\epsilon}$ was always the main term, then we would regain Conjecture 1.1 (actually Conjecture 3.3). Thus Conjecture 1.4 is basically the average version of Montgomery's conjecture when u is restricted to $N^2 \ll u \ll N^{4-2\eta-4\eta'}$; however, as we are using Goldston and Vaughan's results on the variance, we feel this does provide some support for Montgomery's conjecture.

3.4. **Analogue of Theorem 1.5 for $\mathcal{F}_m$.**

Conjecture 3.7. *There exists a $\theta \in [0,1]$ such that for all prime m (or at least a sequence of primes tending to infinity), if $m^2 \ll u \ll m^{4-2\theta}$ then*

$$E(u,m,1)^2 \ \ll\ m^{\theta} \cdot \frac{1}{\phi(m)} \sum_{\substack{a=1 \\ (a,m)=1}}^{m} E(u,m,a)^2. \tag{3.38}$$

For the above conjecture to imply an analogue of Theorem 1.5 for the family $\mathcal{F}_m$ (primitive characters with prime conductor m), we need an analogue of Theorem 1.3 where we do not sum over m. Hooley has conjectured that (1.9), namely

$$V(x,q) \ \sim \ x \cdot (xq)^{\epsilon}, \tag{3.39}$$

holds for some unspecified range of q relative to x. In [Ho] he shows (1.9) is true for almost all $q \in [\frac{Q}{2}, Q]$ with $x(\log x)^{-A} < Q \leq x$, and under GRH the range may be extended to $x^{\frac{4}{5}+\epsilon} < Q \leq x$. This range was extended further by Friedlander and Goldston [FG] to $x^{\frac{3}{4}+\epsilon} < Q \leq x$. If we assume that we can find a sequence of primes m such that (1.9) holds for all u with $m^2 \ll u \ll m^{\sigma}$, then we can prove the analogue of Theorem 1.5 for $\mathcal{F}_m$. From (3.14) and recalling that $B_m(u) = \phi(m)E(u,m,1)$ we have

$$\begin{aligned} S_1(m) &\ll \frac{1}{m}\int_{m^2}^{m^{\sigma}} \phi(m)|E(u,m,1)|u^{-\frac{3}{2}}du + o(1) \\ &\ll \frac{\phi(m)}{m}\int_{m^2}^{m^{\sigma}} \left(m^{\theta}\cdot\frac{1}{\phi(m)}\sum_{\substack{a=1 \\ (a,m)=1}}^{m} E(u,m,a)^2\right)^{\frac{1}{2}} u^{-\frac{3}{2}}du + o(1) \\ &\ll \int_{m^2}^{m^{\sigma}} \left(m^{\theta-1}u(um)^{\epsilon}\right)^{\frac{1}{2}} u^{-\frac{1}{2}}du + o(1) \\ &\ll m^{\frac{\theta}{2}-\frac{1}{2}+\epsilon}\int_{m^2}^{m^{\sigma}} u^{-1}du + o(1) \\ &\ll m^{\frac{\theta}{2}-\frac{1}{2}+\epsilon'} + o(1). \end{aligned} \tag{3.40}$$

As long as $\theta < 1$, $S_1(m)$ is negligible.

We do not need the full strength of 3.7. As Hooley's conjecture (and results towards it) hold with $\log x$ and $\log q$ instead of x^{ϵ} and q^{ϵ}, we may replace m^{θ} by $\frac{m}{\log^C m}$ for C sufficiently large. While Hooley's conjecture gives arbitrarily large support, it is important to note that the difficulty in the proof of Theorem 1.5 was in the error terms in Goldston-Vaughan's bound (Theorem 1.3). Thus we should be careful about our assumptions on the error terms in Hooley's conjecture when m is much smaller than u.

Appendix A. Alternate Form of the Goldston-Vaughan Bound

Lemma A.1. *Let*

$$W(x,q) \ = \ \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \left|\psi(x,q,a) - \frac{\psi(x)}{\phi(q)}\right|^2. \tag{A.1}$$

Assume RH. Then for $Q \gg x^{2\epsilon}$

$$\sum_{q \leq Q} W(x,q) \ \ll \ Qx\log Q + Q^{\frac{7}{4}}x^{\frac{1}{4}+\epsilon} + x^{\frac{3}{2}+\epsilon} \tag{A.2}$$

where the sum is over prime q.

Proof. We have

$$\begin{aligned}
W(x,q) &= \sum_{\substack{a=1\\(a,q)=1}}^{q} \left|\psi(x,q,a) - \frac{\psi(x)}{\phi(q)}\right|^2 \\
&= \sum_{\substack{a=1\\(a,q)=1}}^{q} \left|\psi(x,q,a) - \frac{x}{\phi(q)} + \frac{x}{\phi(q)} - \frac{\psi(x)}{\phi(q)}\right|^2 \\
&\leq 3\sum_{\substack{a=1\\(a,q)=1}}^{q} \left|\psi(x,q,a) - \frac{x}{\phi(q)}\right|^2 + 3\sum_{\substack{a=1\\(a,q)=1}}^{q} \left|\frac{x}{\phi(q)} - \frac{\psi(x)}{\phi(q)}\right|^2 \\
&\leq 3\sum_{\substack{a=1\\(a,q)=1}}^{q} \left|\psi(x,q,a) - \frac{x}{\phi(q)}\right|^2 + 3\sum_{\substack{a=1\\(a,q)=1}}^{q} \left|\frac{O(x^{\frac{1}{2}+\epsilon})}{\phi(q)}\right|^2 \\
&\leq 3\sum_{\substack{a=1\\(a,q)=1}}^{q} \left|\psi(x,q,a) - \frac{x}{\phi(q)}\right|^2 + O\left(\frac{x^{1+2\epsilon}}{\phi(q)}\right). \qquad \text{(A.3)}
\end{aligned}$$

Summing over prime $q \leq Q$ gives

$$\sum_{q\leq Q} W(x,q) \ll \sum_{q\leq Q} V(x,q) + x^{1+2\epsilon}\sum_{q\leq Q}\frac{1}{\phi(q)}; \qquad \text{(A.4)}$$

the proof is completed by using Goldston and Vaughan's bound for the sum of $V(x,q)$. Note we used GRH to give $x - \psi(x) = O(x^{\frac{1}{2}+\epsilon})$, and then $\sum_{q\leq Q}\frac{1}{\phi(q)} \ll \log Q$ as q is prime. As $Q \gg x^{2\epsilon}$, we may replace $x^{1+2\epsilon}$ with Qx. □

APPENDIX B. DIRICHLET CHARACTERS FROM A SQUARE-FREE NUMBER

Fix an r and let $m_1, \dots, m_r$ be distinct odd primes. Let

$$\begin{aligned}
m &= m_1 m_2 \cdots m_r \\
M_1 &= (m_1-1)(m_2-1)\cdots(m_r-1) = \phi(m) \\
M_2 &= (m_1-2)(m_2-2)\cdots(m_r-2). \qquad \text{(B.1)}
\end{aligned}$$

M_2 is the number of primitive characters mod m, each of conductor m. For each $l_i \in [1, m_i-2]$ we have the primitive character discussed in the previous section, χ_{l_i}. A general primitive character mod m is given by a product of these characters:

$$\chi(u) = \chi_{l_1}(u)\chi_{l_2}(u)\cdots\chi_{l_r}(u) \qquad \text{(B.2)}$$

Let $\mathcal{F} = \{\chi : \chi = \chi_{l_1}\chi_{l_2}\cdots\chi_{l_r}\}$. Then $|\mathcal{F}| = M_2$, and we are led to investigating the following sums:

$$\begin{aligned}
S_1 &= \frac{1}{M_2}\sum_p \frac{\log p}{\log(m/\pi)}\widehat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right)p^{-\frac{1}{2}}\sum_{\chi\in\mathcal{F}}[\chi(p)+\overline{\chi}(p)] \\
S_2 &= \frac{1}{M_2}\sum_p \frac{\log p}{\log(m/\pi)}\widehat{\phi}\left(2\frac{\log p}{\log(m/\pi)}\right)p^{-1}\sum_{\chi\in\mathcal{F}}[\chi^2(p)+\overline{\chi}^2(p)] \qquad \text{(B.3)}
\end{aligned}$$

B.1. **The First Sum (m Square-free).** We must study $\sum_{\chi\in\mathcal{F}}\chi(p)$ (the sum with $\overline{\chi}$ is handled similarly). In the previous section we showed

$$\sum_{l_i=1}^{m_i-2}\chi_{l_i}(p) \ = \ \begin{cases} m_i-1-1 & \text{if } p\equiv 1 \bmod m_i \\ -1 & \text{otherwise.}\end{cases} \tag{B.4}$$

Define

$$\delta_{m_i}(p,1) \ = \ \begin{cases} 1 & \text{if } p\equiv 1 \bmod m_i \\ 0 & \text{otherwise.}\end{cases} \tag{B.5}$$

Then

$$\begin{aligned}\sum_{\chi\in\mathcal{F}}\chi(p) &= \sum_{l_1=1}^{m_1-2}\cdots\sum_{l_r=1}^{m_r-2}\chi_{l_1}(p)\cdots\chi_{l_r}(p) \\ &= \prod_{i=1}^{r}\sum_{l_i=1}^{m_i-2}\chi_{l_i}(p) \\ &= \prod_{i=1}^{r}(-1+(m_i-1)\delta_{m_i}(p,1)).\end{aligned} \tag{B.6}$$

Let us denote by $k(s)$ an s-tuple $(k_1,k_2,\dots,k_s)$ with $k_1<k_2<\cdots<k_s$. This is just a subset of $\{1,2,\dots,r\}$. There are 2^r possible choices for $k(s)$. We will use these to expand the above product. Define

$$\delta_{k(s)}(p,1)=\prod_{i=1}^{s}\delta_{m_{k_i}}(p,1). \tag{B.7}$$

If $s=0$ we define $\delta_{k(0)}(p,1)=1$ for all p. Then

$$\prod_{i=1}^{r}(-1+(m_i-1)\delta_{m_i}(p,1))=\sum_{s=0}^{r}\sum_{k(s)}(-1)^{r-s}\delta_{k(s)}(p,1)\prod_{i=1}^{s}(m_{k_i}-1). \tag{B.8}$$

Let $h(p)=2\frac{\log p}{\log(m/\pi)}\widehat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right)\ll||\widehat{\phi}||$. Then

$$\begin{aligned}S_1 &= \sum_{p}^{m^\sigma}\frac{1}{2}h(p)p^{-\frac{1}{2}}\frac{1}{M_2}\sum_{\chi\in\mathcal{F}}[\chi(p)+\overline{\chi}(p)] \\ &= \sum_{p}^{m^\sigma}h(p)p^{-\frac{1}{2}}\frac{1}{M_2}\sum_{s=0}^{r}\sum_{k(s)}(-1)^{r-s}\delta_{k(s)}(p,1)\prod_{i=1}^{s}(m_{k_i}-1) \\ &\ll \sum_{p}^{m^\sigma}p^{-\frac{1}{2}}\frac{1}{M_2}\left(1+\sum_{s=1}^{r}\sum_{k(s)}\delta_{k(s)}(p,1)\prod_{i=1}^{s}(m_{k_i}-1)\right).\end{aligned} \tag{B.9}$$

Observing that $m/M_2\le 3^r$ we see the $s=0$ sum contributes

$$S_{1,0}=\frac{1}{M_2}\sum_{p}^{m^\sigma}p^{-\frac{1}{2}}\ll 3^r m^{\frac{1}{2}\sigma-1}, \tag{B.10}$$

hence negligible for $\sigma < 2$. Now we study

$$S_{1,k(s)} = \frac{1}{M_2}\prod_{i=1}^{s}(m_{k_i}-1)\sum_{p}^{m^\sigma} p^{-\frac{1}{2}}\delta_{k(s)}(p,1). \tag{B.11}$$

The effect of the factor $\delta_{k(s)}(p,1)$ is to restrict the summation to primes $p \equiv 1(m_{k_i})$ for $k_i \in k(s)$. The sum will increase if instead of summing over primes satisfying the congruences we sum over all numbers n satisfying the congruences (with $n \geq 1+\prod_{i=1}^{s} m_{k_i}$). But now that the sum is over integers and not primes, we can use basic uniformity properties of integers to bound it. We are summing integers mod $\prod_{i=1}^{s} m_{k_i}$, so summing over integers satisfying these congruences is basically just $\prod_{i=1}^{s}(m_{k_i})^{-1}$ $\sum_{n=1}^{m^\sigma} n^{-\frac{1}{2}} = \prod_{i=1}^{s}(m_{k_i})^{-1} m^{\frac{1}{2}\sigma}$. We can do this as the sum of the reciprocals from the residue classes of $\prod_{i=1}^{s} m_{k_i}$ differ by at most their first term. Throwing out the first term of the class $1+\prod_{i=1}^{s} m_{k_i}$ makes it have the smallest sum of the $\prod_{i=1}^{s} m_{k_i}$ classes, so adding all the classes and dividing by $\prod_{i=1}^{s} m_{k_i}$ increases the sum.

Hence (recalling $m/M_2 \leq 3^r$)

$$\begin{aligned} S_{1,k(s)} &\ll \frac{1}{M_2}\prod_{i=1}^{s}(m_{k_i}-1)\prod_{i=1}^{s}(m_{k_i})^{-1} m^{\frac{1}{2}\sigma} \\ &\ll 3^r m^{\frac{1}{2}\sigma-1}. \end{aligned} \tag{B.12}$$

Therefore, $\forall s$ the $S_{1,k(s)}$ contribute $3^r m^{\frac{1}{2}\sigma-1}$. There are 2^r choices, yielding

$$S_1 \ll 6^r m^{\frac{1}{2}\sigma-1}, \tag{B.13}$$

which is negligible as m goes to infinity for fixed r if $\sigma < 2$. We cannot let r go to infinity in the arguments above because if m is the product of the first r primes, then for r large,

$$\begin{aligned} \log m &= \sum_{k=1}^{r} \log p \\ &= \sum_{p \leq r} \log p \ \sim \ r \\ &\to \ 6^r \sim m^{\log 6} \ \sim \ m^{1.79}. \end{aligned} \tag{B.14}$$

B.2. **The Second Sum (m Square-free).** We must study $\sum_{\chi\in\mathcal{F}} \chi^2(p)$ (the sum with $\overline{\chi}$ is handled similarly). In the previous section we showed

$$\sum_{l_i=1}^{m_i-2} \chi_{l_i}^2(p) \ = \ \begin{cases} m_i - 1 - 1 & \text{if } p \equiv \pm 1 \bmod m_i \\ -1 & \text{otherwise.} \end{cases} \tag{B.15}$$

Then

$$\begin{aligned}\sum_{\chi\in\mathcal{F}}\chi^2(p) &= \sum_{l_1=1}^{m_1-2}\cdots\sum_{l_r=1}^{m_r-2}\chi_{l_1}^2(p)\cdots\chi_{l_r}^2(p)\\ &= \prod_{i=1}^{r}\sum_{l_i=1}^{m_i-2}\chi_{l_i}^2(p)\\ &= \prod_{i=1}^{r}(-1+(m_i-1)\delta_{m_i}(p,1)+(m_i-1)\delta_{m_i}(p,-1)). \qquad \text{(B.16)}\end{aligned}$$

We now show the Second Sum is negligible for all σ. Instead of having 2^r terms we have 3^r. Let $k(s)$ be as before, and let $j(s)$ be an s-tuple of ±1s. As s ranges from 0 to r we get each of the 3^r possibilities, as for a fixed s, there are $\binom{r}{s}$ choices for $k(s)$, each of these having 2^s choices for $j(s)$. But $\sum_{s=0}^{r}2^s\binom{r}{k} = (1+2)^r$. Let $h(p) = 2\frac{\log p}{\log(m/\pi)}\widehat{\phi}\left(2\frac{\log p}{\log(m/\pi)}\right) \ll ||\widehat{\phi}||$. Define

$$\delta_{k(s)}(p,j(s)) = \prod_{i=1}^{s}\delta_{m_{k_i}}(p,j_i). \qquad \text{(B.17)}$$

Then

$$\sum_{\chi\in\mathcal{F}}\chi^2(p) = \sum_{s=0}^{r}\sum_{k(s)}\sum_{j(s)}(-1)^{r-s}\delta_{k(s)}(p,j(s))\prod_{i=1}^{s}(m_{k_i}-1) \qquad \text{(B.18)}$$

Therefore

$$\begin{aligned}S_2 &= \frac{1}{M_2}\sum_p\frac{\log p}{\log(m/\pi)}\widehat{\phi}\left(2\frac{\log p}{\log(m/\pi)}\right)p^{-1}\sum_{\chi\in\mathcal{F}}[\chi^2(p)+\overline{\chi}^2(p)]\\ &= \frac{1}{M_2}\sum_p h(p)\sum_{s=0}^{r}\sum_{k(s)}\sum_{j(s)}p^{-1}(-1)^{r-s}\delta_{k(s)}(p,j(s))\prod_{i=1}^{s}(m_{k_i}-1)\\ &\ll \frac{1}{M_2}\sum_p\sum_{s=0}^{r}\sum_{k(s)}\sum_{j(s)}p^{-1}\delta_{k(s)}(p,j(s))\prod_{i=1}^{s}(m_{k_i}-1)\\ &= \sum_{s=0}^{r}\sum_{k(s)}\sum_{j(s)}S_{2,k(s),j(s)}. \qquad \text{(B.19)}\end{aligned}$$

The term where $s=0$ is handled easily (recall $m/M_2 \le 3^r$):

$$S_{2,0,0} = \frac{1}{M_2}\sum_p^{m^\sigma}p^{-1} \ll 3^r\frac{\log m^\sigma}{m}. \qquad \text{(B.20)}$$

We would like to handle the terms for $s \neq 0$ analogously as before. The congruences on p from $k(s)$ and $j(s)$ force us to sum only over certain primes mod $\prod_{i=1}^{s}m_{k_i}$, with each prime satisfying $p \ge m_{k_i}\pm1$. We increase the sum by summing over all integers satisfying these congruences. As each congruence class mod $\prod_{i=1}^{s}m_{k_i}$ has basically the same sum, we can bound our sum over primes satisfying the congruences $k(s), j(s)$ by $\prod_{i=1}^{s}(m_{k_i})^{-1}\sum_{n=1}^{m^\sigma}n^{-1} = \prod_{i=1}^{s}(m_{k_i})^{-1}\log m^\sigma$.

There is one slight problem with this argument. Before each prime was congruent to 1 mod each prime m_{k_i}, hence the first prime occurred no earlier than at $1+\prod_{k=1}^{s} m_{k_i}$. Now, however, some primes are congruent to $+1$ mod m_{k_i}, some to -1, and it is possible the first such prime occurs before $\prod_{k=1}^{s} m_{k_i}$.

For example, say the prime is congruent to $+1$ mod 11, and -1 mod $3,5,17$. We want the prime to be greater than $3\cdot 5\cdot 11\cdot 17$, but $3\cdot 5\cdot 17-1$ is congruent to -1 mod $3,5,17$ and $+1$ mod 11. (Fortunately it equals 254, which is composite).

So, for each pair $(k(s),j(s))$ we handle all but the possibly first prime as we did in the First Sum case. We now need an estimate on the possible error for low primes. Fortunately, there is at most one for each pair, and as our sum has a $\frac{1}{p}$, we can expect cancellation if it is large.

Fix now a pair (remember there are at most 3^r pairs). As we never specified the order of the primes m_i, without loss of generality (basically, for notational convenience) we may assume that our prime p is congruent to $+1$ mod $m_{k_1}\cdots m_{k_a}$, and -1 mod $m_{k_{a+1}}\cdots m_{k_s}$.

The contribution to the second sum from the possible low prime in this pair is

$$\frac{1}{M_2}\frac{1}{p}\prod_{i=1}^{s}(m_{k_i}-1). \tag{B.21}$$

How small can p be? The $+1$ congruences imply that $p\equiv 1(m_{k_1}\cdots m_{k_a})$, so p is at least $m_{k_1}\cdots m_{k_a}+1$. Similarly the -1 congruences imply p is at least $m_{k_{a+1}}\cdots m_{k_s}-1$. Since the product of these two lower bounds is greater than $\prod_{i=1}^{s}(m_{k_i}-1)$, at least one must be greater than $(\prod_{i=1}^{s}(m_{k_i}-1))^{\frac{1}{2}}$. Therefore the contribution to the second sum from the possible low prime in this pair is bounded by (remember $m/M_2\le 3^r$)

$$\frac{1}{M_2}\left(\prod_{i=1}^{s}(m_{k_i}-1)\right)^{\frac{1}{2}} \le \frac{m^{\frac{1}{2}}}{M_2}\le 3^r m^{-\frac{1}{2}}. \tag{B.22}$$

Combining this with the estimate for the primes larger than $\prod_{i=1}^{s}(m_{k_i}-1)$ yields

$$S_{2,k(s),j(s)} \ \ll\ 3^r m^{-\frac{1}{2}}+\frac{3^r}{m}\log m^{\sigma}, \tag{B.23}$$

yielding (as there are 3^r pairs)

$$S_2=\sum_{s=0}^{r}\sum_{k(s)}\sum_{j(s)} S_{2,k(s),j(s)} \ll 9^r m^{-\frac{1}{2}}. \tag{B.24}$$

B.3. **Density Function in the Square-free case.**

Theorem B.1 (Density Function for Square-free m)**.** *Let $\widehat{\phi}$ be an even Schwartz function with supp($\widehat{\phi}$) $\subset(-2,2)$. Fix an $r\ge 1$. Let $\mathcal{F}_m=\{\chi:\chi$ is primitive mod $m\}$, where m is a square-free odd integer. Then assuming GRH we have*

$$\frac{1}{\mathcal{F}_m}\sum_{\chi\in\mathcal{F}_m}\ \sum_{\gamma:L(\frac{1}{2}+i\gamma,\chi)=0}\phi\left(\gamma\frac{\log(m/\pi)}{2\pi}\right)=\int_{-\infty}^{\infty}\phi(y)dy+O\left(\frac{1}{\log m}\right). \tag{B.25}$$

We note for future reference the following bounds on the First and Second sums:

Lemma B.2. *Let m be a square-free odd integer with $r = r(m)$ factors. Let $m = \prod_{i=1}^{r} m_i$ and $M_2 = \prod_{i=1}^{r}(m_i - 2)$. Consider the family $\mathcal{F}_m$ of primitive characters mod m. There are M_2 such characters, and the First and Second sums satisfy the following bounds:*

$$\begin{aligned} S_1 &\ll \frac{1}{M_2} 2^r m^{\frac{1}{2}\sigma} \\ S_2 &\ll \frac{1}{M_2} 3^r m^{\frac{1}{2}}. \end{aligned} \tag{B.26}$$

Appendix C. Dirichlet Characters from Square-free Numbers

We now generalize the results of the previous section to consider the family $\mathcal{F}_N$ of all primitive characters whose conductor is an odd square-free integer in $[N, 2N]$. Some of the bounds below can be improved, but as the improvements do not increase the range of convergence, they will only be sketched.

First we calculate the number of primitive characters arising from odd square-free numbers $m \in [N, 2N]$. Let $n = n_1 n_2 \cdots n_r$. Then n contributes $(n_1 - 2) \cdots (n_r - 2)$ characters. On average we might expect this to be (up to a constant) N, and as a positive percent of numbers are square-free, we might expect there to be cN^2 characters.

Instead we prove there are at least $N^2 / \log^2 N$ primitive characters in the family. There are at least $N/\log^2 N + 1$ primes in the interval. For each prime p (except possibly the first) we have $p - 2 \geq N$. Hence there are at least $N \cdot \frac{N}{\log^2 N} = N^2 \log^{-2} N$ primitive characters. Let $M = |\mathcal{F}|$. Then

$$M \geq N^2 \log^{-2} N \quad \Rightarrow \quad \frac{1}{M} \leq \frac{\log^2 N}{N^2}. \tag{C.1}$$

We recall the results from the previous section. Fix an odd square-free number $m \in [N, 2N]$, and say m has $r = r(m)$ factors. Before we divided the First and Second sums by $M_2 = (m_1 - 2) \cdots (m_r - 2)$, as this was the number of primitive characters in our family. Now we divide by M. Hence the contribution to the First and Second sum from this m is

$$\begin{aligned} S_{1,m} &\ll \frac{1}{M} 2^{r(m)} m^{\frac{1}{2}\sigma} \\ S_{2,m} &\ll \frac{1}{M} 3^{r(m)} m^{\frac{1}{2}}. \end{aligned} \tag{C.2}$$

Note that $2^{r(m)} = \tau(m)$, the number of divisors of m. While it is possible to prove

$$\sum_{n \leq x} \tau^l(n) \ll x (\log x)^{2^l - 1} \tag{C.3}$$

the crude bound

$$\tau(n) \leq c(\epsilon) n^{\epsilon} \tag{C.4}$$

yields the same region of convergence. Note $3^{r(m)} \leq \tau^2(m)$. Therefore the contributions to the first sum is majorized by

$$\begin{aligned} S_1 &= \sum_{\substack{m=N \\ m \text{ square-free}}}^{2N} S_{1,m} \\ &\ll \sum_{m=N}^{2N} \frac{1}{M} 2^{r(m)} m^{\frac{1}{2}\sigma} \\ &\ll \frac{1}{M} N^{\frac{1}{2}\sigma} \sum_{m=N}^{2N} \tau(m) \\ &\ll \frac{1}{M} N^{\frac{1}{2}\sigma} c(\epsilon) N^{1+\epsilon} \\ &\ll \frac{\log^2 N}{N^2} N^{\frac{1}{2}\sigma} c(\epsilon) N^{1+\epsilon} \\ &\ll c(\epsilon) N^{\frac{1}{2}\sigma+\epsilon-1} \log^2 N. \end{aligned} \tag{C.5}$$

For $\sigma < 2$, choosing $\epsilon < 1 - \frac{1}{2}\sigma$ yields S_1 goes to zero as N tends to infinity. For S_2 we have

$$\begin{aligned} S_2 &= \sum_{\substack{m=N \\ m \ squarefree}}^{2N} S_{2,m} \\ &\ll \sum_{m=N}^{2N} \frac{1}{M} 3^{r(m)} m^{\frac{1}{2}} \\ &\ll \frac{1}{M} N^{\frac{1}{2}} \sum_{m=N}^{2N} \tau^2(m) \\ &\ll c(\epsilon) \frac{\log^2 N}{N^2} N^{\frac{1}{2}} N^{1+2\epsilon} \\ &\ll c(\epsilon) N^{2\epsilon-\frac{1}{2}} \log^2 N. \end{aligned} \tag{C.6}$$

which converges to zero as N tends to infinity for all σ. Hence we have proved

Theorem C.1 (Dirichlet Characters from Square-free Numbers)**.** *Let $\mathcal{F}_N$ denote the family of primitive Dirichlet characters arising from odd square-free numbers $m \in [N, 2N]$. Denote the conductor of χ by $c(\chi)$. Then $\forall \sigma < 2$*

$$\frac{1}{\mathcal{F}_N} \sum_{\chi \in \mathcal{F}_N} \sum_{\gamma : L(\frac{1}{2}+i\gamma, \chi)=0} \phi\left(\gamma \frac{\log(c(\chi)/\pi)}{2\pi}\right) = \int_{-\infty}^{\infty} \phi(y) dy + O\left(\frac{1}{\log N}\right). \tag{C.7}$$

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