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LOW LYING ZEROS OF FAMILIES OF L-FUNCTIONS

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CONTENTS

1. Introduction

In Iwaniec-Sarnak [IS] the percentage of nonvanishing of central values of families of GL_2 automorphic L-functions was investigated. In this paper we examine the distribution of zeros which are at or near $s=\frac{1}{2}$ (that is the central point) for such families of L-functions. Unlike [IS], most of the results in this paper are conditional, depending on the generalized Riemann Hypothesis (GRH). It is by no means obvious, but on the other hand not surprising, that this allows us to obtain sharper results on the nonvanishing.

The density and the distribution of zeros near $s=\frac{1}{2}$ for the L-functions of certain families $\mathscr{F}$ have been studied recently in Katz-Sarnak [KS1, KS2]. The philosophy and conjectures which emerge assert that for such families, the distributions of the low lying zeros, when we order the L-functions by their conductors (see below), are

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governed by a symmetry group $G(\mathcal{F})$ associated with $\mathcal{F}$. In the case where we can identify the function field analogues and compute the scaling limits of the corresponding monodromies of the family, one arrives at such a symmetry $G(\mathcal{F})$. Examples where this can be done and where the corresponding predictions can be verified are given in [KS2]. One of our aims in this paper is to pursue these conjectures for the L-functions associated with automorphic forms on GL_2 and in one case on GL_3.

• *The Density Conjecture.* — Before stating our results we describe the goal in general terms. Let $\mathcal{F}$ be a family of automorphic forms to be specified later. To any f in $\mathcal{F}$ we associate the L-function

$$L(s, f) = \sum_{1}^{\infty} \lambda_f(n) n^{-s} . \tag{1.1}$$

We assume that $L(s, f)$ is entire and self-dual. The latter means that the corresponding completed function $\Lambda(s, f) = L_\infty(s, f) L(s, f)$ satisfies a functional equation of type

$$\Lambda(s, f) = \varepsilon_f \Lambda(1 - s, f) \tag{1.2}$$

with $\varepsilon_f = \pm 1$. We say the functional equation is even or odd according to $\varepsilon_f = 1$ or -1. The sign ε_f has a considerable impact on the distribution of zeros of $L(s, f)$ near the central point $s = \frac{1}{2}$.

Unless otherwise stated we assume that the Riemann hypothesis holds for each $L(s, f)$ with $f \in \mathcal{F}$ and for all Dirichlet L-functions (including the Riemann zeta-function). Accordingly we denote the nontrivial zeros of $L(s, f)$ by

$$\rho_f = \frac{1}{2} + i\gamma_f . \tag{1.3}$$

They appear in complex pairs. By classical arguments of Riemann it follows that the number of zeros with $|\gamma_f|$ bounded by an absolute large constant is of order $\log c_f$, where $c_f > 1$ is a certain number assigned to f (which we call the analytic conductor of f). We shall give the exact values of c_f in particular cases.

We shall investigate the "one-level" densities (see [KS1], p. 405) of the low lying zeros. To this end we define

$$D(f; \phi) = \sum_{\gamma_f} \phi\left(\frac{\gamma_f}{2\pi} \log c_f\right) \tag{1.4}$$

where $\phi(x)$ is an even function which vanishes rapidly as $|x| \to \infty$. Here, of course, γ_f are counted with the corresponding multiplicities. Throughout the paper ϕ will be a Schwartz class function for which the Fourier transform

$$\widehat{\phi}(y) = \int_{-\infty}^{\infty} \phi(x) e^{-2\pi i x y} dx \tag{1.5}$$

has compact support so that $\phi(x)$ extends to an entire function. Since ϕ is localized, the scaling by $\frac{1}{2\pi}\log c_f$ means that $D(f;\phi)$ with varying ϕ measures the density of zeros of $L(s,f)$ which are with $O(1/\log c_f)$ of the central point $s=\frac{1}{2}$.

In practice it is impossible to evaluate asymptotically the sum (1.4) for a single L-function, because such a sum captures only few zeros (essentially a bounded number of zeros). Therefore we consider various averages over f in $\mathcal{F}$ ordered by the conductor. First we choose the finite subsets

$$\{f\in\mathcal{F}\,;\, c_f=Q\} \tag{1.6}$$

and let $Q\to\infty$. Later, in order to get stronger results, we take the larger sets

$$\{f\in\mathcal{F}\,;\, c_f\leqslant Q\}\ . \tag{1.7}$$

To unify the presentation of both cases we denote by $\mathcal{F}(Q)$ one of the two sets above, and we consider the average (or expectation)

$$E(\mathcal{F}(Q);\phi)=\frac{1}{|\mathcal{F}(Q)|}\sum_{f\in\mathcal{F}(Q)}D(f;\phi)\ . \tag{1.8}$$

We assume that $\mathcal{F}$ has plenty of independent forms (relative to conductors) so that $|\mathcal{F}(Q)|\to\infty$ as $Q\to\infty$. If the family $\mathcal{F}$ is complete in a certain spectral sense, it is reasonable to assume that $E(\mathcal{F}(Q);\phi)$ converges. Precisely, if $\phi\in\mathcal{S}(\mathbf{R})$ with support of $\widehat{\phi}$ compact we may be able to show that

$$\lim_{Q\to\infty}E(\mathcal{F}(Q);\phi)=\int_{-\infty}^{\infty}\phi(x)W(\mathcal{F})(x)dx\ , \tag{1.9}$$

where $W(\mathcal{F})$ is a distribution depending on $\mathcal{F}$. We shall refer to the above statement (with $W(\mathcal{F})$ explicitly given) as the "Density Theorem" for the family $\mathcal{F}$.

The numerous observations and results in [KS1], [KS2] suggest that $W(\mathcal{F})$ depends on $\mathcal{F}$ through a symmetry group $G(\mathcal{F})$ so we shall be writing $W(G)$ in place of $W(\mathcal{F})$. For G a symmetry of type O (that is the scaling limit of orthogonal groups $O(N)$), or SO(even) (that is the scaling limit of $SO(2N)$), or SO(odd) (that is the scaling limit of $SO(2N+1)$), or Sp (that is the scaling limit of $Sp(2N)$), the corresponding densities $W(G)$ are determined in [KS1] on page 409. They are as follows:

$$W(O)(x)=1+\frac{1}{2}\delta_0(x)\ , \tag{1.10}$$

$$W(SO(\text{even}))(x)=1+\frac{\sin 2\pi x}{2\pi x}\ , \tag{1.11}$$

$$W(SO(\text{odd}))(x)=1-\frac{\sin 2\pi x}{2\pi x}+\delta_0(x)\ , \tag{1.12}$$

$$W(Sp)(x) = 1 - \frac{\sin 2\pi x}{2\pi x}. \tag{1.13}$$

Here $\delta_0(x)$ is the Dirac distribution at $x=0$.

• *Statement of Main Results.* — Throughout we assume that k is even and N is squarefree, and we shall recall these assumptions occasionally but not always. Let $H_k^\star(N)$ denote the set of holomorphic cusp forms of weight k which are newforms of level N (see the next section for more details). For any $f \in H_k^\star(N)$ we define its analytic conductor to be

$$c_f = k^2 N. \tag{1.14}$$

We shall consider separately the subsets $H_k^+(N)$ and $H_k^-(N)$ of the forms f for which $\varepsilon_f = 1$ and $\varepsilon_f = -1$, respectively. In particular for $N=1$ we have $\varepsilon_f = i^k$, so $H_k^+(1) = H_k^\star(1)$ if $k \equiv 0 \pmod 4$ and $H_k^-(1) = H_k^\star(1)$ if $k \equiv 2 \pmod 4$. The whole space $S_k(1)$ is spanned by $H_k^\star(1)$, so

$$|H_k^\star(1)| = \dim S_k(1) \sim \frac{k}{12} \tag{1.15}$$

as $k \to \infty$. For $N \neq 1$ we have

$$|H_k^+(N)| \sim |H_k^-(N)| \sim \frac{1}{2}|H_k^\star(N)| \sim \frac{k-1}{24}\varphi(N) \tag{1.16}$$

as $kN \to \infty$ (more precise asymptotics are given in Corollary 2.14). For these families the expectation (see [KS1], page 18) is that G is orthogonal and that the subsets with $\varepsilon_f = 1, \varepsilon_f = -1$ are SO(even), SO(odd), respectively.

Our first results towards the Density Conjecture are

Theorem **1.1.** — *Fix any $\phi \in \mathcal{S}(\mathbf{R})$ with the support of $\widehat{\phi}$ in $(-2, 2)$. Then, as* N *runs over squarefree numbers we have*

$$\lim_{N\to\infty} \frac{1}{|H_k^+(N)|} \sum_{f \in H_k^+(N)} D(f;\phi) = \int_{-\infty}^{\infty} \phi(x) W(SO(\text{even}))(x)\,dx, \tag{1.17}$$

$$\lim_{N\to\infty} \frac{1}{|H_k^-(N)|} \sum_{f \in H_k^-(N)} D(f;\phi) = \int_{-\infty}^{\infty} \phi(x) W(SO(\text{odd}))(x)\,dx, \tag{1.18}$$

$$\lim_{N\to\infty} \frac{1}{|H_k^\star(N)|} \sum_{f \in H_k^\star(N)} D(f;\phi) = \int_{-\infty}^{\infty} \phi(x) W(O)(x)\,dx. \tag{1.19}$$

The first two results follow from Theorem 7.2, while the last one is deduced from the first two by the asymptotics in (1.16).

Remark A. — Here the restriction N to squarefree numbers is made merely for simplifications in the theory of newforms as well as in some technical arguments. It is almost certain that the same densities W(G) as above will appear in the limit as the level N runs to infinity over all integers. Note that for fixed k the ratio $\log c_f / \log |H_k^\star(N)|$ tends to one.

Theorem **1.2.** — *Fix any* $\phi \in \mathcal{S}(\mathbf{R})$ *with the support of* $\widehat{\phi}$ *in* $(-1, 1)$. *Then we have*

$$\textbf{(1.20)} \qquad \lim_{kN\to\infty} \frac{1}{|H_k^+(N)|} \sum_{f\in H_k^+(N)} D(f;\phi) = \int_{-\infty}^{\infty} \phi(x) W(SO(\text{even}))(x)dx\,.$$

$$\textbf{(1.21)} \qquad \lim_{kN\to\infty} \frac{1}{|H_k^-(N)|} \sum_{f\in H_k^-(N)} D(f;\phi) = \int_{-\infty}^{\infty} \phi(x) W(SO(\text{odd}))(x)dx$$

(recall that N *runs over squarefree numbers and* k *runs over even numbers, and in the case* $N = 1$ *we assume* $k \equiv 1 \pm 1 \pmod 4$*, or else the sets* $H_k^\pm(1)$ *are empty, respectively).*

Theorem 1.2 follows from Theorem 5.1 of Section 5.

Note that for the test function in Theorem 1.2 we made a restriction which is twice as strong as that in Theorem 1.1. This restriction has a natural source. The point is that the conductor $c_f = k^2 N$ for the forms in the set $H_k^\star(N)$, with N fixed, is twice as large as the cardinality of the set (on a logarithmic scale). In the next theorem we bring this ratio back to one by performing extra averaging over k. Put

$$\textbf{(1.22)} \qquad M^+(K, N) = \sum_{k\leqslant K} |H_k^+(N)|\,,$$

$$\textbf{(1.23)} \qquad M^-(K, N) = \sum_{k\leqslant K} |H_k^-(N)|\,,$$

and $M^\star(K, N) = M^+(K, N) + M^-(K, N)$. The last satisfies the asymptotics $M^\star(K, N) \sim 2M^+(K, N) \sim 2M^-(K, N)$ as $KN \to \infty$.

Theorem **1.3.** — *Fix any* $\phi \in \mathcal{S}(\mathbf{R})$ *with the support of* $\widehat{\phi}$ *in* $(-2, 2)$. *Then we have*

$$\textbf{(1.24)} \qquad \lim_{KN\to\infty} \frac{1}{M^+(K, N)} \sum_{k\leqslant K} \sum_{f\in H_k^+(N)} D(f;\phi) = \int_{-\infty}^{\infty} \phi(x) W(SO(\text{even}))(x)dx\,.$$

$$\textbf{(1.25)} \qquad \lim_{KN\to\infty} \frac{1}{M^-(K, N)} \sum_{k\leqslant K} \sum_{f\in H_k^-(N)} D(f;\phi) = \int_{-\infty}^{\infty} \phi(x) W(SO(\text{odd}))(x)dx\,.$$

and

$$\lim_{KN\to\infty} \frac{1}{M^\star(K,N)} \sum_{k\leqslant K} \sum_{f\in H_k^\star(N)} D(f;\phi) = \int_{-\infty}^{\infty} \phi(x) W(O)(x) dx\,. \tag{1.26}$$

These results follow from Theorem 8.4 of Section 8.

Finally we consider the family $H_k^{(2)}(N)$ of automorphic forms $sym^2(f)$, where f is from $H_k^\star(N)$ and $sym^2(f)$ denotes the symmetric square representation associated to f (see (3.14)). These are (after an application of Gelbart-Jacquet lifting [GJ]) automorphic forms on GL_3. Shimura [S1] was first to establish analytic properties of the corresponding L-functions $L(s, sym^2(f))$ (see the next section). For any $f\in H_k^\star(N)$ the sign of the functional equation is $\varepsilon_{sym^2(f)}=1$. Examining the functional equation (3.18) we define the analytic conductor of $sym^2(f)$ by

$$c_{sym^2(f)} = k^2N^2\,, \quad \text{if } f\in H_k^\star(N)\,. \tag{1.27}$$

The symmetry group for this family appears to be the scaling limit of symplectic groups Sp (see [KS2], page 19). We are able to verify the Density Conjecture in various ranges. First we prove

Theorem **1.4.** — *Fix any* $\phi\in\mathcal{S}(\mathbf{R})$ *with the support of* $\widehat{\phi}$ *in* $(-\frac{1}{2},\frac{1}{2})$. *Then we have*

$$\lim_{kN\to\infty} \frac{1}{|H_k^\star(N)|} \sum_{f\in H_k^\star(N)} D(\mathrm{sym}^2(f);\phi) = \int_{-\infty}^{\infty} \phi(x) W(Sp)(x) dx\,. \tag{1.28}$$

This result follows from Theorem 5.1.

Introducing further averaging over k (recall that k runs over positive even numbers) we extend the range of test functions considerably, but not as much as in Theorem 1.3. Precisely we prove the following

Theorem **1.5.** — *Fix any* $\phi\in\mathcal{S}(\mathbf{R})$ *with the support of* $\widehat{\phi}$ *in* $(-\frac{3}{2},\frac{3}{2})$. *Then we have*

$$\lim_{K\to\infty} \frac{1}{M^\star(K,N)} \sum_{k\leqslant K} \sum_{f\in H_k^\star(N)} D(\mathrm{sym}^2(f);\phi) = \int_{-\infty}^{\infty} \phi(x) W(Sp)(x) dx\,. \tag{1.29}$$

This result follows from Theorem 9.1.

Remark B. — Theorems 1.2 and 1.4 can be established without recourse to the Riemann hypothesis for Dirichlet L-functions. The reason is that for ϕ with support of $\widehat{\phi}$ in $(-1,1)$ and $(-\frac{1}{2},\frac{1}{2})$ respectively, the main contribution comes only from the diagonal term in the Petersson formula (2.8). However when we extend the support of $\widehat{\phi}$ beyond these segments, new non-diagonal terms contribute to the asymptotics (such additional contributions occur in a similar context in [IS], and earlier in [DFI]).

These terms arise from Kloosterman sums and are of an arithmetic nature. For an evaluation of sums of Kloosterman sums we need uniform asymptotics for primes in arithmetic progressions, and for the latter we apply the Riemann hypothesis for the classical Dirichlet L-functions.

Remark C. — It is interesting that the Riemann hypothesis for the Dirichlet L-functions does not help if one attempts to extend the range in Theorem 1.4. The problem is that in the critical range of primes in question either the phase of the Bessel function is too large in terms of k, or the modulus of Kloosterman sum is too large in terms of N. For this purpose more relevant is the method of Vinogradov for estimating sums over primes and related bilinear forms. His method, together with Weyl's estimates for exponential sums, would probably allow for a small extension of the range in Theorem 1.4 for N fixed. If k is fixed, then one needs a cancellation of $S(p, p; c)$ in a sum over primes p with a modulus c which is as large as p. This can be established by Vinogradov's method together with Burgess' estimates for short character sums and the Riemann hypothesis for elliptic curves (Hasse's estimate). In this way one could get a small extension of the range in Theorem 1.4 for k fixed. Because the improvements are small we do not show the details. Our purpose in making the above remarks is to point out interesting cases for which the classical techniques are better than the Riemann hypothesis.

Remark D. — The extension of our density results to test functions $\phi(x)$ with $\widehat{\phi}(y)$ having larger support is significant. To see the critical features we write by Plancherel theorem

$$\int_{-\infty}^{\infty} \phi(x) W(G)(x) dx = \int_{-\infty}^{\infty} \widehat{\phi}(y) \widehat{W}(G)(y) dy . \tag{1.30}$$

Note that the Fourier transforms of $1, \delta_0(x)$, $(\sin 2\pi x)/2\pi x$ are (in the sense of distributions) $\delta_0(y)$, 1, $\frac{1}{2}\eta(y)$ respectively, where $\eta(y)$ is the characteristic function of the segment $[-1, 1]$, more appropriately $\eta(y) = 1, \frac{1}{2}, 0$ for $|y| < 1, y = \pm 1, |y| > 1$. Hence the Fourier transforms of the densities for the groups O, SO(even), SO(odd) are

$$\widehat{W}(O)(y) = \delta_0(y) + \frac{1}{2} , \tag{1.31}$$

$$\widehat{W}(SO(even))(y) = \delta_0(y) + \frac{1}{2}\eta(y) , \tag{1.32}$$

$$\widehat{W}(SO(odd))(y) = \delta_0(y) - \frac{1}{2}\eta(y) + 1 . \tag{1.33}$$

They all agree in $-1 < y < 1$, but split at $y = \pm 1$. This means that in order to be able to distinguish the families of automorphic L-functions of different parity by looking at the distribution of the low lying zeros one must use the test functions ϕ with the support of $\widehat{\phi}$ larger than $[-1, 1]$. For the group $G = Sp$, we get

$$\widehat{W}(Sp)(y) = \delta_0(y) - \tfrac{1}{2}\eta(y) \tag{1.34}$$

which also has a discontinuity at $y = \pm 1$.

Remark E. — We note that the results in Theorems 1.1, 1.3, 1.5 go well beyond the similar analysis of the pair and higher correlations for the zeros of the Riemann zeta function (Montgomery [Mon], Hejhal [Hej], Rudnick-Sarnak [RS]). The analysis in those works extends only as far as the diagonal terms being the main contribution to the asymptotics. In particular, in as much as our results test the limit (1.9) beyond the diagonal, we feel they lend strong evidence to the truth of the full Density Conjecture.

• *Applications to the non-vanishing of central values.* — There are applications of Density Theorems for counting automorphic forms f in the relevant family $\mathcal{F}$ for which the point $s = \frac{1}{2}$ is a zero of $L(s, f)$ of given order. Put

$$p_m(Q) = \frac{1}{|\mathcal{F}(Q)|}\left|\{f \in \mathcal{F}(Q); \quad \operatorname*{ord}_{s=\frac{1}{2}} L(s, f) = m\}\right| . \tag{1.35}$$

Clearly

$$\sum_{m=0}^{\infty} p_m(Q) = 1 . \tag{1.36}$$

On the other hand, by choosing test functions $\phi(x)$ such that $\phi(x) \geqslant 0$, $\phi(0) = 1$ and the support of $\widehat{\phi}(y)$ compact, one derives from (1.9) and (1.30) that

$$\sum_{m=1}^{\infty} m p_m(Q) < g + \varepsilon \tag{1.37}$$

for any $\varepsilon > 0$, provided Q is sufficiently large, where

$$g = \int_{-\infty}^{\infty} \widehat{\phi}(y)\widehat{W}(G)(y)dy . \tag{1.38}$$

This yields the upper bound, $p_m(Q) < m^{-1}(g + \varepsilon)$ for any $m \geqslant 1$. Moreover, subtracting (1.37) from (1.36), one gets the lower bound $p_0(Q) > 1 - g - \varepsilon$.

Somewhat better estimates follow along the above lines by breaking up the family with respect to the parity of the functional equation. Indeed, if $\varepsilon_f = 1$ for all $f \in \mathcal{F}$

then the order of zero of $L(s, f)$ at $s=\frac{1}{2}$ is always even, so $p_m(Q)=0$ if $2 \nmid m$, and one gets

$$p_0(Q) > \frac{1}{2}(2-g-\varepsilon)\,. \tag{1.40}$$

If $\varepsilon_f = -1$ for all $f \in \mathscr{F}$, then $p_m(Q)=0$ if $2|m$, and one gets

$$p_1(Q) > \frac{1}{2}(3-g-\varepsilon)\,. \tag{1.41}$$

Recall that the integral (1.38) depends on the test function ϕ, and one should make g as small as possible to get the best results. An analysis of the optimal choice for this purpose involves extremizing a quadratic form subject to a linear constraint and is carried out in Appendix A. The Fourier pair

$$\phi(x) = \left(\frac{\sin \pi v x}{\pi v x}\right)^2, \quad \hat{\phi}(y) = \frac{1}{v}\left(1 - \frac{|y|}{v}\right) \qquad \text{if } |y| < v \tag{1.42}$$

yields quite good results. Using (1.31-1.34) and (1.42) we compute $g=g(v)$ in various cases. We get

$$g(v) = \frac{1}{v} + \frac{1}{2} \qquad \text{for } G = O\,. \tag{1.43}$$

$$g(v) = \begin{cases} \frac{1}{v} + \frac{1}{2}, & \text{if } v \leqslant 1 \\ \frac{2}{v} - \frac{1}{2v^2}, & \text{if } v \geqslant 1 \end{cases} \qquad \text{for } G = SO(\text{even})\,, \tag{1.44}$$

$$g(v) = \begin{cases} \frac{1}{v} + \frac{1}{2}, & \text{if } v \leqslant 1 \\ 1 + \frac{1}{2v^2}, & \text{if } v \geqslant 1 \end{cases} \qquad \text{for } G = SO(\text{odd})\,, \tag{1.45}$$

$$g(v) = \begin{cases} \frac{1}{v} - \frac{1}{2}, & \text{if } v \leqslant 1 \\ \frac{1}{2v^2}, & \text{if } v \geqslant 1 \end{cases} \qquad \text{for } G = Sp\,. \tag{1.46}$$

In particular for $v=2$ our estimates (1.37), (1.40), (1.41) yield

$$\sum_{m=1}^{\infty} m p_m(Q) < 1 + \varepsilon, \quad \text{if } G = O\,, \tag{1.47}$$

$$p_0(Q) > \frac{9}{16} - \varepsilon, \quad \text{if } G = SO(\text{even})\,, \tag{1.48}$$

$$p_1(Q) > \frac{15}{16} - \varepsilon, \quad \text{if } G = SO(\text{odd})\,. \tag{1.49}$$

We also have the results (1.20) and (1.21) for $G = SO(\text{even})$ and $G = SO(\text{odd})$ with $v = 1$ which yield $p_0(Q) > \frac{1}{4} - \varepsilon$ and $p_1(Q) > \frac{3}{4} - \varepsilon$, respectively. Moreover, the result (1.28) for $G = Sp$ with $v = \frac{1}{2}$ yields $p_0(Q) > \frac{1}{4} - \varepsilon$ and (1.29) with $v = \frac{3}{2}$ yields

$$p_0(Q) > \frac{8}{9} - \varepsilon, \quad \text{if } G = Sp\,. \tag{1.50}$$

As shown in Appendix A the test function (1.42) is definitely not optimal when $v > 1$ for the symmetries SO(even), SO(odd) and Sp. Consequently the corresponding bounds above can be improved by an absolute positive constant. However, for $G = O$ the function (1.43) is optimal. Nevertheless we can reduce the upper bound (1.47) slightly by roundabout arguments. To this end we split the family according to $\varepsilon_f = \pm 1$, apply the improved version (1.47) to each subfamily separately (i.e. for the groups SO(even), SO(odd)) and add up the results. The precise improvements obtained by using the optimal test function is given in Appendix A. By the above estimates and remarks we conclude

Corollary **1.6.** — *We have*

$$\liminf_{kN\to\infty} \frac{1}{|H_k^+(N)|} |\{f \in H_k^+(N);\ L(\tfrac{1}{2}, f) \neq 0\}| \geqslant \frac{1}{4}\,, \tag{1.51}$$

$$\liminf_{kN\to\infty} \frac{1}{|H_k^-(N)|} |\{f \in H_k^-(N);\ L'(\tfrac{1}{2}, f) \neq 0\}| \geqslant \frac{3}{4}\,. \tag{1.52}$$

Fixing k, we have

$$\liminf_{N\to\infty} \frac{1}{|H_k^\star(N)|} \sum_{f\in H_k^\star(N)} \operatorname*{ord}_{s=\frac{1}{2}} L(s, f) < 1\,, \tag{1.53}$$

$$\liminf_{N\to\infty} \frac{1}{|H_k^+(N)|} |\{f \in H_k^+(N);\ L(\tfrac{1}{2}, f) \neq 0\}| > \frac{9}{16}\,, \tag{1.54}$$

$$\liminf_{N\to\infty} \frac{1}{|H_k^-(N)|} |\{f \in H_k^-(N);\ L'(\tfrac{1}{2}, f) \neq 0\}| > \frac{15}{16}\,. \tag{1.55}$$

Corollary **1.7.** — *Fixing* N *we have*

$$\liminf_{K\to\infty} \frac{1}{M^+(K, N)} |\{f \in H_k^+(N);\ k \leqslant K,\ L(\tfrac{1}{2}, f) \neq 0\}| > \frac{9}{16}\,, \tag{1.56}$$

$$\liminf_{K\to\infty} \frac{1}{M^-(K, N)} |\{f \in H_k^-(N);\ k \leqslant K,\ L'(\tfrac{1}{2}, f) \neq 0\}| > \frac{15}{16}\,. \tag{1.57}$$

Corollary **1.8.** — *We have*

$$\liminf_{kN\to\infty} \frac{1}{|H_k^\star(N)|}|\{f \in H_k^\star(N);\ L(\tfrac{1}{2}, \mathrm{sym}^2(f)) \neq 0\}| \geqslant \frac{1}{4}. \tag{1.58}$$

For N *fixed*

$$\liminf_{K\to\infty} \frac{1}{M^\star(K, N)}|\{f \in H_k^\star(N);\ k \leqslant K,\ L(\tfrac{1}{2}, \mathrm{sym}^2(f)) \neq 0\}| > \frac{8}{9}. \tag{1.59}$$

Remark F. — The Density Conjecture would yield the true values (presumably) for the above limits, that is the value $\frac{1}{2}$ for (1.53) and the value 1 for the other limits.

Remark G. — Various of the constants in the estimates for the limits in the above Corollaries depend crucially on the extensions of the range of the support of $\widehat{\phi}$ beyond the segment $[-1, 1]$ in our density theorems. Without these extensions one would, for example, obtain for the limits (1.54) and (1.56) the weaker inequality $\liminf \geqslant \frac{1}{2}$ in place of $\liminf > \frac{9}{16}$. In [IS] this weaker inequality is established unconditionally. Moreover it is shown that improving this to anything bigger than $\frac{1}{2}$ is intimately connected to the Landau-Siegel zero. Of course, in the above Corollaries such a question is not an issue since we are assuming the Riemann Hypothesis (not only for the L-functions associated with cusp forms, but also with the Dirichlet characters).

Remark H. — By now there are a number of unconditional results related to Corollary 1.6. Kowalski-Michel [KM1] and Vanderkam [Van] established a positive lower bound for (1.55) and later [KM2] their work led to a lower bound of 7/8 for this case. Kowalski-Michel [KM2] also established an upper bound of 13/2 in (1.53) and there are significant improvements in the recent joint works [KMV1], [KMV2], [KMV3].

Remark I. — When specified to the family $H_2^\star(N)$, the Density Conjecture as well as the results of Corollary 1.6 have applications to the estimation of the ranks of the Jacobians of the modular curves $X_0(N)$, see [KS2] for a description of these implications.

• *Quasi Riemann Hypothesis.* — Up to this point we freely assumed the Riemann hypotheses for Dirichlet L-functions as well as for the automorphic L-functions of the family in question. In what follows there will be no tacit assumptions of any Riemann hypothesis. We restrict attention to the family of L-functions associated with Hecke cusp forms for the modular group (i.e. $L(s, f)$, where $f \in H_k^\star(1)$, k even). First note that for $\phi \in \mathscr{S}(\mathbf{R})$ with $\widehat{\phi}$ compact support (as we always assume) the sum (1.4), which now is (see (1.19) with $N = 1$)

$$D(f;\phi) = \sum_{\gamma_f} \phi\left(\frac{\gamma_f}{2\pi}\log k^2\right),$$

makes sense irrespective of the Riemann hypothesis for $L(s, f)$, because ϕ is entire (so the sum (1.4) is well defined even if γ_f is not real).

Theorem 1.3 provides separately asymptotics for two families broken up by the parity of functional equations, and for this reason it relies on the Riemann hypothesis for Dirichlet's L-functions. If we do not break this parity then we can quite easily establish unconditionally a version of Theorem 1.3. For simplicity here we work with a weighted average, rather than with (1.8). That is we maintain the arithmetical weights $L(1, sym^2(f))^{-1}$ which appear naturally in the Petersson formula. In fact some effort was put into removal of these weights in the results previously stated.

Note that (see 10.7))

$$\sum_{k\leqslant K}\frac{4\pi^2}{k-1}\sum_{f\in H_k^\star(1)} L^{-1}(1, \mathrm{sym}^2(f)) = K + O(1). \tag{1.60}$$

By Theorem 10.2 one gets

Theorem **1.9.** — *Fix any* $\phi \in \mathcal{S}(\mathbf{R})$ *with the support of* $\widehat{\phi}$ *in* $(-2, 2)$. *Then we have*

$$\lim_{K\to\infty}\frac{1}{K}\sum_{k\leqslant K}\frac{4\pi^2}{k-1}\sum_{f\in H_k^\star(1)} D(f;\phi)L^{-1}(1, \mathrm{sym}^2(f)) = \int_{-\infty}^{\infty}\phi(x)W(O)(x)dx\,. \tag{1.61}$$

This result, as well as all the strongest previously stated density theorems, are restricted to the test functions ϕ with the support of $\widehat{\phi}$ in $(-2, 2)$. Interestingly, extending this range is closely related to estimates for some classical exponential sums over primes. Precisely consider the following

Hypothesis S. — For any $x \geqslant 1$, $c \geqslant 1$, and a with $(a, c) = 1$, we have

$$\sum_{p\leqslant x,\, p\equiv a(c)} e(2\sqrt{p}/c) \ll x^{\frac{1}{2}+\varepsilon} \tag{1.62}$$

where ε is any positive number and the implied constant depends only on ε.

Theorem **1.10.** — *Assuming Hypothesis* S *the formula (1.61) is valid for* ϕ *with the support of* $\widehat{\phi}$ *in* $(-\frac{22}{9}, \frac{22}{9})$.

This result follows from Theorem 10.3.

Some comments about Hypothesis S are in order. Firstly, what is needed in order to extend the range $(-2, 2)$ is any bound in (1.62) of the type $c^A x^{\alpha+\varepsilon}$ for some constant exponents $A \geqslant 1$ and $\alpha < \frac{3}{4}$. A nontrivial bound for the sum in (1.62) was established by I. M. Vinogradov [Vin]. His exponent was $\alpha = \frac{7}{8}$, while the standard density hypothesis for Dirichlet L-functions provides a bound with $\alpha = \frac{3}{4}$. In fact Hypothesis S (or for that matter any estimate with $\alpha < \frac{3}{4}$) is closely related to basic questions about the distribution of zeros of Dirichlet L-functions (see Appendix C and also [Fuj]).

Theorem 1.10 shows that the classical "GL_1" exponential sums (of analytic type) as in Hypothesis S are intimately connected to GL_2 L-functions. In fact, remarkably Theorem 1.10 (or any extension of the range $(-2, 2)$) strikes at the Riemann hypothesis for GL_2 L-functions. For example, it implies the following quasi Riemann hypothesis.

Corollary **1.11.** — *Assume Hypothesis* S *and that the zeros of any* $L(s, f)$ *for* $f \in H_k^\star(1)$ *are either real or on* $\operatorname{Re} s = \frac{1}{2}$. *Then for* k *sufficiently large*

$$L(s, f) \neq 0 \quad \text{if } s > \frac{10}{11} + \varepsilon \,. \tag{1.63}$$

Remark J. — The above result is effective and thus in principle one could establish it for all k with numerical verification. The assumption about the zeros of $L(s, f)$ off the critical line being real can probably be removed by considering two variable sums in place of that in Hypothesis S. The analysis however is complicated, and will be left for the future.

Remark K. — The implication of Corollary 1.11 that bounds on the classical "GL_1" sums in Hypothesis S imply a quasi Riemann hypothesis for GL_2 L-functions reminds one of the Lang-Weil Theorem [LW] which gives a quasi Riemann hypothesis for zeta functions of varieties over finite fields by using the Riemann hypothesis for curves. We add that our implication is very different to the direct relation between the Riemann hypothesis for $L(s, f)$ and cancellations in the sums

$$\sum_{p \leqslant x} \lambda_f(p) \tag{1.64}$$

since the sum (1.62) does not mention any GL_2 objects.

2. Basic automorphic forms

In this section we gather some standard facts about cusp forms for the Hecke congruence group $\Gamma_0(N)$ that are needed in this paper. We do not provide complete proofs of all results, but in the less standard cases we indicate how to derive them from available sources. For further background, we recommend the following text books [S1], [Miy], [Iwa], and the articles [AL], [Li], [P1]. Moreover we prove a few new results. In particular, using our special orthogonal basis (see Proposition 2.6) we manage to express neatly sums over newforms by complete sums of Petersson type (see Proposition 2.8 and Proposition 2.11).

Throughout k, N are positive integers, k even. The linear space $S_k(N)$ of cusp forms of weight k and level N is a finite dimensional Hilbert space with respect to the Petersson inner product

$$\langle f, g\rangle = \int_{\Gamma_0(N)\backslash \mathbf{H}} f(z)\bar{g}(z) y^{k-2} dxdy . \tag{2.1}$$

Every $f \in S_k(N)$ has the Fourier expansion of type

$$f(z) = \sum_1^{\infty} a_f(n) e(nz) \tag{2.2}$$

where $e(z) = e^{2\pi i z}$ and $a_f(n)$ are complex numbers (the Fourier coefficients). For notational convenience we introduce the normalized coefficients

$$\psi_f(n) = \left(\frac{\Gamma(k-1)}{(4\pi n)^{k-1}}\right)^{\frac{1}{2}} \|f\|^{-1} a_f(n) \tag{2.3}$$

where $\|f\|^2 = \langle f, f\rangle$. These (as proved by Deligne [Del]) satisfy

$$\psi_f(n) \ll \tau(n) \tag{2.4}$$

where $\tau(n)$ is the divisor function and the implied constant depends on f.

Let $\mathscr{B}_k(N)$ be an orthogonal basis of $S_k(N)$, so we have

$$|\mathscr{B}_k(N)| = \dim S_k(N) \asymp \nu(N) k , \tag{2.5}$$

where

$$\nu(N) = [\Gamma_0(1) : \Gamma_0(N)] = N \prod_{p|N} \left(1 + \frac{1}{p}\right) . \tag{2.6}$$

Given k, N we put

$$\Delta_{k,N}(m, n) = \sum_{f \in \mathscr{B}_k(N)} \overline{\psi}_f(m) \psi_f(n). \tag{2.7}$$

This is basis independent; indeed $\Delta_{k,N}(m, n)$ is the n-th Fourier coefficient of the m-th Poincaré series up to some normalizing factors. The key tool for averaging over cusp forms is the following formula of Petersson [P2] (see also [Iwa]).

Proposition **2.1.** — *For any $m, n \geqslant 1$ we have*

$$\Delta_{k,N}(m, n) = \delta(m, n) + 2\pi i^k \sum_{c \equiv 0 (\mathrm{mod}\, N)} c^{-1} S(m, n; c) J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right) \tag{2.8}$$

where $\delta(m, n)$ *is the diagonal symbol of Kronecker,* $J_{k-1}(x)$ *is the Bessel function and*

$$(2.9) \qquad S(m, n; c) = \sum_{d \,(\mathrm{mod}\, c)}^{\star} e\left(\frac{md + n\bar{d}}{c}\right)$$

is the classical Kloosterman sum. Here $\sum^{\star}$ *restricts the summation to the primitive residue classes and* $\bar{d}$ *denotes the multiplicative inverse of d modulo c.*

By virtue of the following estimate (which is essentially due to A. Weil)

$$(2.10) \qquad |S(m, n; c)| \leqslant (m, n, c)^{\frac{1}{2}} c^{\frac{1}{2}} \tau(c) ,$$

and by the crude bound for the Bessel function (recall that $k \geqslant 2$)

$$(2.11) \qquad J_{k-1}(x) \ll \min\left(1, \frac{x}{k}\right) k^{-\frac{1}{3}} ,$$

it is clear that the series on the right side of (2.8) converges absolutely. For m, n relatively small $\Delta_{k,N}(m, n)$ approximates to $\delta(m, n)$. Precisely we derive from (2.8).

Corollary **2.2.** — *For any* $m, n \geqslant 1$ *we have*

$$(2.12) \qquad \Delta_{k,N}(m, n) = \delta(m, n) + O\left(\frac{\tau(N)\,(m, n, N)\tau_3((m, n))}{Nk^{\frac{5}{6}}\,((m, N) + (n, N))^{\frac{1}{2}}} \left(\frac{mn}{\sqrt{mn} + kN}\right)^{\frac{1}{2}} \log 2mn\right)$$

where the implied constant is absolute ($\tau_3(\ell)$ *denotes the corresponding divisor function).*

Proof. — First we prove a general bound for Kloosterman sums

$$(2.13) \qquad |S(m, n; c)| \leqslant (m, n, c) \min\left(\frac{c}{(m, c)}, \frac{c}{(n, c)}\right)^{\frac{1}{2}} \tau(c) .$$

This bound is somewhat stronger than (2.10), nevertheless it can be derived from (2.10) as follows. By multiplicative properties of Kloosterman sums we can assume that $c = p^{\alpha}$ and $n = p^{\beta}$. If $\alpha \leqslant \beta$ then $S(m, p^{\beta}; p^{\alpha}) = S(m, 0; p^{\alpha})$ is the Ramanujan sum which is bounded by $(m, p^{\alpha})(m, n, c)(c/(n, c))^{\frac{1}{2}}$. Suppose $\alpha > \beta$. Then $S(m, p^{\beta}; p^{\alpha}) = p^{\beta} S(mp^{-\beta}, 1; p^{\alpha-\beta})$ if $p^{\beta} | m$, and the sum vanishes if $p^{\beta} \nmid m$. In the first case we get $|S(m, p^{\beta}; p^{\alpha})| \leqslant (m, n, c)(c/n, c)^{\frac{1}{2}} \tau(c)$ by (2.10). Therefore we have that $|S(m, n; c)| \leqslant (m, n, c)(c/(n, c))^{\frac{1}{2}} \tau(c)$ for all m, n. Since we can interchange m with n this completes the proof of (2.13).

The crude bound (2.11) follows from the more precise estimates

$$(2.11') \qquad J_\nu(x) \ll x^{-\frac{1}{4}}(|x-\nu| + \nu^{\frac{1}{3}})^{-\frac{1}{4}} ,$$

$$(2.11'') \qquad J_\nu(x) \ll \frac{1}{\nu!}\left(\frac{x}{2}\right)^\nu \ll \frac{x}{\sqrt{\nu+1}}\left(\frac{ex}{2\nu+1}\right)^\nu ,$$

if $\nu \geqslant 0$ and $x > 0$ (see [GR] and [Wat]). The latter implies

$$(2.11''') \qquad J_{k-1}(x) \ll 2^{-k}x ,$$

if $k \geqslant 2$ and $0 < x \leqslant \frac{k}{3}$.

Now we are ready to prove (2.12). The sum of Kloosterman sums in (2.8) is bounded by

$$\begin{aligned} &\sum_{c\equiv 0 \,(\mathrm{mod}\, N)} (m, n, c)((m, c) + (n, c))^{-\frac{1}{2}} c^{-\frac{1}{2}} \tau(c) \min(1, \sqrt{mn}/ck)k^{-\frac{1}{3}} \\ &\leqslant (m, n, N)((m, N) + (n, N))^{-\frac{1}{2}} N^{-\frac{1}{2}} \tau(N) k^{-\frac{1}{3}} S , \end{aligned}$$

where

$$\begin{aligned} S &= \sum_{b=1}^{\infty} (m, n, b)^{\frac{1}{2}} b^{-\frac{1}{2}} \tau(b) \min(1, \sqrt{mn}/bkN) \\ &\leqslant \sum_{d|(m,n)} \tau(d) \sum_{b=1}^{\infty} \frac{\tau(b)}{\sqrt{b}} \min\left(1, \frac{\sqrt{mn}}{bkN}\right) . \end{aligned}$$

Here the sum over divisors of (m, n) is equal to $\tau_3((m, n))$, and the last sum over b is bounded by

$$\min\left\{\frac{\sqrt{mn}}{kN}, \left(\frac{\sqrt{mn}}{kN}\right)^{\frac{1}{2}} \log 2mn\right\} \ll \left(\frac{mn}{kN}\right)^{\frac{1}{2}} \left(\sqrt{mn} + kN\right)^{-\frac{1}{2}} \log 2mn .$$

Hence (2.12) follows.

One can get slightly better results if $mn \ll k^2N^2$.

Corollary **2.3.** — *For any* $m, n \geqslant 1$ *with* $12\pi\sqrt{mn} \leqslant kN$ *we have*

$$(2.12') \qquad \Delta_{k,N}(m, n) = \delta(m, n) + O\left(\frac{\tau(N)}{2^k N^{\frac{3}{2}}} \frac{(m, n, N)(mn)^{\frac{1}{2}}}{((m, N) + (n, N))^{\frac{1}{2}}} \tau((m, n))\right) .$$

Proof. — This follows along the above lines where we apply (2.11″) in place of (2.11).

Next we express $\Delta_{k,N}(m,n)$ in terms of Hecke eigenvalues of cusp forms which are newforms in the sense of Atkin-Lehner theory [AL].

Letting $H_k^\star(M)$ be the set of newforms of weight k and level M we have the orthogonal decomposition

$$S_k(N) = \bigoplus_{LM=N} \bigoplus_{f\in H_k^\star(M)} S_k(L;f) \tag{2.14}$$

where $S_k(L;f)$ denotes the linear space spanned by the forms

$$f_{|\ell}(z) = \ell^{\frac{k}{2}} f(\ell z) \quad \text{with } \ell | L .$$

Note that the forms $f_{|\ell}$ need not be orthogonal (see Lemma 2.4), nevertheless they are linearly independent. Therefore $\dim S_k(L;f) = \tau(L)$ and

$$\dim S_k(N) = \sum_{LM=N} \tau(L)|H_k^\star(M)| .$$

A newform f of level M possesses a handful of properties. First of all f is an eigenfunction of all the Hecke operators $T_M(n)$ defined by

$$(T_M(n)f)(z) = \frac{1}{\sqrt{n}} \sum_{\substack{ad=n\\(a,M)=1}} \left(\frac{a}{d}\right)^{k/2} \sum_{b \,(\mathrm{mod}\, d)} f\left(\frac{az+b}{d}\right) . \tag{2.15}$$

For $f\in H_k^\star(M)$ we define $\lambda_f(n)$ to be the eigenvalues of $T_M(n)$;

$$T_M(n)f = \lambda_f(n) f \tag{2.16}$$

for all $n \geqslant 1$. We have $a_f(n) = a_f(1)\lambda_f(n)n^{(k-1)/2}$ for all $n \geqslant 1$. Hence $a_f(1) \neq 0$, and we can normalize the newforms by setting

$$a_f(1) = 1 . \tag{2.17}$$

Therefore for all $n \geqslant 1$

$$a_f(n) = \lambda_f(n) n^{(k-1)/2} . \tag{2.18}$$

The Hecke eigenvalues are multiplicative; precisely for any $m, n \geqslant 1$

$$\lambda_f(m)\lambda_f(n) = \sum_{\substack{d|(m,n)\\(d,M)=1}} \lambda_f(mn/d^2) . \tag{2.19}$$

By this multiplicativity the bound (2.4) improves itself to

$$|\lambda_f(n)| \leqslant \tau(n) . \tag{2.20}$$

A newform f of level M is also an eigenfunction of the involution W_M which is defined by

$$(2.21) \qquad (W_M f)(z) = (z\sqrt{M})^{-k} f(-1/zM) .$$

For $f \in H_k^\star(M)$ we put

$$(2.22) \qquad W_M f = \eta_f f$$

with $\eta_f = \pm 1$. If M is squarefree then η_f can be expressed in terms of the Hecke eigenvalue, precisely

$$(2.23) \qquad \eta_f = \mu(M)\lambda_f(M)M^{1/2} .$$

Hence it follows that $\lambda_f^2(M) = M^{-1}$. Actually it is also known that

$$(2.24) \qquad \lambda_f^2(p) = p^{-1} \quad \text{if } p|M .$$

We return to the decomposition (2.14). To complete the goal we need to select an orthonormal basis in the space $S_k(L; f)$ for each newform $f \in H_k^\star(M)$. Such a basis is determined by f up to a unitary transformation of the system $\{f_{|\ell}; \ell|L\}$. First of all we need to compute the inner products $\langle f_{|\ell_1}, f_{|\ell_2}\rangle$. From now on we assume that N is squarefree.

Lemma **2.4.** — *Let* $N = LM$ *be squarefree,* $\ell_1|L$, $\ell_2|L$ *and* $f \in H_k^\star(M)$. *Then*

$$(2.25) \qquad \langle f_{|\ell_1}, f_{|\ell_2}\rangle = \lambda_f(\ell)\nu(\ell)^{-1}\ell^{1/2}\langle f, f\rangle$$

where $\ell = \ell_1\ell_2(\ell_1, \ell_2)^{-2}$ *(recall that* $\nu(\ell)$ *is the multiplicative function given by (2.65) and* $\langle f, g\rangle$ *is the inner product given by (2.1)).*

Proof. — For cusp forms of weight $k = 2$ the formula (2.25) was established by A. Abbes and E. Ullmo (see Lemma 3.2 of [AU]). We follow closely their arguments. We begin by the inner product

$$(2.26) \qquad F(s) = \langle E(z, s) f(\ell_1 z), f(\ell_2 z)\rangle$$

where $E(z, s)$ is the Eisenstein series

$$(2.27) \qquad E(z, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} (\mathrm{Im}\gamma z)^s .$$

By the unfolding method

$$F(s) = \int_0^\infty y^{s+k-2} \left(\int_0^1 f(\ell_1 z) \bar{f}(\ell_2 z) dx \right) dy .$$

Inserting the Fourier expansion (2.2) we get

$$F(s) = (4\pi)^{1-k-s}\Gamma(s+k-1) \sum\sum_{\ell_1 n_1 = \ell_2 n_2} a_f(n_1)\bar{a}_f(n_2)(\ell_1 n_1)^{1-k-s} .$$

Introducing (2.18) and writing $n_1 = \ell'' n$, $n_2 = \ell' n$ with $\ell' = \ell_1/(\ell_1, \ell_2)$, $\ell'' = \ell_2/(\ell_1, \ell_2)$ we get

$$F(s) = (4\pi)^{1-k-s}\Gamma(s+k-1)(\ell_1\ell_2)^{(1-k)/2}[\ell_1, \ell_2]^{-s} R_f(\ell'\ell''; s)$$

where

$$R_f(\ell'\ell''; s) = \sum_n \lambda_f(\ell' n)\lambda_f(\ell'' n) n^{-s} .$$

Note that this depends only on $\ell = \ell'\ell''$ (justifying our notation) and it factors into

$$\Big(\sum_{(n,\ell)=1} \lambda_f^2(n) n^{-s}\Big) \prod_{p|\ell} \Big(\sum_{\alpha=0}^{\infty} \lambda_f(p^{\alpha+1})\lambda_f(p^{\alpha}) p^{-\alpha s}\Big) .$$

For $p|\ell$ (so $p \nmid M$) and $\alpha \geqslant 1$ we have $\lambda_f(p^{\alpha+1}) = \lambda_f(p)\lambda_f(p^{\alpha}) - \lambda_f(p^{\alpha-1})$. Hence

$$\sum_{\alpha=0}^{\infty} \lambda_f(p^{\alpha+1})\lambda_f(p^{\alpha}) p^{-\alpha s} = \lambda_f(p)(1+p^{-s})^{-1} \sum_{\alpha=0}^{\infty} \lambda_f^2(p^{\alpha}) p^{-\alpha s} .$$

Multiplying these series we obtain

$$R_f(\ell; s) = \lambda_f(\ell) \prod_{p|\ell} (1+p^{-s})^{-1} L(s, f\otimes f)$$

where $L(s, f\otimes f)$ is the Rankin-Selberg L-function

$$L(s, f\otimes f) = \sum_1^{\infty} \lambda_f^2(n) n^{-s} . \tag{2.28}$$

Finally, combining the above formulas we arrive at

$$F(s) = (4\pi)^{1-k-s}\Gamma(s+k-1)(\ell_1\ell_2)^{(1-k)/2}[\ell_1, \ell_2]^{-s} \lambda_f(\ell) L(s, f\otimes f) \prod_{p|\ell} (1+p^{-s})^{-1} . \tag{2.29}$$

Taking the residue at $s = 1$ we obtain (2.25).

The formula (2.25) shows that all the forms $f_{|\ell}$ with $\ell|L$ have the same norm

$$\langle f_{|\ell}, f_{|\ell} \rangle = \langle f, f \rangle . \tag{2.30}$$

Moreover, since the residue of $E(z, s)$ at $s = 1$ is equal to $1/\mathrm{vol}(\Gamma_0(N)\backslash\mathbf{H})$ (cf. [Sa]), it follows from (2.29) for $\ell_1 = \ell_2 = 1$ that

$$\langle f, f\rangle = (4\pi)^{-k}\Gamma(k)\frac{\pi}{3}\nu(N)\operatorname*{Res}_{s=1} L(s, f\otimes f)\ . \tag{2.31}$$

Next one can easily check by (2.19) that

$$L(s, f\otimes f) = Z(s, f)\zeta(s)/\zeta_M(s) \tag{2.32}$$

where $\zeta_M(s)$ is the local Riemann zeta function

$$\zeta_M(s) = \sum_{m|M^\infty} m^{-s} = \prod_{p|M}(1 - p^{-s})^{-1} \tag{2.33}$$

and $Z(s, f)$ is defined by

$$Z(s, f) = \sum_1^\infty \lambda_f(n^2)n^{-s}\ . \tag{2.34}$$

Therefore we have

$$\operatorname*{Res}_{s=1} L(s, f\otimes f) = Z(1, f)M/\varphi(M)\ . \tag{2.35}$$

Inserting (2.35) into (2.31) we conclude

Lemma **2.5.** — *If f is a newform of weight k and level* $M|N$, *then*

$$\langle f, f\rangle = (4\pi)^{1-k}\Gamma(k)\frac{\nu(N)\varphi(M)}{12M}Z(1, f)\ . \tag{2.36}$$

Now we are ready to select an orthonormal basis of $S_k(L; f)$, say

$$H_k^\star(L; f) = \{f_d;\ d|L\}\ , \tag{2.37}$$

where f_d are suitable linear combinations of $f_{|\ell}$, say

$$f_d = \sum_{\ell|L} x_d(\ell) f_{|\ell}\ . \tag{2.38}$$

Denote $\delta_f(d_1, d_2) = \langle f_{d_1}, f_{d_2}\rangle/\langle f, f\rangle$. By (2.25) we get

$$\delta_f(d_1, d_2) = \sum_{\ell_1}\sum_{\ell_2} x_{d_1}(\ell_1)\bar{x}_{d_2}(\ell_2)\lambda_f(\ell)\sqrt{\ell}/\nu(\ell)\ .$$

Writing $\ell_1 = a\ell'$, $\ell_2 = a\ell''$ with $(\ell', \ell'') = 1$ we get

$$\delta_f(d_1, d_2) = \sum_a \sum_{(\ell', \ell'')=1}\sum x_{d_1}(a\ell')\bar{x}_{d_2}(a\ell'')\lambda_f(\ell')\lambda_f(\ell'')(\ell'\ell'')^{1/2}/\nu(\ell')\nu(\ell'')\ .$$

Next we relax the condition $(\ell', \ell'') = 1$ by Möbius inversion getting

$$\delta_f(d_1, d_2) = \sum_a \sum_b \mu(b) b \left(\frac{\lambda_f(b)}{\nu(b)}\right)^2 \sum_{\ell'} x_{d_1}(ab\ell')\lambda_f(\ell')\sqrt{\ell'}/\nu(\ell') \sum_{\ell''} \bar{x}_{d_2}(ab\ell'')\lambda_f(\ell'')\sqrt{\ell''}/\nu(\ell'') .$$

Collecting terms with $ab = c$ we get

$$\delta_f(d_1, d_2) = \sum_{c|L} \rho_f(c)\, y_{d_1}(c)\, \bar{y}_{d_2}(c) , \tag{2.39}$$

where $\rho_f(c)$ is the multiplicative function given by

$$\rho_f(c) = \sum_{b|c} \mu(b) b \left(\frac{\lambda_f(b)}{\nu(b)}\right)^2 = \prod_{p|c} \left(1 - p\left(\frac{\lambda_f(p)}{p+1}\right)^2\right) , \tag{2.40}$$

and $y_d(c)$ is the corresponding linear combination of the $x_d(c\ell)$'s,

$$y_d(c) = \sum_{\ell|L} x_d(c\ell)\lambda_f(\ell)\sqrt{\ell}/\nu(\ell) . \tag{2.41}$$

Using the Möbius inversion we transform (2.41) into

$$x_d(\ell) = \sum_{c|L} y_d(c\ell)\mu(c)\lambda_f(c)\sqrt{c}/\nu(c) . \tag{2.42}$$

We require $\delta_f(d_1, d_2)$ to be the diagonal symbol, i.e. the matrix

$$Y = \left(y_d(c)\sqrt{\rho_f(c)}\right) , \quad c|L,\ d|L, \tag{2.43}$$

to be unitary. There are many interesting choices. We take for Y the identity matrix getting

$$x_d(\ell) = \mu(c)\sqrt{c}\lambda_f(c)/\nu(c)\sqrt{\rho_f(d)} \tag{2.44}$$

if $d = c\ell$ and $x_d(\ell) = 0$ otherwise. For this choice we get

$$f_d(z) = \left(\frac{d}{\rho_f(d)}\right)^{1/2} \sum_{c\ell = d} \mu(c)\nu(c)^{-1}\lambda_f(c)\ell^{(k-1)/2} f(\ell z) . \tag{2.45}$$

We have proved the following

Proposition **2.6.** — *Let* $N = LM$ *be squarefree and* $f \in H_k^\star(M)$. *Then the set* $\{f_d; d|L\}$ *with* $f_d(z)$ *given by (2.45) is an orthogonal basis of* $S_k(L; f)$. *Moreover, every* f_d *has the same norm (with respect to the inner product (2.1)) given by (2.36).*

Now we are ready to express $\Delta_{k,\mathrm{N}}(m,n)$ in terms of Hecke eigenvalues for newforms of level M|N. First observe that if $f(z)$ has the Fourier expansion (2.2) with coefficient $a_f(n)$, then so does $f_d(z)$ with coefficients

$$a_{f_d}(n) = \left(\frac{d}{\rho_f(d)}\right)^{1/2} \sum_{\substack{c\ell = d \\ \ell \mid n}} \frac{\mu(c)}{\nu(c)} \lambda_f(c) \ell^{(k-1)/2} a_f\left(\frac{n}{\ell}\right). \tag{2.46}$$

Hence, using our particular basis (2.45) we arrive at

$$\Delta_{k,\mathrm{N}}(m,n) = (4\pi)^{1-k}\Gamma(k-1) \sum_{\mathrm{LM}=\mathrm{N}} \sum_{f\in \mathrm{H}_k^\star(\mathrm{M})} \|f\|^{-2} \sum_{d\mid \mathrm{L}} d\rho_f(d)^{-1}$$

$$\Bigg(\sum_{\substack{c_1\ell_1 = d \\ \ell_1 \mid m}} \frac{\mu(c_1)}{\nu(c_1)} \lambda_f(c_1) \lambda_f\Big(\frac{m}{\ell_1}\Big)\Bigg)\Bigg(\sum_{\substack{c_2\ell_2 = d \\ \ell_2 \mid n}} \frac{\mu(c_2)}{\nu(c_2)} \lambda_f(c_2) \lambda_f\Big(\frac{n}{\ell_2}\Big)\Bigg).$$

To simplify this expression we assume that $(m,n,\mathrm{N})=1$. Then $(\ell_1,\ell_2)=1$ so $c_1 = b\ell_2$ and $c_2 = b\ell_1$ giving

$$\Big(\sum_{\ell_1}\Big)\Big(\sum_{\ell_2}\Big) = \sum_{\substack{b\ell_1\ell_2 = d \\ \ell_1\mid m,\, \ell_2 \mid n}} \left(\frac{\lambda_f(b)}{\nu(b)}\right)^2 \frac{\mu(\ell_1\ell_2)}{\nu(\ell_1\ell_2)} \lambda_f(\ell_1\ell_2) \lambda_f\Big(\frac{m}{\ell_1}\Big)\lambda_f\Big(\frac{n}{\ell_2}\Big).$$

Hence, using the formula (see the definition (2.40))

$$\sum_{b\mid \mathrm{B}} \frac{b}{\rho_f(b)} \left(\frac{\lambda_f(b)}{\nu(b)}\right)^2 = \frac{1}{\rho_f(\mathrm{B})}$$

we find that

$$\sum_{d\mid \mathrm{L}} \frac{d}{\rho_f(d)} \Big(\sum_{\ell_1}\Big)\Big(\sum_{\ell_2}\Big) = \frac{1}{\rho_f(\mathrm{L})} \mathrm{A}_f(m,\mathrm{L}) \mathrm{A}_f(n,\mathrm{L})$$

where

$$\mathrm{A}_f(m,\mathrm{L}) = \sum_{\ell \mid (m,\mathrm{L})} \ell \frac{\mu(\ell)}{\nu(\ell)} \lambda_f(\ell) \lambda_f\left(\frac{m}{\ell}\right).$$

Next we apply (2.19) showing that

$$\mathrm{A}_f(m,\mathrm{L}) = \frac{1}{\nu((m,\mathrm{L}))} \sum_{\delta^2 \mid (m,\delta\mathrm{L})} \mu(\delta)\delta\lambda_f\left(\frac{m}{\delta^2}\right). \tag{2.47}$$

Collecting the above evaluations together with (2.36) we obtain

$$\Delta_{k,\mathrm{N}}(m,n)=\frac{12}{(k-1)\nu(\mathrm{N})}\sum_{\mathrm{LM}=\mathrm{N}}\frac{\mathrm{M}}{\varphi(\mathrm{M})}\sum_{f\in\mathrm{H}_k^\star(\mathrm{M})}\frac{\mathrm{A}_f(m,\mathrm{L})\mathrm{A}_f(n,\mathrm{L})}{\rho_f(\mathrm{L})\mathrm{Z}(1,f)}\,. \tag{2.48}$$

However, this formula can be expressed more naturally in terms of the local zeta function

$$\mathrm{Z}_\mathrm{N}(s,f)=\sum_{\ell\mid\mathrm{N}^\infty}\lambda_f(\ell^2)\ell^{-s}\,. \tag{2.49}$$

We compute by (3.14), (3.15) and (3.16) that

$$\mathrm{Z}_p(1,f)=\begin{cases}\left(1+\frac{1}{p}\right)^{-1}\rho_f(p)^{-1}\,, & \text{if } p\nmid \mathrm{M}\\ \left(1+\frac{1}{p}\right)^{-1}\left(1-\frac{1}{p}\right)^{-1}, & \text{if } p\mid\mathrm{M}\end{cases}.$$

Hence we get

$$\mathrm{Z}_\mathrm{N}(1,f)=\mathrm{MN}/\varphi(\mathrm{M})\nu(\mathrm{N})\rho_f(\mathrm{L})\,. \tag{2.50}$$

Introducing (2.50) into (2.48) we conclude that for N squarefree and $(m,n,\mathrm{N})=1$

$$\Delta_{k,\mathrm{N}}(m,n)=\frac{12}{(k-1)\mathrm{N}}\sum_{\mathrm{LM}=\mathrm{N}}\sum_{f\in\mathrm{H}_k^\star(\mathrm{M})}\mathrm{A}_f(m,\mathrm{L})\mathrm{A}_f(n,\mathrm{L})\frac{\mathrm{Z}_\mathrm{N}(1,f)}{\mathrm{Z}(1,f)}\,. \tag{2.51}$$

To achieve further simplifications we are going to assume that $(mn,\mathrm{N}^2)\mid\mathrm{N}$ in which case (2.51) becomes

Lemma **2.7.** — *Let* N *be squarefree,* $(m,n,\mathrm{N})=1$ *and* $(mn,\mathrm{N}^2)\mid\mathrm{N}$. *Then*

$$\Delta_{k,\mathrm{N}}(m,n)=\frac{12}{(k-1)\mathrm{N}}\sum_{\mathrm{LM}=\mathrm{N}}\sum_{f\in\mathrm{H}_k^\star(\mathrm{M})}\frac{\lambda_f(m)\lambda_f(n)}{\nu((mn,\mathrm{L}))}\frac{\mathrm{Z}_\mathrm{N}(1,f)}{\mathrm{Z}(1,f)}\,. \tag{2.52}$$

Now we proceed to convert (2.52) into formulas for sums over newforms. We begin by considering the arithmetically weighted sums

$$\Delta^\sigma_{k,\mathrm{N}}(m,n)=\sum_{f\in\mathrm{H}_k^\sigma(\mathrm{N})}\lambda_f(m)\lambda_f(n)\mathrm{Z}_\mathrm{N}(1,f)/\mathrm{Z}(1,f) \tag{2.53}$$

where $\sigma=\star,+,-$. Note that $\mathrm{Z}_p(1,f)=(1-p^{-2})^{-1}$ if $p\mid\mathrm{N}$ by (3.14), (3.16), hence

$$\Delta^\sigma_{k,\mathrm{N}}(m,n)=\zeta_\mathrm{N}(2)\sum_{f\in\mathrm{H}_k^\sigma(\mathrm{N})}\lambda_f(m)\lambda_f(n)/\mathrm{Z}(1,f)\,. \tag{2.54}$$

First, by (2.52) one can check directly (using Möbius inversion) the following

Proposition **2.8.** — *Let* N *be squarefree,* $(m, \mathrm{N}) = 1$ *and* $(n, \mathrm{N}^2)|\mathrm{N}$. *Then*

$$\Delta^{\star}_{k,\mathrm{N}}(m, n) = \frac{k-1}{12} \sum_{\mathrm{LM}=\mathrm{N}} \frac{\mu(\mathrm{L})\mathrm{M}}{\nu((n, \mathrm{L}))} \sum_{\ell|\mathrm{L}^\infty} \ell^{-1} \Delta_{k,\mathrm{M}}(m\ell^2, n) . \tag{2.55}$$

Next, if $(n, \mathrm{N}) = 1$ then by (3.5) we can write

$$2\Delta^{\pm}_{k,\mathrm{N}}(m, n) = \Delta^{\star}_{k,\mathrm{N}}(m, n) \pm i^k \mu(\mathrm{N})\sqrt{\mathrm{N}}\Delta^{\star}_{k,\mathrm{N}}(m, n\mathrm{N}) . \tag{2.56}$$

Applying (2.55) we get

Proposition **2.9.** — *Let* N *be squarefree and* $(mn, \mathrm{N}) = 1$. *Then*

$$\begin{aligned} \Delta^{\pm}_{k,\mathrm{N}}(m, n) = {} & \frac{k-1}{24}\mathrm{N} \sum_{\mathrm{LM}=\mathrm{N}} \sum_{\ell|\mathrm{L}^\infty} \frac{\mu(\mathrm{L})}{\ell \mathrm{L}} \Delta_{k,\mathrm{M}}(m\ell^2, n) \\ & \pm i^k \frac{k-1}{24}\sqrt{\mathrm{N}} \sum_{\mathrm{LM}=\mathrm{N}} \sum_{\ell|\mathrm{L}^\infty} \frac{\mu(\mathrm{M})\mathrm{M}}{\ell \nu(\mathrm{L})} \Delta_{k,\mathrm{M}}(m\ell^2, n) . \end{aligned} \tag{2.57}$$

Inserting (2.12) into (2.55) we deduce

Corollary **2.10.** — *Let* N *be squarefree,* $(m, \mathrm{N}) = 1$ *and* $(n, \mathrm{N}^2)|\mathrm{N}$. *Then*

$$\begin{aligned} \Delta^{\star}_{k,\mathrm{N}}(m, n) = {} & \frac{k-1}{12}\varphi(\mathrm{N})\delta(m, n) \\ & + \mathrm{O}(k^{1/6}(mn)^{1/4}(n, \mathrm{N})^{-1/2}\tau^2(\mathrm{N})\tau_3((m, n)) \log 2mn\mathrm{N}) \end{aligned} \tag{2.58}$$

where the implied constant is absolute.

Finally we consider the pure sums

$$\Delta^{\sigma}_{k,\mathrm{N}}(n) = \sum_{f \in \mathrm{H}^{\sigma}_k(\mathrm{N})} \lambda_f(n) \tag{2.59}$$

where $\sigma = \star, +, -$. If $(n, \mathrm{N}) = 1$, we get by (3.5)

$$2\Delta^{\pm}_{k,\mathrm{N}}(n) = \Delta^{\star}_{k,\mathrm{N}}(n) \pm i^k \mu(\mathrm{N})\sqrt{\mathrm{N}}\Delta^{\star}_{k,\mathrm{N}}(n\mathrm{N}) . \tag{2.60}$$

Summing $m^{-1}\Delta^{\star}_{k,\mathrm{N}}(m^2, n)$ over all $(m, \mathrm{N}) = 1$ we remove the arithmetical weights $\mathrm{Z}_{\mathrm{N}}(1, f)/\mathrm{Z}(1, f)$ completely getting by (2.55).

Proposition **2.11.** — *Let* N *be squarefree and* $(n, \mathrm{N}^2)|\mathrm{N}$. *Then*

$$\Delta^{\star}_{k,\mathrm{N}}(n) = \frac{k-1}{12} \sum_{\mathrm{LM}=\mathrm{N}} \frac{\mu(\mathrm{L})\mathrm{M}}{\nu((n, \mathrm{L}))} \sum_{(m,\mathrm{M})=1} m^{-1} \Delta_{k,\mathrm{M}}(m^2, n) . \tag{2.61}$$

Here the innermost series over m converges by virtue of the holomorphy of the symmetric square L-function (see the next section). The convergence, however, is not absolute and for this reason the formula (2.61) is not quite practical, especially if one expands each $\Delta_{k,\mathrm{M}}(m^2, n)$ into sums of Kloosterman sums by Petersson's formula (2.8). The problem looks like that with character sums of large conductor in which case the application of Poisson's summation would transform the sum into a worse position from the point of view of estimation. Here the large m's reduce considerably the efficiency of the Kloosterman sums expansion in the variable n. Moreover the terms with large L can cause some loss of power. Therefore, to balance these losses, we split

$$\Delta^{\star}_{k,\mathrm{N}}(n) = \Delta'_{k,\mathrm{N}}(n) + \Delta^{\infty}_{k,\mathrm{N}}(n) \tag{2.62}$$

where

$$\Delta'_{k,\mathrm{N}}(n) = \frac{k-1}{12} \sum_{\substack{\mathrm{LM}=\mathrm{N}\\ \mathrm{L}\leqslant\mathrm{X}}} \frac{\mu(\mathrm{L})\mathrm{M}}{\nu((n,\mathrm{L}))} \sum_{\substack{(m,\,\mathrm{M})=1\\ m\leqslant\mathrm{Y}}} m^{-1}\Delta_{k,\mathrm{M}}(m^2, n) \tag{2.63}$$

and $\Delta^{\infty}_{k,\mathrm{N}}(n)$ is the complementary sum. Here $\mathrm{X}, \mathrm{Y} \geqslant 1$ are two parameters at our disposal. In applications we shall choose X and Y relatively small.

In the complementary sum $\Delta^{\infty}_{k,\mathrm{N}}(n)$ we express the terms $\Delta_{k,\mathrm{M}}(m^2, n)$ back in terms of Hecke eigenvalues for newforms, and then estimate $\Delta^{\infty}_{k,\mathrm{N}}(n)$ by using various bounds for the eigenvalues and relevant L-functions (rather than for Kloosterman sums). We also estimate sums of type

$$\sum_{(q,\,n\mathrm{N})=1} \Delta^{\infty}_{k,\mathrm{N}}(nq)a_q \tag{2.64}$$

for some complex coefficients a_q. We could get quite strong results (almost best possible) for any coefficients, but of restricted support which is not acceptable. To allow a larger support (practically unlimited) we are going to assume that the sequence $\mathscr{A} = (a_q)$ has the following property;

$$\sum_{(q,\,n\mathrm{N})=1} \lambda_f(q)a_q \ll (nk\mathrm{N})^{\varepsilon} \tag{2.65}$$

for every $f \in \mathrm{H}^{\star}_k(\mathrm{M})$ with $\mathrm{M}|\mathrm{N}$ the implied constant depending only on ε.

For example this property holds true for the two sequences

$$a_q = p^{-1/2}\log p, \qquad \text{if } q = p \leqslant \mathrm{Q} \tag{2.66}$$

$$a_q = p^{-1/2}\log p, \qquad \text{if } q = p^2 \leqslant \mathrm{Q} \tag{2.67}$$

with $a_q = 0$ elsewhere. This follows from the Riemann hypothesis for $L(s,f)$ and $L(s, \mathrm{sym}^2(f))$ respectively, provided $\log Q \ll \log kN$. The above two examples are all we use in this paper.

Lemma **2.12.** — *Let* N *be squarefree and* $(n, N^2)|N$. *Suppose the sequence* $\mathcal{A} = (a_q)$ *satisfies (2.65). Then we have*

$$\sum_{(q,\, nN)=1} \Delta_{k,N}^{\infty}(nq) a_q \ll (n, N)^{-1/2} kN(X^{-1} + Y^{-1/2})(nkNXY)^{\varepsilon} \tag{2.68}$$

where the implied constant depends only on ε.

Proof. — By Lemma 2.7 we write

$$\Delta_{k,N}^{\infty}(nq) = \sum_{\substack{KLM=N \\ L>X}} \frac{\mu(L)}{\nu((n, KL))} \sum_{f \in H_k^{\infty}(M)} \lambda_f(nq)$$
$$+ \sum_{\substack{KLM=N \\ L\leqslant X}} \frac{\mu(L)}{\nu((n, KL))} \sum_{f \in H_k^{\star}(M)} \lambda_f(nq) r_f(KM; Y)$$

where

$$r_f(KM; Y) = \frac{Z_{KM}(1,f)}{Z(1,f)} \sum_{\substack{(m,\, KM)=1 \\ m>Y}} m^{-1} \lambda_f(m^2) .$$

By the Riemann hypothesis for $L(s, \mathrm{sym}^2(f))$ we get

$$r_f(KM; Y) \ll Y^{-1/2}(kKMY)^{\varepsilon} .$$

Moreover we have $|\lambda_f(n)| \ll \tau(n)(n, N)^{-1/2}$ by (2.20) and (2.24). Hence Lemma 2.12 follows by the hypothesis (2.65).

Taking one term $q = 1$ from (2.68) we get

$$\Delta_{k,N}^{\infty}(n) \ll (n, N)^{-1/2} kN(X^{-1} + Y^{-1/2})(nkNXY)^{\varepsilon} . \tag{2.69}$$

On the other hand, applying (2.12) to every $\Delta_{k,M}(m^2, n)$ in (2.63) we derive

$$\Delta'_{k,N}(n) = \frac{k-1}{12} \frac{\varphi(N)}{\sqrt{n}} \left\{ 1 + O\left(\frac{\tau(N)N}{\varphi(N)X} \right) \right\}$$
$$+ O\left(k^{-1/3} \left(\frac{nXY^2}{(n, N)N} \right)^{1/2} (nkNXY)^{\varepsilon} \right) \tag{2.70}$$

where the first term exists only if $n = m^2 \leqslant Y^2$ and $(n, N) = 1$. Adding (2.69) to (2.70) and choosing $X = Y^{1/2} = n^{-1/7}k^{8/21}N^{3/7}$ we get

Proposition **2.13.** — *Let* N *be squarefree and* $(n, N^2)|N$. *Then*

$$\Delta^\star_{k,N}(n) = \frac{k-1}{12}\frac{\varphi(N)}{\sqrt{n}} + O\left((n, N)^{-1/2} n^{1/6}(kN)^{2/3}\right) \tag{2.71}$$

where the main term exists only if $n = m^2$ *and* $(n, N) = 1$ *and the implied constant is absolute.*

In particular for $n = 1$ and $n = N$ Proposition 2.13 gives us asymptotic formulas for the number of newforms.

Corollary **2.14.** — *Let* $k \geqslant 2$ *be even and* N *be squarefree. Then*

$$|H^\star_k(N)| = \frac{k-1}{12}\varphi(N) + O((kN)^{2/3}), \tag{2.72}$$

and for $N \neq 1$ *we have*

$$|H^\pm_k(N)| = \frac{k-1}{24}\varphi(N) + O((kN)^{5/6}). \tag{2.73}$$

Remarks. — Observe that $\Delta^\star_{k,N}(n)$ captures squares ($n = m^2$), while $\Delta_{k,N}(1, n)$ captures the diagonal ($n = 1$), provided n is small relative to the dimension. In view of these features one can say that the set of newforms $H^\star_k(N)$ is quite a partial selection from a complete orthogonal basis of $S_k(N)$. However this picture changes drastically for large n as the set $H^\star_k(N)$ shows stronger orthogonality than any basis of $S_k(N)$ can offer (in the asymptotic sense).

In our main applications the approximate formula (2.70) is not strong enough. Applying Proposition 2.1 to every $\Delta_{k,M}(m^2, n)$ in (2.63) we get the exact formula in terms of Kloosterman sums.

Proposition **2.15.** — *Let* N *be squarefree and* $(n, N^2)|N$. *Then*

$$\Delta'_{k,N}(n) = \frac{k-1}{12}\frac{\varphi(N)}{\sqrt{n}} + \frac{k-1}{12}\sum_{\substack{LM=N\\ L\leqslant X}}\ \sum_{\substack{(m,M)=1\\ m\leqslant Y}} \frac{\mu(L)M}{\nu((n,L))m}$$
$$2\pi i^k \sum_{c\equiv 0(M)} c^{-1} S(m^2, n; c) J_{k-1}\left(\frac{4\pi m\sqrt{n}}{c}\right) \tag{2.74}$$

where the first term is present only if $n = m^2$ *with* $m \leqslant Y$ *and* $(n, N) = 1$.

3. Automorphic L-functions

For any $f \in H_k^\star(N)$ the Hecke L-function is defined by

$$\text{(3.1)} \qquad L(s,f) = \sum_{1}^{\infty} \lambda_f(n) n^{-s} .$$

This has an Euler product $L(s,f) = \prod_p L_p(s,f)$ with the local factors

$$\text{(3.2)} \qquad L_p(s,f) = (1 - \lambda_f(p)p^{-s} + \chi_0(p)p^{-2s})^{-1}$$

where χ_0 denotes the principal character to modulus N. Define the local factor at $p = \infty$ by

$$\text{(3.3)} \qquad L_\infty(s,f) = \left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma\left(s + \frac{k-1}{2}\right)$$

Then the complete product $\Lambda(s,f) = L_\infty(s,f)L(s,f)$ is entire and it satisfies the functional equation

$$\text{(3.4)} \qquad \Lambda(s,f) = \varepsilon_f \Lambda(1-s,f)$$

with the root number $\varepsilon_f = i^k \eta_f = \pm 1$, where η_f is the eigenvalue of the involution W_N, so by (2.23)

$$\text{(3.5)} \qquad \varepsilon_f = i^k \mu(N) \lambda_f(N) N^{1/2} .$$

The local factors of $L(s,f)$ factor further as

$$\text{(3.6)} \qquad L_p(s,f) = \left(1 - \alpha_f(p)p^{-s}\right)^{-1} \left(1 - \beta_f(p)p^{-s}\right)^{-1}$$

where $\alpha_f(p)$, $\beta_f(p)$ are complex numbers with $\alpha_f(p)+\beta_f(p) = \lambda_f(p)$ and $\alpha_f(p)\beta_f(p) = \chi_0(p)$. For $p \nmid N$ we have $\alpha_f(p) = \overline{\beta}_f(p)$, whence $|\alpha_f(p)| = |\beta_f(p)| = 1$ (the Ramanujan conjecture [Del]). For all p and $m \geqslant 0$ we have

$$\text{(3.7)} \qquad \lambda_f(p^m) = \sum_{0 \leqslant \ell \leqslant m} \alpha_f(p)^\ell \beta_f(p)^{m-\ell} .$$

The local factor at infinity factors as (by the duplication formula for the gamma function)

$$\text{(3.8)} \qquad L_\infty(s,f) = \left(\frac{2^k}{8\pi}\right)^{1/2} \left(\frac{\sqrt{N}}{\pi}\right)^s \Gamma\left(\frac{s}{2} + \frac{k-1}{4}\right) \Gamma\left(\frac{s}{2} + \frac{k+1}{4}\right) .$$

For any $f \in H_k^\star(N)$ the Rankin-Selberg L-function is defined by

$$(3.9) \qquad L(s, f \otimes f) = \sum_1^\infty \lambda_f^2(n) n^{-s}.$$

This has the Euler product $L(s, f \otimes f) = \prod_p L_p(s, f \otimes f)$ with the local factors

$$(3.10) \qquad L_p(s, f \otimes f) = (1 - \alpha_f^2(p) p^{-s})^{-1} (1 - \alpha_f(p) \beta_f(p) p^{-s})^{-2} (1 - \beta_f^2(p) p^{-s})^{-1} (1 - \alpha_f(p) \beta_f(p)\, p^{-2s}).$$

Note that $L_p(s, f \otimes f) = (1 - p^{-s-1})^{-1}$ if $p|N$. The complete product is defined by

$$(3.11) \qquad \Lambda(s, f \otimes f) = \left(\frac{\sqrt{N}}{2\pi}\right)^{2s} \Gamma(s)\Gamma(s + k - 1)\zeta(2s)\zeta_N(s)\zeta_N(2s)^{-1} L(s, f \otimes f)\,.$$

This satisfies the functional equation (see [Li], Theorem 10)

$$(3.12) \qquad \Lambda(s, f \otimes f) = \Lambda(1 - s, f \otimes f)\,.$$

Closely related to the Rankin-Selberg L-function is

$$(3.13) \qquad Z(s, f) = \sum_1^\infty \lambda_f(n^2) n^{-s}\,.$$

We have already referred to these functions in Section 2.

Next we define the symmetric square L-function by

$$(3.14) \qquad L(s, \mathrm{sym}^2(f)) = \zeta(2s)\, \zeta_N(2s)^{-1} Z(s, f)\,.$$

This has the Euler product $L(s, \mathrm{sym}^2(f)) = \prod_p L_p(s, \mathrm{sym}^2(f))$ with

$$(3.15) \qquad L_p(s, \mathrm{sym}^2(f)) = (1 - \alpha_f^2(p) p^{-s})^{-1} (1 - \alpha_f(p)\beta_f(p) p^{-s})^{-1} (1 - \beta_f^2(p) p^{-s})^{-1}$$

if $p \nmid N$ and

$$(3.16) \qquad L_p(s, \mathrm{sym}^2(f)) = (1 - p^{-s-1})^{-1}$$

if $p|N$. Shimura [S1] proved that $L(s, \mathrm{sym}^2(f))$ is entire (since f has trivial nebentypus). In fact the complete product

$$(3.17) \qquad \Lambda(s, \mathrm{sym}^2(f)) = \pi^{-\frac{3s}{2}} \Gamma\left(\frac{s+1}{2}\right) \Gamma\left(\frac{s+k-1}{2}\right) \Gamma\left(\frac{s+k}{2}\right) N^s L(s, \mathrm{sym}^2(f))$$

is entire and it satisfies the functional equation

$$(3.18) \qquad \Lambda(s, \mathrm{sym}^2(f)) = \Lambda(1 - s, \mathrm{sym}^2(f))$$

(which follows by (3.12), the functional equation for the Riemann zeta function and the duplication formula for the gamma function).

Remarks. — Notice that the functional equation (3.18) has three gamma factors, which is consistent with the fact that $\mathrm{sym}^2(f)$ is an automorphic form on GL(3), but only two of these involve k. Therefore (as far as the conductor goes) $\mathrm{sym}^2(f)$ looks rather like a form on GL(2) in the k aspect; it also looks like a form on GL(2) in the N aspect. But for various estimations $\mathrm{sym}^2(f)$ is harder than f, because of lacunarity of the involved Fourier coefficients.

For simplification (in a few minor places, see (4.23), (4.24)) we shall appeal to analytic properties of

$$Z(s,f\otimes f)=\sum_1^\infty \lambda_f^2(n^2)n^{-s}\,. \tag{3.19}$$

This is essentially equal to the Rankin-Selberg L-function associated with $\mathrm{sym}^2(f)$ on GL(3). That is

$$Z(s,f\otimes f)=L(s,\,\mathrm{sym}^2(f)\otimes\mathrm{sym}^2(f))V(s,f) \tag{3.20}$$

where $V(s,f)$ is an Euler product which converges absolutely in $\mathrm{Re}\, s>\frac{1}{2}$, while $L(s,\,\mathrm{sym}^2 f\otimes\mathrm{sym}^2 f)$ has analytic continuation to $\mathbf{C}$ save for a pole at $s=1$ [JP–SS]. Of course, the latter is expected to satisfy the Riemann hypothesis as well.

4. Explicit formulas

Let $\Lambda(s)=\prod_p L_p(s)$ be an Euler product with local factors of type

$$L_p(s)=(1-\alpha_1(p)p^{-s})^{-1}\ldots(1-\alpha_m(p)p^{-s})^{-1} \tag{4.1}$$

where $|\alpha_j(p)|\leqslant 1$ for all $p\neq\infty$, and

$$L_\infty(s)=AQ^s\Gamma\left(\frac{s}{2}+\alpha_1\right)\ldots\Gamma\left(\frac{s}{2}+\alpha_m\right) \tag{4.2}$$

with $A\neq 0$ a complex number, $Q>0$ and $\alpha_1,\ldots,\alpha_m\geqslant 0$. The condition $|\alpha_j(p)|\leqslant 1$ corresponds to assuming the Ramanujan conjectures hold for the automorphic form of which $\Lambda(s)$ is the standard L-function. For a treatment of the explicit formula in general and without this assumption see [RS]. Suppose $\Lambda(s)$ is entire of order one and it satisfies the functional equation

$$\Lambda(s)=\varepsilon\Lambda(1-s) \tag{4.3}$$

with $\varepsilon = \pm 1$. Take a holomorphic function $G(s)$ in the strip $-1 \leqslant \operatorname{Re} s \leqslant 2$ which satisfies

(4.4) $$G(s) = G(1-s)$$

(4.5) $$s^2 G(s) \ll 1.$$

Let $\rho = \frac{1}{2} + i\gamma$ run over the zeros of $\Lambda(s)$ with the corresponding multiplicity. All of them are in the strip $0 \leqslant \operatorname{Re} s \leqslant 1$. By Cauchy's theorem for contour integrals and by the functional equations (4.3), (4.4) we get

$$\sum_\rho G(\rho) = \frac{1}{2\pi i}\int_{(2)} 2G(s)\frac{\Lambda'}{\Lambda}(s)ds.$$

By $\Lambda(s) = \prod_p L_p(s)$ this splits into (the so called explicit formula)

(4.6) $$\sum_\rho G(\rho) = \sum_p H(p),$$

where $H(p)$ are the corresponding local integrals. For $p \neq \infty$ we get by (4.1)

(4.7) $$H(p) = -2\sum_{\nu=1}^{\infty}\Big(\sum_j \alpha_j^\nu(p)\Big)F(p^\nu)\log p$$

where $F(y)$ is the inverse Mellin transform of $G(s)$

(4.8) $$F(y) = \frac{1}{2\pi i}\int_{(\frac{1}{2})} G(s)y^{-s}ds.$$

For $p = \infty$ we get by (4.2)

(4.9) $$H(\infty) = 2F(1)\log Q + \sum_j F_j$$

where

(4.10) $$F_j = \frac{1}{2\pi i}\int_{(\frac{1}{2})} \psi\left(\frac{s}{2} + \alpha_j\right)G(s)ds$$

and $\psi(z) = \Gamma'(z)/\Gamma(z)$.

We apply the above formulas for

$$G(s) = \phi\left(\left(s - \frac{1}{2}\right)\frac{\log R}{2\pi i}\right)$$

where $R > 1$ and $\phi(x)$ is an even function of Schwartz class whose Fourier transform $\widehat{\phi}(y)$ has compact support. Then

$$F(y) = \widehat{\phi}\left(\frac{\log y}{\log R}\right) / \sqrt{y} \log R .$$

For this test function the explicit formula (4.6) becomes

$$\sum_{\rho} \phi\left(\frac{\gamma}{2\pi} \log R\right) = \frac{A}{\log R} - 2 \sum_{p} \sum_{\nu=1}^{\infty} \left(\sum_{j} \alpha_j^{\nu}(p)\right) \widehat{\phi}\left(\frac{\nu \log p}{\log R}\right) p^{-\nu/2} \frac{\log p}{\log R} \tag{4.11}$$

where

$$A = 2\widehat{\phi}(0) \log Q + \sum_{j} A_j \tag{4.12}$$

with

$$A_j = \int_{-\infty}^{\infty} \psi\left(\alpha_j + \frac{1}{4} + \frac{2\pi i x}{\log R}\right) \phi(x) dx . \tag{4.13}$$

By the approximate formula $\psi(a + bi) + \psi(a - bi) = 2\psi(a) + O(a^{-2}b^2)$ (which holds for a, b real, $a > 0$, see (8.363.3) of [GR]) we derive

$$A_j = \widehat{\phi}(0) \psi\left(\alpha_j + \frac{1}{4}\right) + O(((\alpha_j + 1) \log R)^{-2}) . \tag{4.14}$$

Note that $\psi\left(\alpha + \frac{1}{4}\right) = \log \alpha + O(1)$ if $\alpha \geqslant \frac{1}{4}$, so

$$A = \widehat{\phi}(0) \log(\alpha_1 \ldots \alpha_m Q^2) + O(1) . \tag{4.15}$$

In particular for $\Lambda(s) = \Lambda(s, f)$ with $f \in H_k^{\star}(N)$ we get from (3.8) that the contribution to the explicit formula (4.11) of the local factor at the infinite place is $A/\log R$ with

$$A = \widehat{\phi}(0) \log k^2 N + O(1) . \tag{4.16}$$

For $\Lambda(s) = \Lambda(s, \mathrm{sym}^2(f))$ with $f \in H_k^{\star}(N)$ we get from (3.17) that

$$A = \widehat{\phi}(0) \log k^2 N^2 + O(1) . \tag{4.17}$$

Lemma **4.1.** — *Let* ϕ *be an even function of Schwartz class on* $\mathbf{R}$ *whose Fourier transform* $\widehat{\phi}$ *has compact support. Then for* $f \in H_k^{\star}(N)$ *we have*

$$D(f; \phi) = \widehat{\phi}(0) \frac{\log k^2 N}{\log R} + \frac{1}{2} \phi(0) + O\left(\frac{\log \log 3N}{\log R}\right) \tag{4.18}$$

$$-\sum_{p|N}\lambda_f(p)\widehat{\phi}\left(\frac{\log p}{\log R}\right)\frac{2\log p}{\sqrt{p}\log R}$$
$$-\sum_{p|N}\lambda_f(p^2)\widehat{\phi}\left(\frac{2\log p}{\log R}\right)\frac{2\log p}{p\log R},$$

$$\text{(4.19)}\qquad D(\mathrm{sym}^2(f);\phi)=\widehat{\phi}(0)\frac{\log k^2N^2}{\log R}-\frac{1}{2}\phi(0)+O\left(\frac{\log\log 3N}{\log R}\right)$$
$$-\sum_{p|N}\lambda_f(p^2)\widehat{\phi}\left(\frac{\log p}{\log R}\right)\frac{2\log p}{\sqrt{p}\log R}$$
$$-\sum_{p|N}(\lambda_f(p^4)-\lambda_f(p^2))\widehat{\phi}\left(\frac{2\log p}{\log R}\right)\frac{2\log p}{p\log R},$$

where the implied constant depends only on the test function ϕ.

Proof. — First we consider $D(f;\phi)$. For all $\nu\geqslant 0$ we have $|\sum\alpha_j^\nu(p)|\leqslant 2$. Using (3.7) we compute

$$\sum\alpha_j(p)=\alpha_f(p)+\beta_f(p)=\lambda_f(p)$$

$$\sum\alpha_j^2(p)=\alpha_f^2(p)+\beta_f^2(p)=\lambda_f(p^2)-\chi_0(p).$$

Moreover for $p|N$ we have $\lambda_f^2(p)=p^{-1}$. Hence estimating all the terms with $\nu\geqslant 3$ and the terms with $\nu=1,2$ for $p|N$ trivially the explicit formula (4.11) simplifies to

$$D(f;\phi)=\widehat{\phi}(0)\frac{\log k^2N}{\log R}-\sum_{p|N}\lambda_f(p)\widehat{\phi}\left(\frac{\log p}{\log R}\right)\frac{2\log p}{\sqrt{p}\log R}$$
$$-\sum_{p|N}(\lambda_f(p^2)-1)\widehat{\phi}\left(\frac{2\log p}{\log R}\right)\frac{2\log p}{p\log R}+O\left(\frac{1+\xi(N)}{\log R}\right)$$

where $\xi(N)$ is the additive function defined by

$$\text{(4.19')}\qquad \xi(N)=\sum_{p|N}p^{-1}\log p\ll\log\log 3N.$$

Next by the Prime Number Theorem

$$\text{(4.20)}\qquad \sum_p\widehat{\phi}\left(\frac{2\log p}{\log R}\right)\frac{2\log p}{p\log R}=\frac{1}{2}\phi(0)+O\left(\frac{1}{\log R}\right).$$

These estimates yield (4.18).

The arguments for $D(\mathrm{sym}^2(f);\phi)$ are similar. If $p \nmid N$ we use (3.7) getting

$$\sum \alpha_j(p) = \alpha_f^2(p) + \alpha_f(p)\beta_f(p) + \beta_f^2(p) = \lambda_f(p^2)$$

$$\sum \alpha_j^2(p) = \alpha_f^4(p) + \alpha_f^2(p)\beta_f^2(p) + \beta_f^4(p) = \lambda_f(p^4) - \lambda_f(p^2) + 1 .$$

If $p|N$ we use (3.16) getting $\sum \alpha_j(p) = p^{-1}$ and $\sum \alpha_j^2(p) = p^{-2}$. Hence (4.19) follows from (4.11) by the same trivial estimations.

We shall see that the first sums over primes in (4.18), (4.19), namely

$$P(f;\phi) = \sum_{p \nmid N} \lambda_f(p)\widehat{\phi}\left(\frac{\log p}{\log R}\right)\frac{2\log p}{\sqrt{p}\log R}, \tag{4.21}$$

$$P^{(2)}(f;\phi) = \sum_{p \nmid N} \lambda_f(p^2)\widehat{\phi}\left(\frac{\log p}{\log R}\right)\frac{2\log p}{\sqrt{p}\log R}, \tag{4.22}$$

contribute to the main terms of $D(f;\phi)$ and $D(\mathrm{sym}^2(f);\phi)$, respectively. However, the second sums over primes in (4.18), (4.19) are quite small. Indeed, by the Riemann hypothesis for $L(s,\mathrm{sym}^2(f))$ and $L(s,\mathrm{sym}^2(f)\otimes\mathrm{sym}^2(f))$ (see the last paragraph of the previous section) it follows that

$$\sum_{p \nmid N} \lambda_f(p^2)\widehat{\phi}\left(\frac{2\log p}{\log R}\right)\frac{2\log p}{p\log R} \ll \frac{\log\log kN}{\log R}, \tag{4.23}$$

$$\sum_{p \nmid N} \lambda_f(p^4)\widehat{\phi}\left(\frac{2\log p}{\log R}\right)\frac{2\log p}{p\log R} \ll \frac{\log\log kN}{\log R}. \tag{4.24}$$

We use the individual estimates (4.23), (4.24), although they are conditional, since we are anyway assuming the Riemann hypothesis for other purposes. However we should emphasize that one can establish sufficiently good estimates on average with respect to f without appealing to the Riemann hypothesis for $L(s,\mathrm{sym}^2(f))$ and $L(s,\mathrm{sym}^2(f)\otimes\mathrm{sym}^2(f))$. For example this can be done by one of the following ways. The first one is direct; one finds that (4.23), (4.24) are within the scope of the Petersson formula (2.8), provided $\widehat{\phi}$ is restricted as in the main theorems (no other conditions are required). The second way is more involved, but it produces (4.23), (4.24) (on average over f) for $\widehat{\phi}$ with any compact support. This method uses crude estimates (rather than the Riemann hypothesis) for higher symmetric power L-functions; specifically for $L(s,\mathrm{sym}^2(f)\otimes\mathrm{sym}^2(f))$ in the case of $D(f;\phi)$ and for $L(s,\mathrm{sym}^4(f)\otimes\mathrm{sym}^4(f))$ in the case of $D(\mathrm{sym}^2(f);\phi)$. In the latter case the required estimate is not yet established so the results are conditional. Nevertheless we present these arguments in considerable generality in Appendix B, since they are useful in various contexts.

Employing (4.23) and (4.24) we have that

$$D(f;\phi)=\widehat{\phi}(0)\frac{\log k^2N}{\log R}+\frac{1}{2}\phi(0)-P(f;\phi)+O\left(\frac{\log\log kN}{\log R}\right) \tag{4.25}$$

$$D(\mathrm{sym}^2(f);\phi)=\widehat{\phi}(0)\frac{\log k^2N^2}{\log R}-\frac{1}{2}\phi(0)-P^{(2)}(f;\phi)+O\left(\frac{\log\log kN}{\log R}\right) \tag{4.26}$$

where the implied constant depends only on the test function ϕ.

We still have a free hand to choose the scaling parameter R. By the above analysis (at the infinite place) it seems that the natural choices are $R=k^2N$ and $R=k^2N^2$, which we call the analytic conductors of f and $\mathrm{sym}^2(f)$, respectively. However, we prefer to locate R only up to a positive constant factor, i.e. we assume that

$$R\asymp k^2N \qquad \text{for } f\in H_k^\star(N)\ , \tag{4.29}$$

$$R\asymp k^2N^2 \qquad \text{for } \mathrm{sym}^2(f)\ . \tag{4.30}$$

In this way we retain a slight flexibility which will help us to perform an averaging over k in Section 8 (and perhaps over N if one so desired).

Note that the implied constants in (4.29), (4.30) do not affect the asymptotic formulas (4.25), (4.26) (they are washed out by the existing error term), therefore these formulas simplify to

$$D(f;\phi)=E(\phi)-P(f;\phi)+O\left(\frac{\log\log kN}{\log kN}\right)\ , \tag{4.31}$$

$$D(\mathrm{sym}^2(f);\phi)=E^{(2)}(\phi)-P^{(2)}(f;\phi)+O\left(\frac{\log\log kN}{\log kN}\right)\ , \tag{4.32}$$

where

$$E(\phi)=\widehat{\phi}(0)+\frac{1}{2}\phi(0)\ , \tag{4.33}$$

$$E^{(2)}(\phi)=\widehat{\phi}(0)-\frac{1}{2}\phi(0)\ . \tag{4.34}$$

We can write these functionals as

$$E(\phi)=\int_{-\infty}^{\infty}\phi(x)\left(1+\frac{1}{2}\delta_0(x)\right)dx \tag{4.35}$$

and

$$E^{(2)}(\phi)=\int_{-\infty}^{\infty}\phi(x)\left(1-\frac{1}{2}\delta_0(x)\right)dx\ . \tag{4.36}$$

Remarks. — The integral representation (4.35) already exhibits part of the expected distribution (see (1.10)) of low lying zeros of an individual L-function and one might think that the contribution of $P(f;\phi)$ is negligible. However, this is not true. We shall see that the distribution does change (for very low lying zeros, precisely if the support of $\widehat{\phi}(y)$ exceeds $[-1, 1]$).

5. Density theorems limited

In this section we give preliminary estimates of the sums

$$(5.1)\qquad \mathscr{B}_k^{\sigma}(\phi) = \sum_{f\in H_k^{\sigma}(N)} D(f;\phi)$$

for $\sigma = \star, +, -$, and

$$(5.2)\qquad \mathscr{B}_k^{(2)}(\phi) = \sum_{f\in H_k^{\star}(N)} D(\mathrm{sym}^2(f);\phi)\,.$$

By (3.5) the sums (5.1) with $\sigma = \pm$ split into

$$(5.3)\qquad 2\mathscr{B}_k^{\pm}(\phi) = \mathscr{B}_k^{\star}(\phi) \pm \mathscr{B}_k^{\#}(\phi)\,,$$

where

$$(5.4)\qquad \mathscr{B}_k^{\#}(\phi) = i^k\mu(N)\sqrt{N} \sum_{f\in H_k^{\star}(N)} \lambda_f(N)D(f;\phi)\,.$$

We shall treat $\mathscr{B}_k^{\star}(\phi)$, $\mathscr{B}_k^{\#}(\phi)$ and $\mathscr{B}_k^{(2)}(\phi)$ separately.

First we insert (4.31) into (5.1) and use (2.72) getting

$$(5.5)\qquad \mathscr{B}_k^{\star}(\phi) = \frac{k-1}{12}\varphi(N)E(\phi) - \mathscr{P}_k^{\star}(\phi) + O\left(k\varphi(N)\frac{\log\log kN}{\log kN}\right)\,,$$

where

$$(5.6)\qquad \mathscr{P}_k^{\star}(\phi) = \sum_{f\in H_k^{\star}(N)} P(f;\phi)\,.$$

Similarly we derive for $N \neq 1$ that (use $|\lambda_f(N)| = N^{-1/2}$)

$$(5.7)\qquad \mathscr{B}_k^{\#}(\phi) = -\mathscr{P}_k^{\#}(\phi) + O\left(k\varphi(N)\frac{\log\log kN}{\log kN}\right)\,,$$

where

$$(5.8)\qquad \mathscr{P}_k^{\#}(\phi) = i^k\mu(\mathrm{N})\sqrt{\mathrm{N}} \sum_{f\in \mathrm{H}_k^\star(\mathrm{N})} \lambda_f(\mathrm{N})\mathrm{P}(f;\phi)\,.$$

Next we insert (4.32) into (5.2) and use (2.72) getting

$$(5.9)\qquad \mathscr{B}_k^{(2)}(\phi) = \frac{k-1}{12}\varphi(\mathrm{N})\mathrm{E}^{(2)}(\phi) + \mathrm{O}\left(\frac{\log\log k\mathrm{N}}{\log k\mathrm{N}}\right)\,,$$

where

$$(5.10)\qquad \mathscr{P}_k^{(2)}(\phi) = \sum_{f\in \mathrm{H}_k^\star(\mathrm{N})} \mathrm{P}^{(2)}(f;\phi)\,.$$

It remains to evaluate $\mathscr{P}_k^\star(\phi)$, $\mathscr{P}_k^{\#}(\phi)$ and $\mathscr{P}_k^{(2)}(\phi)$. We write

$$(5.11)\qquad \mathscr{P}_k^\star(\phi) = \sum_{p\nmid \mathrm{N}} \Delta_{k,\mathrm{N}}^\star(p)\widehat{\phi}\left(\frac{\log p}{\log \mathrm{R}}\right)\frac{2\log p}{\sqrt{p}\log \mathrm{R}}\,,$$

$$(5.12)\qquad \mathscr{P}_k^{\#}(\phi) = i^k\mu(\mathrm{N})\sqrt{\mathrm{N}}\sum_{p\nmid \mathrm{N}} \Delta_{k,\mathrm{N}}^\star(p\mathrm{N})\widehat{\phi}\left(\frac{\log p}{\log \mathrm{R}}\right)\frac{2\log p}{\sqrt{p}\log \mathrm{R}}\,,$$

$$(5.13)\qquad \mathscr{P}_k^{(2)}(\phi) = \sum_{p\nmid \mathrm{N}} \Delta_{k,\mathrm{N}}^\star(p^2)\widehat{\phi}\left(\frac{\log p}{\log \mathrm{R}}\right)\frac{2\log p}{\sqrt{p}\log \mathrm{R}}\,.$$

Throughout $\Delta_{k,\mathrm{N}}'(n)$ denotes the partial sum (2.63) with the cut-off parameters

$$(5.14)\qquad \mathrm{X} = \mathrm{Y} = (k\mathrm{N})^\varepsilon$$

where ε is a positive constant which is sufficiently small.

First we replace $\Delta_{k,\mathrm{N}}^\star(n)$ by $\Delta_{k,\mathrm{N}}'(n)$ in (5.11), (5.12), (5.13) with $n = p, p\mathrm{N}, p^2$, respectively. In each case this replacement produces an error $\mathrm{O}(k\varphi(\mathrm{N})/\log k\mathrm{N})$ which is derived from the estimation (2.68). Then we apply the formula (2.74) for each $\Delta_{k,\mathrm{N}}'(n)$ with relevant n. The first term of (2.74) does not appear if $n = p$, $p\mathrm{N}$, and it equals $(k-1)\varphi(\mathrm{N})/12p$ if $n = p^2$. This term contributes to (5.13) at most $\mathrm{O}(k\varphi(\mathrm{N})/\log k\mathrm{N})$. Having made the above insertions we are left with the following;

$$(5.15)\qquad \mathscr{P}_k^\star(\phi) = \frac{k-1}{12}\sum_{\substack{\mathrm{LM}=\mathrm{N}\\ \mathrm{L}\leqslant \mathrm{X}}} \mu(\mathrm{L})\mathrm{M} \sum_{\substack{(m,\mathrm{M})=1\\ m\leqslant \mathrm{Y}}} m^{-1} \sum_{c\equiv 0(\mathrm{M})} c^{-1}\mathrm{Q}_k^\star(m;c) + \mathrm{O}\left(\frac{k\varphi(\mathrm{N})}{\log k\mathrm{N}}\right)\,,$$

$$(5.16)\qquad \mathscr{P}_k^{\#}(\phi) = i^k\frac{k-1}{12}\sqrt{\mathrm{N}}\sum_{\substack{\mathrm{LM}=\mathrm{N}\\ \mathrm{L}\leqslant \mathrm{X}}} \frac{\mu(\mathrm{M})\mathrm{M}}{\nu(\mathrm{L})} \sum_{\substack{(m,\mathrm{M})=1\\ m\leqslant \mathrm{Y}}} m^{-1} \sum_{c\equiv 0(\mathrm{M})} c^{-1}\mathrm{Q}_k^{\#}(m;c) + \mathrm{O}\left(\frac{k\varphi(\mathrm{N})}{\log k\mathrm{N}}\right)\,,$$

(5.17) $$\mathscr{P}_k^{(2)}(\phi) = \frac{k-1}{12} \sum_{\substack{LM=N \\ L\leqslant X}} \mu(L)M \sum_{\substack{(m,M)=1 \\ m\leqslant Y}} m^{-1} \sum_{c\equiv 0(M)} c^{-1} Q_k^{(2)}(m;c) + O\left(\frac{k\varphi(N)}{\log kN}\right) .$$

Here the terms $Q_k^\sigma(m;c)$ denote the following sums of Kloosterman sums

(5.18) $$Q_k^\star(m;c) = 2\pi i^k \sum_{p\nmid N} S(m^2, p; c) J_{k-1}\left(\frac{4\pi m}{c}\sqrt{p}\right) \widehat{\phi}\left(\frac{\log p}{\log R}\right) \frac{2\log p}{\sqrt{p}\log R} ,$$

(5.19) $$Q_k^\#(m;c) = 2\pi i^k \sum_{p\nmid N} S(m^2, pN; c) J_{k-1}\left(\frac{4\pi m}{c}\sqrt{pN}\right) \widehat{\phi}\left(\frac{\log p}{\log R}\right) \frac{2\log p}{\sqrt{p}\log R} ,$$

(5.20) $$Q_k^{(2)}(m;c) = 2\pi i^k \sum_{p\nmid N} S(m^2, p^2; c) J_{k-1}\left(\frac{4\pi m p}{c}\right) \widehat{\phi}\left(\frac{\log p}{\log R}\right) \frac{2\log p}{\sqrt{p}\log R} .$$

Interlude. — First we give quick estimates of $Q_k^\sigma(m;c)$ by applying (2.13) for individual Kloosterman sums and (2.11‴) for the Bessel function. For the latter we must secure the condition $x \leqslant \frac{k}{3}$, which translates into restrictions for the test function ϕ. Suppose $\widehat{\phi}$ has support in $(-v, v)$, so the sums $Q_k^\sigma(m;c)$ run over primes $p \leqslant P = R^{v'}$ with $v' < v$. Recall that $m \leqslant Y$ and $c \equiv 0 \pmod M$, so $c \geqslant M \geqslant NX^{-1}$. Therefore we require

(5.21) $$12\pi XYP^{1/2} \leqslant kN, \qquad \text{if } \sigma = \star ,$$

(5.22) $$12\pi XYP^{1/2} \leqslant kN^{1/2}, \qquad \text{if } \sigma = \# ,$$

(5.23) $$12\pi XYP \leqslant kN, \qquad \text{if } \sigma = (2) .$$

We obtain

$$Q_k^\star(m;c) \ll 2^{-k} P c^{\varepsilon - 1/2} ,$$
$$Q_k^\#(m;c) \ll 2^{-k} P c^{\varepsilon - 1/2} ,$$
$$Q_k^{(2)}(m;c) \ll 2^{-k} P^{3/2} c^{\varepsilon - 1/2} .$$

Hence

$$\mathscr{P}_k^\star(\phi) \ll 2^{-k} k^2 P N^{\varepsilon - 1/2} + k\varphi(N)/\log kN ,$$
$$\mathscr{P}_k^\#(\phi) \ll 2^{-k} k^2 P N^{\varepsilon} + k\varphi(N)/\log kN ,$$
$$\mathscr{P}_k^{(2)}(\phi) \ll 2^{-k} k^2 P^{3/2} N^{\varepsilon - 1/2} + k\varphi(N)/\log kN .$$

We require $\mathscr{P}_k^\sigma(\phi)$ be bounded by $k\varphi(N)/\log kN$, so that these sums would have no contribution to the main terms of $\mathscr{B}_k^\sigma(\phi)$. Recalling (4.29) one can see that our requirements are satisfied if $v=1+\log N/2\log kN$, $v=1$, $v=1/2$, respectively. Therefore we have proved

Theorem **5.1.** — *The Density Conjecture holds for the families* $H_k^\star(N)$, $H_k^\pm(N)$, $H_k^{(2)}(N)$ *for any test function* $\phi(x)$ *of Schwartz class whose Fourier transform* $\widehat{\phi}(y)$ *has support in* $(-v, v)$ *with* $v=1+\log N/2\log kN$, $v=1$, $v=1/2$, *respectively.*

In the next section we estimate the sums $\mathscr{P}_k^\star(\phi)$, $\mathscr{P}_k^\#(\phi)$, $\mathscr{P}_k^{(2)}(\phi)$ more precisely. We shall take advantage of the summation over primes which so far was hardly exploited (except for the easy sum (4.20) and for technical simplifications in various other places). Consequently we shall allow larger support of $\widehat{\phi}$, and we shall see that in the extended ranges the sums $\mathscr{P}_k^\#(\phi)$, $\mathscr{P}_k^{(2)}(\phi)$ do in fact contribute to the main terms of $\mathscr{B}_k^\pm(\phi)$, $\mathscr{B}_k^{(2)}(\phi)$, respectively.

6. Sums of Kloosterman Sums

We are going to execute the summation over primes in (5.18), (5.19) and (5.20) by means of the Riemann hypothesis for Dirichlet L-functions in the following form

$$\sum_{p\leqslant x}\chi(p)\log p=\delta_\chi x+O(x^{1/2}(\log cx)^2)\ ,$$

where χ is any character to modulus c, and δ_χ is the indicator of the principal character, the implied constant being absolute. For any integers m, n we have

$$\sum_{\substack{p\leqslant x\\ p\nmid c}}S(m,np;c)\log p=\frac{1}{\varphi(c)}\sum_{\chi\ (\mathrm{mod}\ c)}\Big(\sum_{a\ (\mathrm{mod}\ c)}\chi(a)S(m,an;c)\Big)\Big(\sum_{p\leqslant x}\overline{\chi}(p)\log p\Big)$$

$$=\frac{1}{\varphi(c)}\sum_{\chi\ (\mathrm{mod}\ c)}G_\chi(m)G_\chi(n)\{\delta_\chi x+O(x^{1/2}(\log cx)^2)\}$$

where

$$G_\chi(n)=\sum_{a\ (\mathrm{mod}\ c)}\chi(a)e\Big(\frac{an}{c}\Big)$$

is a Gauss sum. For the principal character this becomes the Ramanujan sum

$$R(n;c)=\sideset{}{^\star}\sum_{a\ (\mathrm{mod}\ c)}e\Big(\frac{an}{c}\Big)=\sum_{d|(c,n)}\mu\Big(\frac{c}{d}\Big)d\,.$$

By the orthogonality of characters we have

$$\sum_{\chi\ (\mathrm{mod}\ c)}|G_\chi(n)|^2=\varphi(c)^2\ .$$

From the above formulas we conclude

Lemma **6.1.** — *For any integers m, n and $x \geqslant 2$ we have*

$$\sum_{\substack{p \leqslant x \\ p \nmid c}} S(m, np; c) \log p = \frac{x}{\varphi(c)} R(m; c) R(n; c) + O(\varphi(c) x^{1/2} (\log cx)^2) . \tag{6.1}$$

If c and n have a large common divisor, we can reduce the error term in (6.1) by employing characters of a smaller modulus.

Lemma **6.2.** — *Let* M *be such that* $M|(c, n)$ *and* $(M, m) = 1$. *Then*

$$\sum_{\substack{p \leqslant x \\ pM \nmid c}} S(m, np; c) \log p = \frac{x}{\varphi(c)} R(m; c) R(n; c) + O\left(\varphi(\frac{c}{M}) x^{1/2} (\log cx)^2 \right) . \tag{6.2}$$

Here the main term vanishes unless $(M, c/M) = 1$ *in which case*

$$\frac{1}{\varphi(c)} R(m; c) R(n; c) = \frac{\mu(M)}{\varphi(c/M)} R\left(m; \frac{c}{M}\right) R\left(\frac{n}{M}; \frac{c}{M}\right) . \tag{6.3}$$

Proof. — Put $c = c'M$ and $n = n'M$. The sum on the left side of (6.2) is equal to

$$\frac{1}{\varphi(c)} \sum_{\chi \ (\mathrm{mod}\ c')} \Big(\sum_{a \ (\mathrm{mod}\ c')} \chi(a) S(m, an; c) \Big) \Big(\sum_{p \leqslant x} \chi(p) \log p \Big)$$

$$= \frac{1}{\varphi(c)} \sum_{\chi \ (\mathrm{mod}\ c')} \mu(M) \chi(M) G_\chi(m) G_\chi(m') \{ \delta_\chi x + O(x^{1/2} (\log c'x)^2) \}$$

$$= \frac{x}{\varphi(c')} \mu(M) \chi_0(M) R(m; c') R(n'; c') + O(\varphi(c') x^{1/2} (\log c'x)^2)$$

by the same arguments which were used to prove (6.1). Here, of course, the main terms must agree with that in (6.1) so this formula completes the proof of (6.2) and (6.3).

Similarly we derive

Lemma **6.3.** — *For any integer m and $x \geqslant 2$ we have*

$$\sum_{\substack{p \leqslant x \\ p \nmid c}} S(m^2, p^2; c) e\left(\frac{2mp}{c}\right) \log p = s_m(c) x + O(\tau(c) c x^{1/2} (\log cx)^2) \tag{6.4}$$

where $s_m(c)$ is the multiplicative function given by $s_m(p) = -1$ if $p|m$, $s_m(p^\alpha) = 0$ if $p|m$, $\alpha \geqslant 2$, $s_m(p) = (p-1)^{-1}$ if $p \nmid m$, $s_m(p^\alpha) = 0$ if $p \nmid m$, $2 \nmid \alpha$, $\alpha \geqslant 3$, $s_m(p^\alpha) = p^{\alpha/2}$ if $p \nmid m$, $2|\alpha$.

Proof. — The sum in (6.4) is equal to

$$\frac{1}{\varphi(c)}\sum_{\chi \ (\mathrm{mod}\ c)} \mathrm{T}_m(\chi)\sum_{p\leqslant x}\overline{\chi}(p)\log p = \frac{1}{\varphi(c)}\sum_{\chi \ (\mathrm{mod}\ c)} \mathrm{T}_m(\chi)\{\delta_\chi x + \mathrm{O}(x^{1/2}(\log cx)^2)\},$$

where

$$\begin{aligned}\mathrm{T}_m(\chi) &= \sum_{a \ (\mathrm{mod}\ c)} \chi(a)\mathrm{S}(m^2, a^2; c)e\left(\frac{2am}{c}\right)\\ &= \sum\sum_{a,\, d \ (\mathrm{mod}\ c)} \chi(ad)e\left(\frac{d(a+m)^2}{c}\right) = \sum_{a \ (\mathrm{mod}\ c)} \chi(a-m)\mathrm{G}_\chi(a^2)\,.\end{aligned}$$

For the principal character $\chi=\chi_0$ we get a sum of Ramanujan sums

$$\begin{aligned}\mathrm{T}_m(\chi_0) &= \sum_{(a-m,\, c)=1} \mathrm{R}(a^2, c) = \sum_{\substack{b\mid c\\ (b,\, m)=1}} \mu\left(\frac{c}{b}\right) b \sum_{\substack{(a-m,\, c)=1\\ b\mid a^2}} 1\\ &= \varphi(c)\sum_{\substack{b\mid c\\ (b,\, m)=1}} \mu\left(\frac{c}{b}\right)\frac{b}{\varphi(b)}\sum_{\substack{a \ (\mathrm{mod}\ b)\\ b\mid a^2}} 1 = \varphi(c)s_m(c)\,.\end{aligned}$$

To estimate $\mathrm{T}_m(\chi)$ for all characters we write

$$\varphi(c)\mathrm{T}_m(\chi) = \sum\sum_{a,\, b \ (\mathrm{mod}\ c)} \chi((a-bm)b)\mathrm{G}_\chi(a^2)$$

by changing variables $a \longrightarrow a\bar{b}$, $d \longrightarrow db^2$ and summing over b (mod c), $(b, c)=1$. Hence

$$\begin{aligned}\sum_{\chi \ (\mathrm{mod}\ c)} |\mathrm{T}_m(\chi)| &\leqslant \frac{1}{\varphi(c)}\sum_{a \ (\mathrm{mod}\ c)}\sum_{\chi \ (\mathrm{mod}\ c)}\Big|\sum_{b \ (\mathrm{mod}\ c)} \chi((a-bm)b)\Big||\mathrm{G}_\chi(a^2)|\\ &\leqslant c^{1/2}\left(\sum_{a \ (\mathrm{mod}\ c)}\sum_{\chi \ (\mathrm{mod}\ c)}\Big|\sum_{b \ (\mathrm{mod}\ c)} \chi((a-bm)b)\Big|^2\right)^{1/2}\\ &\leqslant (c\varphi(c)t(c))^{1/2}\,.\end{aligned}$$

Here $t(c)$ is the number of solutions to $(a-b_1m)b_1 \equiv (a-b_2m)b_2$ (mod c) in a, b_1, b_2 (mod c) with $(b_1b_2, c)=1$. This congruence can be written as $(a-m(b_1+b_2))(b_1-b_2)\equiv 0$ (mod c), whence it is easy to see that

$$t(c) \leqslant \sum_{g\mid c} g \sum^{\star}\sum^{\star}_{b_1\equiv b_2 \ (\mathrm{mod}\ g)} 1 \leqslant c\varphi(c)\tau(c)\,.$$

Hence

$$\frac{1}{\varphi(c)}\sum_{\chi\ (\mathrm{mod}\ c)}|\mathrm{T}_m(\chi)| \leqslant c\tau(c)\,.$$

This completes the proof of Lemma 6.3.

The multiplicative function $s_m(c)$ in the main term of (6.4) vanishes unless $c = ab^2$ with $(a, b) = 1$, a squarefree and $(b, m) = 1$, in which case

$$s_m(c) = \mu((a, m))\frac{\varphi((a, m))}{\varphi(a)}b\,. \tag{6.4'}$$

For the modulus c of the above type the condition $c \equiv 0 \pmod{\mathrm{M}}$ with M squarefree and $(m, \mathrm{M}) = 1$ determines uniquely the factorization $\mathrm{M} = \mathrm{M}_1\mathrm{M}_2$ such that $\mathrm{M}_1 | a$ and $\mathrm{M}_2 | b$.

In the formulas (6.1), (6.2), (6.4) the conditions $p \nmid c$, $p\mathrm{M} \nmid c$ can be dropped because these missing terms are absorbed by the existing error terms. Next we remove the terms with $p \nmid \mathrm{N}$ by trivial estimation. Moreover, we change $\log p$ into $p^{-1/2}\log p$ by partial summation. In particular we get the following relations

$$\sum_{\substack{p\leqslant x\\ p\nmid \mathrm{N}}} \mathrm{S}(m^2, p; c)\frac{\log p}{\sqrt{p}} = 2\frac{\mu(c)}{\varphi(c)}\mathrm{R}(m^2; c)x^{1/2} + \mathrm{O}(c(cx)^{\varepsilon}) \tag{6.5}$$

by Lemma 6.1,

$$\sum_{\substack{p\leqslant x\\ p\nmid \mathrm{N}}} \mathrm{S}(m^2, p\mathrm{N}; c)\frac{\log p}{\sqrt{p}} = 2\frac{\mu(\mathrm{M})}{\varphi(b)}\mathrm{R}(m^2; b)\mathrm{R}(\mathrm{L}; b)x^{1/2} + \mathrm{O}(b(bx)^{\varepsilon}) \tag{6.6}$$

where $\mathrm{LM} = \mathrm{N}$, $c = b\mathrm{M}$ and the main term exists only if $(b, \mathrm{M}) = 1$ by Lemma 6.2, and

$$\sum_{\substack{p\leqslant x\\ p\nmid \mathrm{N}}} \mathrm{S}(m^2, p^2; c)e\left(\frac{2mp}{c}\right)\log p = s_m(c)x + \mathrm{O}(cx^{1/2}(cx)^{\varepsilon}) \tag{6.7}$$

by Lemma 6.3.

Remark. — To the Kloosterman sum $\mathrm{S}(m^2, p^2, c)$ we attached the additive character $e(2mp/c)$. This character emerges from the phase of Bessel functions $\mathrm{J}_{k-1}(4\pi mp/c)$ in the Petersson formula in the context of the symmetric square L-functions (see Section 9); it fits nicely to the angle of the particular Kloosterman sum.

Now we are ready to estimate the sums $\mathrm{Q}_k^{\sigma}(m; c)$. Suppose $\widehat{\phi}(y)$ has support in $(-v, v)$ so that the sums $\mathrm{Q}_k^{\sigma}(m; c)$ run over primes $p \leqslant \mathrm{P} = \mathrm{R}^{v'}$ with some $v' < v$. By

(5.18) and (6.5) we get

$$\textbf{(6.9)}\qquad Q_k^\star(m;c) = -\frac{4\pi i^k}{\log R}\int_0^\infty \left\{2\frac{\mu(c)}{\varphi(c)}R(m^2;c)x^{1/2} + O(c(cx)^\varepsilon)\right\} d\,J_{k-1}\left(\frac{4\pi m\sqrt{x}}{c}\right)\widehat{\phi}\left(\frac{\log x}{\log R}\right).$$

Here the first term equals (by partial integration and a change of variable.)

$$\textbf{(6.10)}\qquad V_k^\star(m;c) = 2i^k\frac{\mu(c)cR(m^2;c)}{\varphi(c)m\log R}\int_0^\infty J_{k-1}(y)\widehat{\phi}\left(2\frac{\log(cy/4\pi m)}{\log R}\right)dy.$$

The error term is bounded by

$$c(cP)^\varepsilon\int_0^P\left(\frac{4\pi m\sqrt{x}}{c}\left|J'_{k-1}\left(\frac{4\pi m\sqrt{x}}{c}\right)\right| + \left|J_{k-1}\left(\frac{4\pi m\sqrt{x}}{c}\right)\right|\right)\frac{dx}{x}$$

$$= 2c(cP)^\varepsilon\int_0^z(|J'_{k-1}(y)| + |J_{k-1}(y)|y^{-1})dy$$

where $z = 4\pi m\sqrt{P}/c$. Using the recurrences $2J'_\nu(y) = J_{\nu-1}(y) - J_{\nu+1}(y)$ and $2\nu y^{-1}J_\nu(y) = J_{\nu-1}(y) + J_{\nu+1}(y)$ this is bounded by $c(cP)^\varepsilon\gamma_k(z)z$, where

$$\textbf{(6.11)}\qquad \gamma_k(z) = z^{-1}\int_0^z(|J_{k-2}(y)| + |J_{k-1}(y)| + |J_k(y)|)dy.$$

For any $k \geqslant 2$ and $z > 0$ we have (see (2.11′))

$$\textbf{(6.12)}\qquad \gamma_k(z) \ll k^{-1/2}$$

Moreover we have (see (2.11‴))

$$\textbf{(6.13)}\qquad \gamma_k(z) \ll 2^{-k}, \qquad \text{if } 3z \leqslant k.$$

Define $\widetilde{\gamma}_k(z) = 2^{-k}$ if $3z \leqslant k$ and $\widetilde{\gamma}_k(z) = k^{-1/2}$ otherwise so $\gamma_k(z) \ll \widetilde{\gamma}_k(z)$ for any $k \geqslant 2$ and $z > 0$. By the above estimates we conclude that

$$\textbf{(6.14)}\qquad Q_k^\star(m;c) = V_k^\star(m;c) + O(\widetilde{\gamma}_k(z)mP^{1/2}(ckN)^\varepsilon)$$

where $V_k^\star(m;c)$ is given by (6.10) and $z = 4\pi m\sqrt{P}/c$, the implied constant depending only on ε.

Similarly we derive by (5.19) and (6.6) that

$$\textbf{(6.15)}\qquad Q_k^\#(m;c) = V_k^\#(m;c) + O(\widetilde{\gamma}_k(z)mL(P/N)^{1/2}(ckN)^\varepsilon)$$

where

$$V_k^{\#}(m;c) = 2i^k \frac{c\mu(M)R(L;b)R(m^2;b)}{m\sqrt{N}\varphi(b)\log R} \int_0^\infty J_{k-1}(y)\widehat{\phi}\left(\frac{2\log(cy/4\pi m\sqrt{N})}{\log R}\right) dy, \tag{6.16}$$

with $c = b\mathrm{M}$, $\mathrm{LM} = \mathrm{N}$ and $z = 4\pi m\sqrt{\mathrm{PN}}/c$, the implied constant depending only on ε.

Remarks. — For very large c we can do better by applying (2.13). Indeed for $\sigma = \star, +, -$ we derive

$$Q_k^{\sigma}(m;c) \ll 2^{-k}\tau(c)c^{-1/2}m^2\mathrm{LP} \tag{6.17}$$

by (2.11‴), provided $c \geqslant 12\pi k m(\mathrm{N}+\mathrm{P})$. Hence we can refine the error terms in (6.14), (6.15) by writing $(k\mathrm{N})^{\varepsilon}(\log 2c)^{-2}$ in place of $(ck\mathrm{N})^{\varepsilon}$. This slight refinement will help to resolve the convergence problem in the forthcoming summation over c.

7. Density theorems extended

We proceed with the asymptotic evaluations of $\mathscr{P}_k^{\star}(\phi)$ and $\mathscr{P}_k^{\#}(\phi)$.

First notice that the main term in (6.14) is absorbed by the error term, because the integral in (6.10) is bounded by $\gamma_k(z)z$. Therefore

$$Q_k^{\star}(m;c) \ll \widetilde{\gamma}_k(z)m\mathrm{P}^{1/2}(k\mathrm{N})^{\varepsilon}(\log 2c)^{-2}. \tag{7.1}$$

Assume (5.21). Then (7.1) holds with $\widetilde{\gamma}_k(z) = 2^{-k}$ giving

$$\mathscr{P}_k^{\star}(\phi) \ll k\varphi(\mathrm{N})/\log k\mathrm{N} \tag{7.2}$$

by (5.15). Therefore there is no essential contribution to $\mathscr{B}_k^{\star}(\phi)$ of the sum over primes $\mathscr{P}_k^{\star}(\phi)$. We get

$$\mathscr{B}_k^{\star}(\pi) = \frac{k-1}{12}\varphi(\mathrm{N})\left\{\widehat{\phi}(0) + \frac{1}{2}\varphi(0) + \mathrm{O}\left(\frac{\log\log k\mathrm{N}}{\log \mathrm{KN}}\right)\right\} \tag{7.3}$$

by inserting (7.2) into (5.5), provided (5.21) holds. This last condition holds if $\widehat{\phi}$ is supported in $(-v, v)$ with

$$v = 2(\log k\mathrm{N})/\log k^2\mathrm{N}. \tag{7.4}$$

We have proven

Theorem **7.1.** — *The Density Conjecture holds for the family* $\mathrm{H}_k^{\star}(\mathrm{N})$ *for any test function* $\phi(x)$ *of Schwartz class whose Fourier transform* $\widehat{\phi}(y)$ *has support in* $(-v, v)$ *with* v *given by (7.4).*

Next we estimate $\mathscr{P}_k^{\#}(\phi)$. This time we do not impose the condition (5.22), because it limits the support of $\widehat{\phi}$ to $(-1, 1)$, that case having already been established in Theorem 5.1. Consequently we use (6.12) in the whole range (so one should not expect sharp estimates in the k-aspect). By (5.16), (6.15) (with $(ckN)^{\varepsilon}$ replaced by $(kN)^{\varepsilon}(\log 2c)^{-2}$) and (6.16) we derive

$$\mathscr{P}_k^{\#}(\phi) = \frac{k-1}{6} \sum_{\substack{LM=N \\ L\leqslant X}} \frac{M}{\nu(L)} \sum_{\substack{(m,M)=1 \\ m\leqslant Y}} m^{-2} \sum_{(b,M)=1} R(L;b)R(m^2;b)\varphi(b)^{-1} \tag{7.5}$$

$$\int_0^{\infty} J_{k-1}(y)\widehat{\phi}\left(\frac{2\log(by\sqrt{N}/4\pi mL)}{\log R}\right)\frac{dy}{\log R} + O((kP)^{1/2}(kN)^{\varepsilon}).$$

By the definition (1.5) and the formula (see (6.561.14) of [GR])

$$\int_0^{\infty} J_{k-1}(y)y^s dy = 2^s\Gamma\left(\frac{k+s}{2}\right)/\Gamma\left(\frac{k-s}{2}\right)$$

we find that the integral in (7.5) equals

$$\int_{-\infty}^{\infty} \phi(x\log R)\left(\frac{2\pi mL}{b\sqrt{N}}\right)^{4\pi ix}\frac{\Gamma(\frac{k}{2}-2\pi ix)}{\Gamma(\frac{k}{2}+2\pi ix)}dx$$

(after changing $x \to x\log R$). Next we interchange the integration over x with the summation over b, however for the convergence we introduce a parameter $\varepsilon > 0$ getting

$$\sum_b \int_0^{\infty} = \lim_{\varepsilon\to 0}\int_{-\infty}^{\infty} \phi(x\log R)\chi(\varepsilon+4\pi ix)\left(\frac{2\pi mL}{\sqrt{N}}\right)^{4\pi ix}\frac{\Gamma(\frac{k}{2}-2\pi ix)}{\Gamma(\frac{k}{2}+2\pi ix)}dx$$

where

$$\chi(s) = \sum_{(b,M)=1} R(L;b)R(m^2;b)\varphi(b)^{-1}b^{-s}.$$

Using the multiplicativity of Ramanujan sums, we compute

$$\chi(s) = \zeta(s+1)\zeta_M(s+1)^{-1}\alpha_{mL}(s)\beta_{(m,L)}(1-s)$$

where

$$\alpha_d(s) = \prod_{p\mid d}\left(1-\frac{1}{p^s}\right)\left(1+\frac{1}{(p-1)p^s}\right)^{-1}$$

$$\beta_d(s) = \prod_{p\mid d}(p^s+1).$$

We need $\chi(s)$ for $s \ll (\log R)^{-1}$. For this purpose the Laurent expansion of $\chi(s)$ near $s=0$ is useful

$$\begin{aligned}\zeta(s+1) &= s^{-1} + O(1)\\ \zeta_M(s+1)^{-1} &= \frac{\varphi(M)}{M}\{1 + O(|s|\xi(M))\}\\ \alpha_d(s) &= \delta(d,1) + O(|s|\log d)\\ \beta_d(1-s) &= \nu(d)\{1 + O(|s|\log d)\}\end{aligned}$$

where $\xi(M)$ is given by (4.19′) so $\xi(M) \ll \log\log 3M$, and $\delta(d, 1)$ is the diagonal symbol. Gathering the above approximations we get

$$\chi(s) = \frac{\varphi(N)}{sN}\delta(mL, 1) + O\left(\frac{\varphi(M)}{M}\log(mL\log 3N)\right).$$

Moreover we have (use (8.322) of [GR])

$$\Gamma\left(\frac{k+s}{2}\right) = \Gamma\left(\frac{k-s}{2}\right)\left(\frac{k}{2}\right)^{s}\left\{1 + O\left(\frac{|s|}{k}\right)\right\}.$$

Hence we deduce that

$$\sum_b \int_0^\infty = \delta(mL, 1)\frac{\varphi(N)}{N}\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}\phi(x\log R)\left(\frac{k\sqrt{N}}{4\pi}\right)^{4\pi i x}\frac{dx}{\varepsilon - 4\pi i x}$$
$$+ O\left(\frac{\varphi(M)}{M}\frac{\log(mL\log 3N)}{\log R}\right).$$

Let $A = k^2N/16\pi^2$ so that the last integral becomes

$$-\int_{-\infty}^{\infty}\phi(x\log R)\sin(2\pi x\log A)\frac{dx}{4\pi x + i\varepsilon}$$
$$+\int_{-\infty}^{\infty}\phi(\varepsilon x\log R)\cos(2\pi\varepsilon x\log A)\frac{dx}{(4\pi x)^2 + 1}.$$

Now we can take the limit as $\varepsilon \to 0$ getting

$$-\int_{\infty}^{\infty}\phi(x)\sin\left(2\pi x\frac{\log A}{\log R}\right)\frac{dx}{4\pi x} + \frac{1}{4}\phi(0).$$

For $R \asymp k^2N$ this is half of

$$-\int_{-\infty}^{\infty}\phi(x)\frac{\sin 2\pi x}{2\pi x}dx + \frac{1}{2}\phi(0) + O\left(\frac{1}{\log R}\right).$$

Introducing this into (7.5) we arrive at

$$(7.6)\qquad \mathscr{P}_k^{\#}(\phi) = -\frac{k-1}{12}\varphi(\mathrm{N})\left(\int_{-\infty}^{\infty}\phi(x)\frac{\sin 2\pi x}{2\pi x}dx - \frac{1}{2}\phi(0)\right) + \mathrm{O}\left(k\varphi(\mathrm{N})\frac{\log\log k\mathrm{N}}{\log k\mathrm{N}} + (k\mathrm{P})^{1/2}(k\mathrm{N})^{\varepsilon}\right).$$

Here the second error term is absorbed by the first one if $\mathrm{P} \ll k\mathrm{N}^2(k\mathrm{N})^{-3\varepsilon}$, which is satisfied if $\widehat{\phi}$ is supported in $(-v, v)$ with

$$(7.7)\qquad v = \frac{\log k\mathrm{N}^2}{\log k^2\mathrm{N}}.$$

This value of v is smaller than that in (7.4) for $\mathscr{P}_k^{\star}(\phi)$. Adding, or subtracting $\mathscr{P}_k^{\#}(\phi)$ from $\mathscr{B}_k^{\star}(\phi)$ we get

$$(7.8)\qquad \mathscr{B}_k^{\pm}(\phi) = \frac{k-1}{24}\varphi(\mathrm{N})\left\{\widehat{\phi}(0) + \frac{1\pm 1}{2}\phi(0) \pm \int_{-\infty}^{\infty}\phi(x)\frac{\sin 2\pi x}{2\pi x}dx\right\} + \mathrm{O}\left(k\varphi(\mathrm{N})\frac{\log\log k\mathrm{N}}{\log k\mathrm{N}}\right).$$

This asymptotic formula translates into

Theorem **7.2.** — *The Density Conjecture holds true for the families* $\mathrm{H}_k^{+}(\mathrm{N})$, $\mathrm{H}_k^{-}(\mathrm{N})$ *with the densities* W(SO(even))(x), W(SO(odd))(x) *given by (1.11), (1.12) respectively, for any test function* $\phi(x)$ *of Schwartz class whose Fourier transform* $\widehat{\phi}(y)$ *has support in* $(-v, v)$ *with* v *given by (7.7).*

8. Averaging over the weight

First we explain the features which allow us to extend the range of the Density Theorems 7.1 and 7.2. We begin by the error terms in (6.1) and (6.2). These terms are relatively small, so we have treated them crudely by use of partial summation and the magnitude of the Bessel function alone. The problem is that for $x \gg k$ the Bessel function $\mathrm{J}_{k-1}(x)$ behaves like $e^{ix}/\sqrt{x}$ (and slightly differently in the transition range), and we may regard $\mathrm{J}_{k-1}(4\pi\sqrt{mn}/c)$ as a continuous analog of the Kloosterman sum $\mathrm{S}(m, n; c)$ (see an interesting case in [CI]). Therefore, in this range the oscillating factor $e(2\sqrt{mn}/c)$ should be better treated by arithmetical means such as characters. We shall see this precise oscillating behaviour after averaging over k.

Proposition **8.1.** — *Fix a real valued function* $h \in \mathscr{C}_0^\infty(\mathrm{R}^+)$ *and* $\mathrm{K} \geqslant 1$. *For* $a = 0, 2$ *and* $x > 0$, *we have*

$$4\sum_{k \equiv a(4)} h\left(\frac{k-1}{\mathrm{K}}\right) \mathrm{J}_{k-1}(x) = h_\mathrm{K}(x) - i^a g_\mathrm{K}(x), \tag{8.1}$$

where

$$\begin{aligned} h_\mathrm{K}(x) &= h\left(\frac{x}{\mathrm{K}}\right) + \mathrm{O}\left(\frac{x}{\mathrm{K}^3}\right) \\ &= h\left(\frac{x}{\mathrm{K}}\right) + \frac{x}{6\mathrm{K}^3} h'''\left(\frac{x}{\mathrm{K}}\right) + \mathrm{O}\left(\frac{x^2}{\mathrm{K}^6}\right), \\ g_\mathrm{K}(x) &= \frac{\mathrm{K}}{\sqrt{x}} \mathrm{Im}\left(e^{ix - \pi i/4} \hbar\left(\frac{\mathrm{K}^2}{2x}\right)\right) + \mathrm{O}\left(\frac{x}{\mathrm{K}^4}\right) \end{aligned}$$

with

$$\hbar(v) = \int_0^\infty \frac{h(\sqrt{u})}{\sqrt{2\pi u}} e^{iuv} du, \tag{8.2}$$

the implied constants depending only on h.

Proof. — See Lemma 5.8 of [Iwa]. The important point is that $h_\mathrm{K}(x)$, $g_\mathrm{K}(x)$ do not depend on a, precisely we have

$$\begin{aligned} h_\mathrm{K}(x) &= -2i\mathrm{K} \int_{-\infty}^{\infty} \widehat{h}(t\mathrm{K}) \sin(x \sin 2\pi t) dt, \\ g_\mathrm{K}(x) &= -2\mathrm{K} \int_{-\infty}^{\infty} \widehat{h}(t\mathrm{K}) \cos(x \cos 2\pi t) dt. \end{aligned}$$

Corollary **8.2.** — *Let the conditions be as in Proposition 8.1. Then*

$$2\sum_{k \equiv 0(2)} h\left(\frac{k-1}{\mathrm{K}}\right) \mathrm{J}_{k-1}(x) = h\left(\frac{x}{\mathrm{K}}\right) + \mathrm{O}\left(\frac{x}{\mathrm{K}^3}\right)$$

$$2\sum_{k \equiv 0(2)} i^k h\left(\frac{k-1}{\mathrm{K}}\right) \mathrm{J}_{k-1}(x) = -\frac{\mathrm{K}}{\sqrt{x}} \mathrm{Im}\left(\overline{\zeta}_8 e^{ix} \hbar\left(\frac{\mathrm{K}^2}{2x}\right)\right) + \mathrm{O}\left(\frac{x}{\mathrm{K}^4}\right),$$

where $\zeta_8 = e^{2\pi i/8}$.

Remarks. — The oscillating term $\overline{\zeta}_8 e^{ix} \hbar(\mathrm{K}^2/2x)$ does not appear in the first formula while the leading term $h(x/\mathrm{K})$ is absent in the second formula. Since $\hbar(v)$ is a Schwartz function, the oscillating term is significant only if $x \gg \mathrm{K}^2$.

Let $K \geqslant 3$ and N be a positive squarefree number. Put

(8.3) $$\mathcal{A}^{\sigma}(K, N) = \sum_{k \equiv 0(2)} \frac{24}{k-1} h\left(\frac{k-1}{K}\right) \sum_{f \in H_k^{\sigma}(N)} D(f; \phi)$$

for $\sigma = \star, +, -$. Recall that the inner sum in (8.3) is the averaged density which was considered in the last three sections. Here and hereafter we maintain the same notation for all relevant quantities, and we suppress the subscript k after the averaging over the weight is performed. For example (8.3) becomes

(8.4) $$\mathcal{A}^{\sigma}(K, N) = \sum_{k \equiv 0(2)} \frac{24}{k-1} h\left(\frac{k-1}{K}\right) \mathcal{B}_k^{\sigma}(N).$$

Remember that for $N = 1$ the sets $H_k^{\pm}(1)$ are empty unless $k \equiv 3 \pm 1 \pmod 4$ respectively, in which cases they are $H_k^{\star}(1)$. In any case the sum $\mathcal{A}^{\sigma}(K, N)$ should be naturally normalized by

(8.5) $$A^{\sigma}(K, N) = \sum_{k \equiv 0(2)} \frac{24}{k-1} h\left(\frac{k-1}{K}\right) |H_k^{\sigma}(N)|.$$

By Corollary 2.14 we get (for all N, but consider the case $N = 1$ separately)

(8.6) $$A^{\sigma}(K, N) = \widehat{h}(0) K \varphi(N) + O((KN)^{2/3}).$$

The asymptotic formulas for $\mathcal{B}_k^{\sigma}(\phi)$ which were established for all k extend to $\mathcal{A}^{\sigma}(K, N)$ by linearity as far as the main terms are concerned. The early error terms remain the same after averaging. However the later error terms which resulted from estimates for individual Bessel functions can be now improved (in the k aspect) due to the explicit formula (8.1) for the sum of Bessel functions. The improvements in question appear in estimates for $Q_k^{\star}(\phi)$ and $Q_k^{\#}(\phi)$ (the asymptotics for other terms are valid for any ϕ without conditions on the support of $\widehat{\phi}$ except for being compact).

First we consider $\mathcal{A}^{\star}(K, N)$. For this we have $i^k J_{k-1}(x)$ in (5.18) replaced by the sum

(8.7) $$I(x) = \sum_{k \equiv 0(2)} 2 i^k h\left(\frac{k-1}{K}\right) J_{k-1}(x).$$

with $x = 4\pi m \sqrt{p}/c$. Assuming

(8.8) $$P^{1/2} \ll K^2 N (KN)^{-4\varepsilon}$$

we have $x \ll K^{2-\varepsilon}$, so by Corollary 8.2

(8.9) $$I(x) \ll x K^{-4}$$

in the range of summation in (5.18). Moreover using the formula $2J'_{k-1}(x)=J_{k-2}(x)-J_k(x)$ we find by (8.1) that

$$4I'(x) = \sum_{k\equiv 0(2)} 4i^k h\left(\frac{k-1}{K}\right)[J_{k-2}(x)-J_k(x)]$$
$$= h\left(\frac{x+1}{K}\right) - h\left(\frac{x-1}{K}\right) + O\left(\frac{x}{K^4}\right) = \frac{2}{K}h'\left(\frac{x}{K}\right) + O\left(\frac{x}{K^4}\right)$$

if $x \ll K^{2-\varepsilon}$. We will only need the bound

(**8.10**) $$I'(x) \ll K^{-1}.$$

By (6.9) with $i^k J_{k-1}(x)$ replaced by $I(x)$ we get

$$Q^\star(m;c) = 2\frac{\mu(c)cR(m^2;c)}{\varphi(c)m\log R}\int_0^\infty I(y)\widehat{\phi}\left(2\frac{\log(cy/4\pi m)}{\log R}\right)dy$$
$$+ O\left(c(cP)^\varepsilon\int_0^z (|I'(y)| + |I(y)|y^{-1})dy\right)$$

where $z = 4\pi m\sqrt{P}/c$. By (8.9) and (8.10) we deduce that

(**8.11**) $$Q^\star(m;c) \ll K^{-1}mP^{1/2}(KN)^\varepsilon(\log 2c)^{-2}.$$

Actually we also use directly (2.13) for very large c to refine $(cKN)^\varepsilon$ into $(KN)^\varepsilon$ $(\log 2c)^{-2}$ as we did for (7.1). Note that (8.11) is like (7.1) with $\gamma_k(z) \ll k^{-1}$, which is sharper by a factor of $k^{-1/2}$ than the bound (6.12) (we are in the range $z \ll K^{2-\varepsilon}$, so (6.13) is not available). Inserting (8.11) into (5.15) and using the assumption (8.8) we arrive at

(**8.11′**) $$\mathscr{P}^\star(\phi) \ll K\varphi(N)/\log KN.$$

This estimate shows that the sums over primes $\mathscr{P}_k^\star(\phi)$ do not contribute to the main term of $\mathscr{A}^\star(K,N)$. We are left with

(**8.12**) $$\mathscr{A}^\star(K,N) = \widehat{h}(0)K\varphi(N)\left\{\widehat{\phi}(0) + \frac{1}{2}\phi(0)\right\} + O\left(K\varphi(N)\frac{\log\log KN}{\log KN}\right)$$

provided P satisfies (8.8). This last condition holds if $\widehat{\phi}$ is supported in $(-2,2)$. In view of (8.6), (4.33) and (4.35) we can state the formula (8.12) as

Theorem **8.3.** — *Let ϕ be a Schwartz function with the Fourier transform $\widehat{\phi}$ supported in $(-2,2)$. Then*

(**8.13**) $$\frac{\mathscr{A}^\star(K,N)}{A^\star(K,N)} = \int_{-\infty}^{\infty}\phi(x)W(O)(x)dx + O\left(\frac{\log\log kN}{\log KN}\right)$$

where the implied constant depends only on the test function ϕ.

Similarly we consider $\mathcal{A}^{\pm}(K, N)$. This splits into

$$2\mathcal{A}^{\pm}(K, N) = \mathcal{A}^{\star}(K, N) \pm \mathcal{A}^{\#}(K, N), \tag{8.14}$$

where $\mathcal{A}^{\star}(K, N)$ has been treated above and $\mathcal{A}^{\#}(K, N)$ corresponds to $\mathcal{B}_k^{\#}(\phi)$ with $i^k J_{k-1}(x)$ replaced by the sum

$$J(x) = \sum_{k \equiv 0(2)} 2h\left(\frac{k-1}{K}\right) J_{k-1}(x) \tag{8.15}$$

with $x = 4\pi m\sqrt{pN}/c$ (see (5.19)). By Corollary 8.2

$$J(x) = h\left(\frac{x}{K}\right) + O\left(\frac{x}{K^3}\right). \tag{8.16}$$

Hence, following the arguments which led us to (7.6), we arrive at

$$\mathcal{P}^{\#}(\phi) = -\widehat{h}(0)K\varphi(N)\left(\int_{-\infty}^{\infty}\phi(x)\frac{\sin 2\pi x}{2\pi x}dx - \frac{1}{2}\phi(0)\right) + O\left(K\varphi(N)\frac{\log\log kN}{\log kN} + K^{-1}P^{1/2}(KN)^{\varepsilon}\right). \tag{8.17}$$

Here the second error term is absorbed by the first one if $P^{1/2} \ll K^2N(KN)^{-4\varepsilon}$, this condition coincides with (8.8). Adding, or subtracting (8.17) from (8.12) we can state the results as

Theorem **8.4.** — *Let* ϕ *be a Schwartz function with the Fourier transform* $\widehat{\phi}$ *supported in* $(-2, 2)$. *Then*

$$\frac{\mathcal{A}^{\pm}(K, N)}{A^{\pm}(K, N)} = \int_{-\infty}^{\infty}\phi(x)W^{\pm}(x)dx + O\left(\frac{\log\log KN}{\log KN}\right), \tag{8.18}$$

where $W^{+}(x) = W(SO(\text{even}))(x)$, $W^{-}(x) = W(SO(\text{odd}))(x)$, *and the implied constant depends only on the test function* ϕ.

9. The symmetric square

In this section we are going to evaluate asymptotically

$$\mathcal{A}^{(2)}(K, N) = \sum_{k \equiv 0(2)} \frac{24}{k-1} h\left(\frac{k-1}{K}\right) \sum_{f \in H_k^{\star}(N)} D(\text{sym}^2(f); \phi). \tag{9.1}$$

Recall that the innermost sums $\mathcal{B}_k^{(2)}(\phi)$ were treated in Section 5, but only for a small range of support of $\widehat{\phi}$, the result being

$$\mathcal{B}_k^{(2)}(\phi) = \frac{k-1}{12}\varphi(\mathrm{N})\left\{\widehat{\phi}(0) - \frac{1}{2}\phi(0) + \mathrm{O}\left(\frac{\log\log \mathrm{KN}}{\log \mathrm{KN}}\right)\right\} \tag{9.2}$$

if $\widehat{\phi}$ is supported in $\left(-\frac{1}{2}, \frac{1}{2}\right)$. Our goal is to establish a formula for $\mathcal{A}^{(2)}(\mathrm{K}, \mathrm{N})$ which is valid for ϕ with the support of $\widehat{\phi}$ in $(-v, v)$ for v strictly larger than one (in the K aspect). The key point of having $v > 1$ is that an additional contribution enters in (9.2). This will come from the sums (5.20) which we are able to evaluate more precisely due to an important feature of the sum of Bessel functions (8.7).

Gathering together (5.9), (5.17), 5.20) we obtain

$$\begin{aligned} \mathcal{A}^{(2)}(\mathrm{K}, \mathrm{N}) = \widehat{h}(0)\mathrm{K}\varphi(\mathrm{N})\left(\widehat{\phi}(0) - \frac{1}{2}\phi(0)\right) \\ - \mathcal{P}^{(2)}(\phi) + \mathrm{O}\left(\mathrm{K}\varphi(\mathrm{N})\frac{\log\log \mathrm{KN}}{\log \mathrm{KN}}\right) \end{aligned} \tag{9.3}$$

where

$$\mathcal{P}^{(2)}(\phi) = \sum_{\substack{\mathrm{LM}=\mathrm{N} \\ \mathrm{L}\leqslant \mathrm{X}}} \mu(\mathrm{L})\mathrm{M} \sum_{\substack{(m,\mathrm{M})=1 \\ m\leqslant \mathrm{Y}}} m^{-1} \sum_{c\equiv 0(\mathrm{M})} c^{-1}\mathrm{Q}^{(2)}(m;c), \tag{9.4}$$

$$\mathrm{Q}^{(2)}(m;c) = 2\pi \sum_{p\nmid \mathrm{N}} \mathrm{S}(m^2, p^2; c)\mathrm{I}\left(\frac{4\pi m p}{c}\right)\widehat{\phi}\left(\frac{\log p}{\log \mathrm{R}}\right)\frac{2\log p}{\sqrt{p}\log \mathrm{R}}. \tag{9.5}$$

and $\mathrm{I}(x)$ is the sum of Bessel functions (8.7). By Corollary 8.2

$$\mathrm{I}(x) = -\frac{\mathrm{K}}{\sqrt{x}}\mathrm{Im}\left\{\overline{\zeta}_8 e^{ix}\hbar\left(\frac{\mathrm{K}^2}{2x}\right)\right\} + \mathrm{O}\left(\frac{x}{\mathrm{K}^4}\right). \tag{9.6}$$

Inserting (9.6) into (9.5) and estimating the error terms by means of (2.10) we get

$$\mathrm{Q}^{(2)}(m;c) = -\frac{2\mathrm{K}}{\log \mathrm{R}}\left(\frac{\pi c}{m}\right)^{1/2}\mathrm{T}(m;c) + \mathrm{O}\left(\mathrm{K}^{-4}c^{-1/2}\mathrm{P}^{3/2}(c\mathrm{KN})^{\varepsilon}\right) \tag{9.7}$$

where

$$\mathrm{T}(m;c) = \sum_{p\nmid \mathrm{N}} \mathrm{S}(m^2, p^2; c)\mathrm{Im}\left\{\overline{\zeta}_8 e\left(\frac{2mp}{c}\right)\hbar\left(\frac{c\mathrm{K}^2}{8\pi m p}\right)\right\}\widehat{\phi}\left(\frac{\log p}{\log \mathrm{R}}\right)\frac{\log p}{p}. \tag{9.8}$$

Since $\hbar(v) \ll v^{-\mathrm{A}}$ for any $\mathrm{A} \geqslant 0$ (with the implied constant depending on A) we estimate the contribution of primes $p \leqslant \mathrm{P}_0 = c\mathrm{K}^2/m\mathrm{H}$ trivially by $\mathrm{O}(c\mathrm{H}^{-\mathrm{A}})$, where $\mathrm{H} \geqslant 1$ will be chosen soon. In the remaining range $P_0 < p < P$ we apply (6.7) getting by partial summation (recall that $h^\star(v)$ is of Schwartz class)

$$T(m;c) = s_m(c)\int_{P_0}^{P} \operatorname{Im}\left\{\overline{\zeta}_8 \hbar\left(\frac{cK^2}{8\pi m y}\right)\right\}\widehat{\phi}\left(\frac{\log y}{\log R}\right)\frac{dy}{y} + O\left(cH^{-A} + cP_0^{-1/2}(cP_0)^{\varepsilon}\right).$$

Since $c \leqslant mHPK^{-2}$ the above error term is $\ll c^{-1/2}(\log 2c)^{-2}K^{-3}P(P^2H^{-A} + m^2H^2p^{4\varepsilon})$. Taking $H = P^{\varepsilon}$ this is bounded by $O(c^{-1/2}(\log 2c)^{-2}K^{-3}P(KNP)^{6\varepsilon})$. The integral $\int_{P_0}^{P}$ can be extended to $\int_0^\infty$ up to the same error term as above. Putting

$$h^\star(v) = \operatorname{Im}(\overline{\zeta}_8\hbar(v)) = \int_0^\infty \frac{h(\sqrt{u})}{\sqrt{2\pi u}}\sin\left(uv - \frac{\pi}{4}\right)du \tag{9.9}$$

we have shown that

$$T(m;c) = s_m(c)\int_0^\infty h^\star\left(\frac{cK^2}{8\pi m y}\right)\widehat{\phi}\left(\frac{\log y}{\log R}\right)\frac{dy}{y} + O(c^{-1/2}(\log 2c)^{-2}k^{-3}P(KN)^{\varepsilon}).$$

for any $\varepsilon > 0$. Inserting this into (9.7) we arrive at

$$\begin{aligned} Q^{(2)}(m;c) = &-\frac{2K}{\log R}\left(\frac{\pi c}{m}\right)^{1/2} s_m(c)\int_0^\infty h^\star\left(\frac{cK^2}{8\pi m y}\right)\widehat{\phi}\left(\frac{\log y}{\log R}\right)\frac{dy}{y} \\ &+ O(P(K\log 2c)^{-2}(KN)^{\varepsilon} + P^{3/2}K^{-4}c^{-1/2}(cKN)^{\varepsilon}). \end{aligned} \tag{9.10}$$

Next inserting (9.10) into (9.4) we get

$$\begin{aligned} \mathscr{P}^{(2)}(\phi) = \frac{-2\sqrt{\pi}K}{\log R}\sum_{\substack{LM=N\\ L\leqslant X}}\mu(L)M\sum_{\substack{(m,M)=1\\ m\leqslant Y}} m^{-3/2}\int_0^\infty S\left(\frac{K^2}{8\pi m y}\right)\widehat{\phi}\left(\frac{\log y}{\log R}\right)\frac{dy}{y} \\ + O((K^{-2}P + N^{-1/2}K^{-4}P^{3/2})(KN)^{\varepsilon}) \end{aligned} \tag{9.11}$$

where

$$S(w) = \sum_{c\equiv 0 \pmod M} c^{-1/2}s_m(c)h^\star(cw). \tag{9.12}$$

Note that the series (9.12) depends on m and M. By (6.4′)

$$S(w) = \sum_{M_1M_2=M}\ \sideset{}{^\flat}\sum_{(a,M_1)=1}\frac{\mu((a,m))\varphi((a,m))}{\varphi(aM_1)\sqrt{aM_1}}\sum_{(b,amM_1)=1} h^\star(ab^2M_1M_2^2w) \tag{9.13}$$

where $\sum^\flat$ restricts the variable a to squarefree numbers.

First we give a trivial estimate for $S(w)$. Using $h^\star(v) \ll (1+v)^{-1}$ we get

$$\sum_{(b,amM_1)=1} h^\star(ab^2M_1M_2^2w) \ll (aM_1M_2^2w)^{-1/2}.$$

Hence

$$\text{(9.14)} \qquad S(w) \ll w^{-1/2} M^{-1} \prod_{p|mM} \left(1 + \frac{1}{p}\right)$$

Next we give a more precise treatment of $S(w)$. The innermost sum in (9.13) runs over positive integers b with $(b, amM_1) = 1$. Extending this summation to all integers we write

$$\sum_{(b,\, amM_1)=1} h^\star = \frac{1}{2} \sum_{\substack{b \in \mathbf{Z} \\ (b,\, amM_1)=1}} h^\star - \frac{1}{2} h^\star(0)\delta(1, amM_1)$$

where the subtracted term comes from $b = 0$. Note that $h^\star(0) = -\pi^{-1/2}\widehat{h}(0)$ by (9.9). Accordingly $S(w)$ splits into

$$\text{(9.16)} \qquad S(w) = \frac{1}{2} T(w) + \frac{1}{2\sqrt{\pi}} \widehat{h}(0)\delta(1, m)$$

where $T(w)$ is defined by (9.13) with the innermost sum replaced by

$$\text{(9.17)} \qquad \sum_{\substack{b \in \mathbf{Z} \\ (b,\, amM_1)=1}} h^\star(ab^2 M_1 M_2 w) = \sum_{\alpha | amM_1} \mu(\alpha) \sum_{b \in \mathbf{Z}} h^\star(\alpha^2 ab^2 M_1 M_2^2 w).$$

For the sum over all $b \in \mathbf{Z}$ we apply the Poisson formula

$$\text{(9.18)} \qquad \sum_{b \in \mathbf{Z}} h^\star(\alpha^2 ab^2 M_1 M_2^2 w) = \frac{1}{\alpha M_2 \sqrt{aM_1 w}} \sum_{b \in \mathbf{Z}} \check{h}\left(\frac{b}{\alpha M_2 \sqrt{aM_1 w}}\right)$$

where $\check{h}(t)$ is the Fourier transform of $h^\star(v^2)$. By (9.9) we have

$$\begin{aligned} \check{h}(t) &= \int_{-\infty}^{\infty} h^\star(v^2) e(vt)\, dv \\ &= 2\sqrt{\frac{2}{\pi}} \int_0^\infty h(u) \left(\int_{-\infty}^{\infty} \sin\left(u^2 v^2 - \frac{\pi}{4}\right) \cos(2\pi vt)\, dv \right) du \\ &= \frac{4}{\sqrt{\pi}} \int_0^\infty \frac{h(u)}{u} \left(\int_0^\infty (\sin v^2 - \cos v^2) \cos(2\pi vt/u)\, dv \right) du \\ &= 2\sqrt{2} \int_0^\infty \frac{h(u)}{u} \sin\left((2\pi t/u)^2\right) du \end{aligned}$$

by (3.691.5) and (3.691.7) of [GR]. Note that $\check{h}(0) = 0$. Moreover this formula shows that $\check{h}(t) \ll |t|^{-A}$ with any $A > 0$. Using this bound with $A = 1 + \theta$ we derive from

(9.18) that

$$\sum_{b\in\mathbf{Z}} h^\star(\alpha^2 ab^2 M_1 M_2^2 w) \ll (\alpha^2 a M_1 M_2^2 w)^\theta.$$

Inserting this into (9.17) we get

$$\sum_{\substack{b\in\mathbf{Z}\\(b,\,amM_1)=1}} h^\star(ab^2 M_1 M_2^2 w) \ll \tau(amM_1)(m^2a^3M_1^3M_2^2w)^\theta.$$

Next inserting this into (9.13) we get

$$\textbf{(9.19)}\qquad T(w) \ll \tau^2(m)(mM\sqrt{w})^\theta$$

for any θ with $0 < \theta < \frac{1}{3}$. Finally we obtain by (9.16)

$$\textbf{(9.20)}\qquad S(w) = \frac{1}{2\sqrt{\pi}}\widehat{h}(0)\delta(1, m) + O(\tau^2(m)(mM\sqrt{w})^\theta).$$

Now we are ready to complete the evaluation of $\mathscr{P}^{(2)}(\phi)$ by applying the estimates (9.14), (9.20) in different ranges of the integral in (9.11). We use (9.14) for $w = K^2/8\pi my$ with $8\pi my \leqslant z^2$, say, showing that this part contributes to $\mathscr{P}^{(2)}(\phi)$ at most

$$\textbf{(9.21)}\qquad \frac{\nu(N)z}{N\log R} \ll z\frac{\log\log 3N}{\log R}.$$

In the remaining range $8\pi my > z^2$ we use (9.20) showing that this part contributes to $\mathscr{P}^{(2)}(\phi)$

$$\textbf{(9.22)}\qquad \frac{\widehat{h}(0)K}{\log R}\sum_{\substack{LM=N\\L\leqslant X}} \mu(L)M\int_{z^2}^\infty \widehat{\phi}\left(\frac{\log y}{\log R}\right)\frac{dy}{y} + O\left(\frac{KN}{\log R}\left(\frac{KN}{z}\right)^\theta\right).$$

Here we have $\sum \mu(L)M = \varphi(N) + O(NX^{-1/2})$ and

$$\int_{z^2}^\infty \widehat{\phi}\left(\frac{\log y}{\log R}\right)\frac{dy}{y} = (\log R)\int_{2\log z/\log R}^\infty \widehat{\phi}(y)dy.$$

We choose $z = KN$ and $R = K^2N^2$ getting

$$\textbf{(9.23)}\qquad \mathscr{P}^{(2)}(\phi) = -\widehat{h}(0)K\varphi(N)\int_1^\infty \widehat{\phi}(y)dy + O\left(KN\frac{\log\log 3N}{\log KN}\right)$$
$$+O((K^{-2}P + N^{-1/2}K^{-4}P^{3/2})(KN)^\varepsilon).$$

The first error term above is admissible and it does not depend on the support of $\widehat{\phi}(y)$. If $\widehat{\phi}(y)$ is supported in $(-v, v)$, then $P = (KN)^{2v'}$ with $v' < v$. Hence the second error term in (9.23) is bounded by KN/ log KN if

$$v=\frac{\log \mathrm{K}^3\mathrm{N}}{2\log \mathrm{KN}}. \tag{9.24}$$

Note that

$$\int_1^\infty \widehat{\phi}(y)dy=\frac{1}{2}\phi(0)-\frac{1}{2}\int_{-1}^{1}\widehat{\phi}(y)dy=\frac{1}{2}\phi(0)-\int_{-\infty}^{\infty}\phi(x)\frac{\sin 2\pi x}{2\pi x}dx$$

by Plancherel's theorem. Finally introducing (9.23) into (9.3) we conclude that

$$\mathscr{A}^{(2)}(\mathrm{K},\mathrm{N})=\widehat{h}(0)\mathrm{K}\varphi(\mathrm{N})\int_{-\infty}^{\infty}\left(1-\frac{\sin 2\pi x}{2\pi x}\right)\phi(x)dx+\mathrm{O}\left(\mathrm{KN}\frac{\log\log \mathrm{KN}}{\log \mathrm{KN}}\right) \tag{9.25}$$

provided $\widehat{\phi}$ is supported in $(-v, v)$ with v given by (9.24). Dividing by $\mathrm{A}^{(2)}(\mathrm{K},\mathrm{N})=\mathrm{A}^*(\mathrm{K},\mathrm{N})$ (see (8.3) and (8.6)) we can state the result as

Theorem **9.1.** — *Let k, N be positive integers, k even, N squarefree. Let ϕ be a Schwartz function with the Fourier transform $\widehat{\phi}$ supported in $(-v, v)$ with v given by (9.24). Then*

$$\frac{\mathscr{A}^{(2)}(\mathrm{K},\mathrm{N})}{\mathrm{A}^{(2)}(\mathrm{K},\mathrm{N})}=\int_{-\infty}^{\infty}\phi(x)\mathrm{W}(Sp(\infty))(x)dx+\mathrm{O}\left(\frac{(\log\log \mathrm{KN})^2}{\log \mathrm{KN}}\right) \tag{9.26}$$

where the implied constant depends only on the test function ϕ.

Remarks. — For $v=v(\mathrm{K},\mathrm{N})$ given by (9.24) we have $v(\mathrm{K},\mathrm{N})\to\frac{3}{2}$ as $\mathrm{K}\to\infty$ and N is fixed.

10. Further improvements and a quasi Riemann hypothesis

Extending the density theorems for ϕ with the support of $\widehat{\phi}$ larger than $(-2, 2)$ does not seem to be within reach of Riemann hypothesis for automorphic L-functions. In this section we do not assume any unproved hypothesis except for a bound for certain exponential sums over primes in arithmetic progressions.

Hypothesis S. — For any $x\geqslant 1$, $c\geqslant 1$ and a with $(a, c)=1$ we have

$$\sum_{\substack{p\leqslant x\\ p\equiv a \ (\mathrm{mod}\ c)}} e\left(\frac{2\sqrt{p}}{c}\right)\ll c^{\mathrm{A}}x^{\alpha+\varepsilon} \tag{10.1}$$

where α, A are constants with $\mathrm{A}\geqslant 0$, $\frac{1}{2}\leqslant\alpha\leqslant\frac{3}{4}$ and ε is any positive number, the implied constant depending only on ε.

We believe Hypothesis S is true with $A = 0$ and $\alpha = \frac{1}{2}$ (see Appendix C), however the critical fact is that one may have (10.1) with $\alpha < \frac{3}{4}$. With this estimate we are going to establish the desired extension of the Density Theorems.

For simplicity of exposition we restrict our consideration to the modular group, i.e. we take $N = 1$. In this special case the whole space $S_k = S_k(1)$ is spanned by the set $H_k = H_k^\star(1)$ of Hecke eigencuspforms. By (2.3), (2.18), (2.36) and (3.14) the trace

$$\Delta_k(m, n) = \sum_{f \in H_k} \overline{\psi}_f(m) \psi_f(n) \tag{10.1}$$

becomes

$$\Delta_k(m, n) = \frac{2\pi^2}{k-1} \sum_{f \in H_k} \frac{\lambda_f(m)\lambda_f(n)}{L(1, \operatorname{sym}^2(f))}. \tag{10.2}$$

As in Section 8 we perform averaging over k of type

$$\mathscr{B}(m, n) = \sum_{k \text{ even}} 2h\left(\frac{k-1}{K}\right) \Delta_k(m, n) \tag{10.3}$$

where the weight function $h(u)$ is smooth of compact support in $\mathbf{R}^+$. Notice we do not break the averaging into classes $k \equiv a \pmod 4$ (see (3.4) and (3.5)) as here we are not interested to capture the sign of the functional equation

$$(2\pi)^{-s}\Gamma\left(\frac{k-1}{2} + s\right) L(s, f) = i^k (2\pi)^{1-s} \Gamma\left(\frac{k+1}{2} - s\right) L(1-s, f). \tag{10.4}$$

Also we do not remove the arithmetical weights $L(1, \operatorname{sym}^2(f))^{-1}$ (which are quite natural from the point of view of spectral theory). Our goal is to evaluate asymptotically the following sum

$$\mathscr{B}(K) = \sum_{k \text{ even}} \frac{4\pi^2}{k-1} h\left(\frac{k-1}{K}\right) \sum_{f \in H_k} D(f; \phi) L^{-1}(1, \operatorname{sym}^2(f)), \tag{10.5}$$

where ϕ is a test function of Schwartz class with $\widehat{\phi}$ supported in $(-v, v)$. Our goal is to allow $v > 2$. It is natural to normalize $\mathscr{B}(K)$ by dividing by

$$B(K) = \sum_{k \text{ even}} \frac{4\pi^2}{k-1} h\left(\frac{k-1}{K}\right) \sum_{f \in H_k} L^{-1}(1, \operatorname{sym}^2(f)). \tag{10.6}$$

We have

$$\Delta_k(1, 1) = \frac{2\pi^2}{k-1} \sum_{f \in H_k} L^{-1}(1, \operatorname{sym}^2(f)) = 1 + O(2^{-k}) \tag{10.7}$$

by Corollary 2.3, which yields

$$\text{(10.8)} \qquad B(K) = \widehat{h}(0)K + O(1).$$

Here and hereafter the implied constants may depend on h. We shall assume that $\widehat{h}(0) \neq 0$.

The Petersson formula (see (2.8))

$$\Delta_k(m, n) = \delta(m, n) + 2\pi i^k \sum_{c=1}^{\infty} S(m, n; c) J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)$$

yields

$$\text{(10.9)} \qquad \mathscr{B}(m, n) = \delta(m, n)H + 2\pi \sum_{c=1}^{\infty} c^{-1} S(m, n; c) I\left(\frac{4\pi\sqrt{mn}}{c}\right)$$

where

$$\text{(10.10)} \qquad H = \sum_{k \text{ even}} 2h\left(\frac{k-1}{K}\right) = \widehat{h}(0)K + O(1)$$

and $I(x)$ is the corresponding sum of Bessel functions (see (8.7)) for which we have

$$\text{(10.11)} \qquad I(x) = -\frac{K}{\sqrt{x}} \operatorname{Im}\left(\overline{\zeta}_8 e^{ix} \hbar\left(\frac{K^2}{2x}\right)\right) + O\left(\frac{x}{K^4}\right)$$

by Corollary 8.2, where $\hbar(v)$ is given by the Fourier integral (8.2). Using (2.10) to estimate the error terms we arrive at

Lemma **10.1.** — *For any positive numbers m, n we have*

$$\text{(10.12)} \qquad \mathscr{B}(m, n) = \widehat{h}(0)K\delta(m, n) + O(\delta(m, n) + (mn)^{1/2}K^{-4})$$

$$-\pi^{1/2}(mn)^{-1/4} K \operatorname{Im}\left(\overline{\zeta}_8 \sum_{c=1}^{\infty} c^{-1/2} S(m, n; c) e\left(\frac{2\sqrt{mn}}{c}\right) \hbar\left(\frac{cK^2}{8\pi\sqrt{mn}}\right)\right).$$

If $mn \ll K^{4-\delta}$ the last term involving $\hbar(v)$ can be deleted (because it is absorbed by the second error term).

Now we are ready to evaluate $\mathscr{B}(K)$. To this end recall the formula (4.31) (for $N = 1$ and $R = K^2$, see (4.29)). This result is essentially unconditional, except for the secondary estimate (4.23) which was deduced assuming the Riemann hypothesis for $L(s, \mathrm{sym}^2(f))$. In this section we do not make use of any Riemann hypothesis, so we provide an unconditional proof of (4.23) on average over $f \in H_k$ in Appendix B. Precisely we shall show that if $\widehat{\phi}$ has compact support (no matter how large)

$$\sum_{f\in H_k} L^{-1}(1, \mathrm{sym}^2(f)) \sum_p \lambda_f(p^2)\widehat{\phi}\left(\frac{\log p}{\log k}\right)\frac{\log p}{p} \ll k \tag{10.13}$$

where the implied constant depends on ϕ.

Inserting (4.31) into (10.5) we get

$$\mathscr{B}(K) = \widehat{h}(0)K\{\widehat{\phi}(0) + \frac{1}{2}\phi(0)\} - \mathscr{P}(\phi) + O\left(K\frac{\log\log K}{\log K}\right) \tag{10.14}$$

where

$$\mathscr{P}(\phi) = \sum_p \mathscr{B}(p, 1)\widehat{\phi}\left(\frac{\log p}{2\log k}\right)\frac{\log p}{\sqrt{p}\log k}. \tag{10.15}$$

It remains to estimate $\mathscr{P}(\phi)$ (there will be no contribution to the main term). First we record an immediate result for test functions ϕ with $\widehat{\phi}$ supported in $(-2, 2)$. In this case p runs up to $P \ll K^{4-\delta}$ so $\mathscr{B}(p, 1) \ll p^{1/2}K^{-4}$ by Lemma 10.1 giving $\mathscr{P}(\phi) \ll 1$. Therefore we established the following

Theorem **10.2.** — *Let ϕ be a Schwartz function with the Fourier transform $\widehat{\phi}$ supported in $(-2, 2)$. Then*

$$\frac{\mathscr{B}(K)}{B(K)} = \int_{-\infty}^{\infty} \phi(x)W(O)(x)dx + O\left(\frac{\log\log K}{\log K}\right) \tag{10.16}$$

where the implied constant depends only on the weight and the test functions h, ϕ.

Next we appeal to Hypothesis S. By (10.12) and (10.15) we derive

$$\mathscr{P}(\phi) \ll PK^{-4} + K\sum_c c^{-1/2}\sum_{a(c)}^{\star} |S(a, 1; c)| \left|\sum_{p\equiv a(c)} e\left(\frac{2\sqrt{p}}{c}\right) \check{h}\left(\frac{cK^2}{8\pi\sqrt{p}}\right)\widehat{\phi}\left(\frac{\log p}{2\log K}\right)\frac{\log p}{p^{3/4}}\right|.$$

The Fourier integral (8.2) implies that both $\check{h}(v)$ and $\check{h}'(v)$ are rapidly decaying. Therefore applying Hypothesis S with exponents α, A we derive

$$\mathscr{P}(\phi) \ll PK^{-4} + P^{\alpha+A/2+5/8}K^{-2A-9/2+\varepsilon}. \tag{10.17}$$

If $\widehat{\phi}$ is supported in $(-v, v)$, then $P = K^{2v'}$ with $v' < v$ so taking

$$v = \frac{8A + 22}{8\alpha + 4A + 5} \tag{10.18}$$

we find that the bound (10.17) is O(K). This, together with (10.14), yield

Theorem **10.3.** — *Assume Hypothesis* S *with exponents* $A \geqslant 0$ *and* $\frac{1}{2} \leqslant \alpha \leqslant \frac{3}{4}$. *Then the formula (10.16) is valid for* ϕ *with the support of* $\widehat{\phi}$ *in* $(-v, v)$ *where* v *is given by (10.18).*

Notice that $v = v(\alpha, A) = 2 + 4(3-4\alpha)(8\alpha + 4A + 5)^{-1} > 2$, if $\alpha < \frac{3}{4}$. For $\alpha = \frac{1}{2}$ and $A = 0$ we get $v = \frac{22}{9}$.

As an example we take the test function

$$\phi(x) = \left(\frac{\sin \pi\eta x}{\pi\eta x}\right)^2 \tag{10.19}$$

with $0 < \eta < v$, its Fourier transform being

$$\widehat{\phi}(y) = \frac{1}{\eta}\left(1 - \frac{|y|}{\eta}\right) \tag{10.20}$$

if $|y| \leqslant \eta$ and $\widehat{\phi}(y) = 0$ if $|y| > \eta$. For this choice Theorem 10.3 reads

$$\sum_{k \text{ even}} \frac{4\pi^2}{k-1} h\left(\frac{k-1}{K}\right) \sum_{f \in H_k} \sum_{\gamma_f} \left(\frac{\sin(\eta\gamma_f \log K)}{\eta\gamma_f \log K}\right)^2 L^{-1}(1, \operatorname{sym}^2(f)) = \left(\frac{1}{2} + \frac{1}{\eta}\right) \widehat{h}(0) K + O\left(K \frac{\log\log K}{\log K}\right). \tag{10.21}$$

Now suppose all zeros of all $L(s, f)$ are either real or lie on the critical line (that is the Riemann hypothesis with possible exceptions for real zeros). In other words γ_f is either imaginary or real. This assumption ensures that all terms of the sum (10.21) are real and non-negative. Dropping all but one term we derive that

$$\left(\frac{\sin(\eta\gamma_f \log k)}{\eta\gamma_f \log k}\right)^2 \ll L(1, \operatorname{sym}^2(f)) k^2$$

(here we could put $k = K$ as soon as k is selected). This inequality is interesting for $\rho_f = \frac{1}{2} + i\gamma_f > \frac{1}{2}$, since it implies

$$k^{\eta(2\rho_f - 1)} \ll L(1, \operatorname{sym}^2(f))(k \log k)^2 \ll k^{2+\varepsilon}.$$

Hence $\rho_f \leqslant \frac{1}{2} + \frac{1}{\eta} + \varepsilon$ for any $\varepsilon > 0$, provided k is sufficiently large in terms of ε. We can choose η arbitrarily close to v showing that $\rho_f \leqslant \beta + \varepsilon$, where

$$\beta = 1 - \frac{3-4\alpha}{11+4A}. \tag{10.22}$$

For $\alpha = \frac{1}{2}$ and $A = 0$ we get $\beta = \frac{10}{11}$. This establishes

Corollary **10.4.** — *Assume Hypothesis* S *with exponents* $A \geqslant 0$ *and* $\frac{1}{2} \leqslant \alpha \leqslant \frac{3}{4}$. *Assume all the zeros of* $L(s,f)$ *for any Hecke cuspform* f *with respect to the modular group are either real or lie on the critical line. Then* $L(s,f) \neq 0$ *if* $s > \beta + \varepsilon$ *for any* $\varepsilon > 0$ *and all* $k \geqslant k_0$, *where the constant* $k_0 = k_0(\varepsilon, \alpha, A)$ *is effectively computable.*

Appendix A: A related extremal problem

In previous Sections we faced particular instances of the following extremization problem. We are given a weight $W(x)$ on $\mathbf{R}$ whose Fourier transform $\widehat{W}(\xi)$ is known only partially, say in the interval $[-2, 2]$. The problem is to determine

$$\text{(A.1)} \qquad \inf_{\phi} \frac{\int_{-\infty}^{\infty} \phi(x) W(x) dx}{\phi(0)},$$

the infimum being taken over all $\phi \geqslant 0$ for which support $(\widehat{\phi}) \subset [-2, 2]$. We assume further that $\phi \in L^1(\mathbf{R})$. Examples of $W(x)$ are the densities in (1.10) to (1.13).

In Section 1 (1.42) we used the test function

$$\text{(A.2)} \qquad \phi(x) = \left(\frac{\sin 2\pi x}{2\pi x}\right)^2$$

in (A.1). We show below that it yields almost but not optimal results.

As it stands (A.1) looks like a linear program problem. However as is pointed out by Gallagher [Gal] it follows from a theorem of Ahiezer and the Paley-Wiener theorem that the admissible functions ϕ in (A.1) coincide with (or have the form) $\phi(z) = |h(z)|^2$ where h is an entire function of exponential type 1 and $h \in L^2(\mathbf{R})$. That is

$$\text{(A.3)} \qquad \widehat{\phi}(\xi) = (g \star \breve{g})(\xi)$$

where

$$\text{(A.4)} \qquad \breve{g}(\xi) = \overline{g(-\xi)}, \quad \text{support } (g) \subset [-1, 1], \ g \in L^2[-1, 1].$$

By the Plancherel Theorem the problem (A.1) is equivalent to minimization of

$$\text{(A.5)} \qquad R(g) = \frac{\int_{-2}^{2} \widehat{W}(\xi)(g \star \breve{g})(\xi) d\xi}{\int_{-2}^{2} (g \star \breve{g})(\xi) d\xi}$$

with respect to $g \in L^2[-1, 1]$. For what we have in mind $\widehat{W}(\xi)$ takes the form

$$\text{(A.6)} \qquad \widehat{W} = \delta_0 + m(\xi)$$

for $|\xi| \leqslant 2$, $m(\xi)$ being a real piecewise continuous function on $[-2, 2]$ which moreover is even in ξ. This is the form of the problem which we examine.

Define the self-adjoint operator K: $L^2[-1, 1] \to L^2[-1, 1]$ by

$$\text{(A.7)} \qquad Kg(x) = \int_{-1}^{1} m(x-y)g(y)dy.$$

The functional R takes the form

$$\text{(A.8)} \qquad R(g) = \frac{\langle (I+K)g, g\rangle}{|\langle g, 1\rangle|^2}.$$

So the minimization problem is that of a quadratic form subject to a linear constraint. Since $W \geqslant 0$, $R(g) \geqslant 0$ for any g, and so $I+K \geqslant 0$. Now K is compact and self-adjoint so it has eigenvalues $\lambda_j, j = 1, \ldots,$ with $|\lambda_j| \to 0$. From the above we have $-1 \leqslant \lambda_j$, for $j = 1, 2, \ldots$.

It may happen that -1 is an eigenvalue or equivalently that the finite dimensional kernel $N = \ker(I+K) \neq \{0\}$. In this case if there is $g \in N$ such that $\langle g, 1\rangle \neq 0$, then clearly $R(g) = 0$, and the minimization in question yields the value 0.

If we are not in this singular case (which will happen if $\lambda_1 > -1$ which is what occurs in our applications), then $\ker(I+K)$ is orthogonal to 1. Hence by Fredholm theory, $1 \in \mathrm{Image}(I+K)$. That is there is an $f_0 \in N^{\perp}$ such that

$$\text{(A.9)} \qquad (I+K)f_0 = 1.$$

Moreover since $I+K > 0$ on $N^{\perp}$ we see that

$$\text{(A.10)} \qquad A = \langle (I+K)f_0, f_0\rangle = \langle 1, f_0\rangle$$

is positive.

Proposition **A.1.** — *In the nonsingular case*

$$\text{(A.11)} \qquad \inf_{g \in L^2[-1,1]} R(g) = \frac{1}{A},$$

and it is attained by f_0 which satisfies (A.9). Moreover if $\lambda_1 > -1$, then this minimizer is unique (in fact the solution to (A.9) is unique).

Proof. — Let $g \in L^2[-1, 1]$ with the normalization $\langle 1, g\rangle = A$ (which we can assume when $\langle 1, g\rangle \neq 0$). Then writing $g = f_0 + h$, we have $\langle h, 1\rangle = 0$, i.e.

$$\text{(A.12)} \qquad \langle h, (I+K)f_0\rangle = 0.$$

Hence

$$\begin{aligned} R(g) &= \frac{\langle f_0 + h, (I+K)(f_0+h)\rangle}{A^2} \\ &= \frac{1}{A} + \frac{\langle h, (I+K)(h)\rangle}{A^2} + 2\frac{\langle h, (I+K)(f_0)\rangle}{A^2} \geqslant \frac{1}{A}. \end{aligned}$$

So as long as we are not in the singular situation, the minimizer is given by (A.9), which is a standard Fredholm equation of the second kind. It can be solved in a number of ways.

We can now answer our main question:

Corollary **A.2.** — *The function given in (A.2) is optimal if and only if*

$$\int_{-1}^{1} m(x-y)dy$$

is independent of x.

Proof. — If ϕ is as the above, the corresponding g is constant on $[-1, 1]$. Now $g = f_0$ is constant according to (A.9), iff the constant function is an eigenfunction of K. This is equivalent to the statement of the corollary.

We apply Corollary A.2 to our weights in (1.10) to (1.13). Firstly, since the property

$$\int_{-\infty}^{\infty} \phi(x)W(x)dx = 0$$

with $\phi \geqslant 0$ and $\phi \in L^1(\mathbf{R})$ implies that $\phi \equiv 0$, it follows that $\lambda_1 > -1$ (in the corresponding eigenvalue problem). That is $(I + K)$ is invertible. Hence we don't have to worry about the singular case, and the unique minimizer f_0 satisfies the equation

$$(I+K)f_0 = 1. \tag{\textbf{A.13}}$$

The functions m for our densities are as follows:

$$\begin{aligned} m(\mathrm{SO(even)})(\xi) &= \frac{1}{2}I_{[-1,1]}(\xi) \\ m(\mathrm{SO(odd)})(\xi) &= 1 - \frac{1}{2}I_{[-1,1]}(\xi) \\ m(Sp)(\xi) &= -\frac{1}{2}I_{[-1,1]}(\xi) \\ m(\mathrm{O})(\xi) &= \frac{1}{2}. \end{aligned} \tag{\textbf{A.14}}$$

According to Corollary A.2 in all cases except the last (i.e. for O), the function $m(\xi)$ is not constant on $[-2, 2]$ and it follows that except in the last case the function (A.2) is not the minimizer. This establishes the comments prior to Corollary 1.6.

It is not difficult to determine the extremal functions for the m's in (A.14). J. Vanderkam first pointed out to us these functions which he obtained by a direct analysis of the functional (A.1) (private communication in 1998).

We must solve the equation (A.9), that is

$$f_0(x) + \int_{-1}^{1} m(x-y) f_0(y)dy = 1, \tag{A.15}$$

where m is any one of the functions in (A.14). Since m is even and the solution f_0 is unique, it follows that f_0 is an even function of x. In particular for SO(even) it satisfies

$$\frac{1}{2}\int_0^1 f_0(y)dy + \frac{1}{2}\int_0^{1-x} f_0(y)dy + f_0(x) = 1, \tag{A.16}$$

for $0 \leqslant x \leqslant 1$. For SO(odd) it satisfies

$$\frac{3}{2}\int_0^1 f_0(y)dy - \frac{1}{2}\int_0^{1-x} f_0(y)dy + f_0(x) = 1, \tag{A.17}$$

for $0 \leqslant x \leqslant 1$. Trying trigonometric functions (i.e. Fourier series) one finds that

$$f_0(x) = \frac{\cos(\frac{x}{2} - \frac{\pi+1}{4})}{\sqrt{2}\sin(\frac{1}{4}) + \sin(\frac{\pi+1}{4})}, \ 0 \leqslant x \leqslant 1, \tag{A.18}$$

solves (A.16), while its even extension to $[-1, 1]$ solves (A.15). Applying (A.10) and Proposition A.1 then yields the minimum. A calculation then shows that

$$\inf(\mathrm{SO(even)}) = \frac{3 + \cos(\frac{1}{4})}{8} = 0.8645\ldots \tag{A.19}$$

Similarly one finds that

$$f_0(x) = \frac{\cos(\frac{x}{2} + \frac{\pi-1}{4})}{3\sin(\frac{\pi+1}{4}) - 2\sin(\frac{\pi-1}{4})}, \tag{A.20}$$

for $0 \leqslant x \leqslant 1$, solves (A.17). It leads to the value

$$\inf(\mathrm{SO(odd)}) = \frac{5 + \cos(\frac{1}{4})}{8} = 1.1145\ldots \tag{A.21}$$

In particular these solutions allow us to conclude that

(a) (1.54) and (1.56) of Section 1 hold with $> \frac{9}{16}$ replaced by

$$\geqslant \frac{13 - \cot(\frac{1}{4})}{16} = 0.5678\ldots$$

(b) (1.55) and (1.57) of Section 1 hold with $> \frac{15}{16}$ replaced by

$$\geqslant \frac{19 - \cot(\frac{1}{4})}{16} = 0.94275\ldots$$

(c) (1.53) of Section 1 holds with < 1 replaced by

$$\leqslant \frac{4 + \cot(\frac{1}{4})}{8} = 0.9895\ldots.$$

Appendix B: Estimates for sums of eigenvalues

Let $\mathcal{H}$ be a subset of $H_k^\star(N)$ and ν be a positive integer. For any $f, g \in \mathcal{H}$ we put

(B.1) $$Z^{(\nu)}(s, f\times g) = \sum_1^\infty \lambda_f(n^\nu)\lambda_g(n^\nu)n^{-s}.$$

Suppose $Z^{(\nu)}(s, f\times g)$ has analytic continuation to $\operatorname{Re} s \geqslant 1-\alpha$ and it satisfies

(B.2) $$|Z^{(\nu)}(s, f\times g)| \leqslant Z|s|^d$$

on the line $\operatorname{Re} s = 1-\alpha$ for almost all $f, g \in \mathcal{H}$, the number of exceptional pairs being $O(H)$, where $H = |\mathcal{H}| \geqslant 1$ and $Z \geqslant 1$. Moreover $\alpha = \alpha(\nu) > 0$, $d = d(\nu) > 0$ and the implied constant in $O(H)$ depending only on ν.

Proposition **B.1.** — *Let the conditions be as above. Then for any complex numbers a_p with $|a_p| \leqslant 1$ we have*

$$\sum_{f\in\mathcal{H}} \Big|\sum_{p\leqslant X} \lambda_f(p^\nu)\, a_p\Big|^2 \ll (ZX^{-\alpha r} + H^{-1} + X^{-2})^{1/r}\, HX^2(\log X)^{\nu(\nu+2)}$$

for any integer $r \geqslant 2$, the implied constant depending only on ν.

Proof. — By Hölder's inequality our sum, say S, satisfies

$$S^{r/2} \leqslant H^{r/2-1} \sum_{f\in\mathcal{H}} \Big|\sum_{p\leqslant X} \lambda_f(p^\nu)a_p\Big|^r.$$

By the multiplicativity of $\lambda_f(n^\nu)$ we have

$$\left(\sum_{p\leqslant X}\lambda_f(p^\nu)a_p\right)^r = \sum_{n\leqslant N}\lambda_f(n^\nu)a_n + O(r^2(\nu+1)^r NX^{-1})$$

where

$$a_n = \mu^2(n)\sum_{\substack{p_1\cdots p_r=n\\ p_1,\ldots,p_r\leqslant X}} a_{p_1}\ldots a_{p_r}$$

for $n \leqslant N = X^r$, and the error term is the contribution of relevant numbers which are not squarefree. Note that $|a_n| \leqslant r!$. Hence

$$S^{r/2} \leqslant r!H^{r/2-1}\sum_{n\leqslant N}\left|\sum_{f\in\mathcal{H}} c_f\lambda_f(n^\nu)\right| + O(r^2(\nu+1)^r H^{r/2}NX^{-1})$$

with $|c_f| = 1$. Next by Cauchy's inequality

$$S^r \ll r^{2r}H^{r-2}N\sum_{f,g\in\mathcal{H}}\sum V(f\times g) + r^4(\nu+1)^{2r}H^rN^2X^{-2},$$

where

$$V(f\times g) = \sum_n \omega(n)\lambda_f(n^\nu)\lambda_g(n^\nu)$$

and $\omega(n)$ is a smooth majorant of the characteristic function of the interval $1 \leqslant n \leqslant N$, for example $\omega(n) = e^{1-n/N}$. We have

$$V(f\times g) \ll N(\log N)^{\nu(\nu+2)}$$

by the estimation $|\lambda_f(n^\nu)\lambda_g(n^\nu)| \leqslant \tau^2(n^\nu)$ for any $f, g \in \mathcal{H}$. However, if (f, g) is not an exceptional pair, then it follows by contour integration and the crude bound (B.2) that

$$V(f\times g) \ll ZN^{1-\alpha}.$$

These estimates yield

$$S^r \ll r^{2r}H^{r-2}N\{H^2ZN^{1-\alpha} + HN(\log N)^{\nu(\nu+2)}\} + r^4(v+1)^{2r}H^rN^2X^{-2}.$$

This completes the proof of Proposition B.1.

Corollary **B.2.** — *Let* H_k *be the set of Hecke cusp forms with respect to the modular group. For any complex numbers* a_p *with* $|a_p| \leqslant 1$ *we have*

$$\textbf{(B.3)}\qquad \sum_{f\in H_k}\left|\sum_{p\leqslant X}\lambda_f(p^2)a_p\right|^2 \ll kX^2(\log X)^{-\log\log\log X}$$

provided X *satisfies*

(B.4) $$(\log k)^{\sqrt{\log k}} \leqslant X \leqslant \exp(k^{1/\log\log k})\,.$$

Proof. — From Proposition B.1 we obtain the bound

$$(k^{A-1}X^{-\alpha r} + k^{-1} + X^{-2})^{1/r} k X^2 (\log X)^8$$

where α, A are absolute positive constants. This yields (B.3) in the range (B.4) on taking $r = 2 + [A \log k/\alpha \log X]$.

Proof of (10.13). — The partial sum of (10.13) with $12\pi p \leqslant k$ satisfies

$$\frac{k-1}{4\pi^2} \sum_{12\pi p \leqslant k} \Delta_k(p^2, 1)\widehat{\phi}\left(\frac{\log p}{\log k}\right)\frac{\log p}{p} \ll 2^{-k}k^2$$

because $\Delta_k(p^2, 1) \ll 2^{-k}p$ by (2.12′). The remaining partial sum satisfies

$$\sum_{f \in H_k} L^{-1}(1, \mathrm{sym}^2(f)) \sum_{12\pi p > k} \lambda_f(p^2)\widehat{\phi}\left(\frac{\log p}{\log k}\right)\frac{\log p}{p} \ll k$$

by applying (B.3) and the bound $L^{-1}(1, \mathrm{sym}^2(f)) \ll \log k$ of [HL]. Adding these estimates we get (10.13).

Appendix C: Comments on Hypothesis S

Due to the striking consequences of Hypothesis S we shed some light on its genesis. We present the heuristic arguments which led us to (1.62). Also by considering such sums in a broader context we show that such hypotheses can be quite delicate. Indeed a closely related sum is shown to be of size $x^{3/4}$ (recall that improving on $\frac{3}{4}$ is crucial for the applications). For simplicity we make no restriction of the relevant exponential sums to residue classes a (mod c).

We begin by considering general sums of the type

(C.1) $$S_q(X) = \sum_n a_n\, e(-2\sqrt{nq})\phi\left(\frac{n}{X}\right)$$

where $\phi(x)$ is a fixed smooth function compactly supported on $\mathbf{R}^+$, q, X are positive parameters and (a_n) is an arithmetically defined sequence of complex numbers satisfying

(C.2) $$a_n \ll n^{\varepsilon}, \quad \varepsilon > 0.$$

Tacitly we assume that X and qX are large, because otherwise the exponential factor $e(-2\sqrt{nq})$ plays no significant role. Assume that the associated zeta function

$$\text{(C.3)} \qquad A(s) = \sum_{n=1}^{\infty} a_n n^{-s}$$

has a holomorphic continuation to $\operatorname{Re}(s) > \frac{1}{2}$ except for a possible pole of finite order at $s = 1$. Assume further that

$$\text{(C.4)} \qquad A(s) \ll |s|^{\varepsilon}, \quad \text{if } \operatorname{Re}(s) = \sigma$$

for any $\frac{1}{2} < \sigma < 1$ and any $\varepsilon > 0$, the implied constant depending on σ and ε. We have

$$S_q(X) = \frac{1}{2\pi i} \int_{(\sigma)} A(s)B(s)ds + \operatorname*{Res}_{s=1} A(s)B(s)$$

where B(s) is given by the Mellin integral

$$B(s) = \int_0^{\infty} \phi\left(\frac{x}{X}\right) e(-2\sqrt{xq})x^{s-1}dx.$$

This can be evaluated asymptotically by the method of stationary phase. If $s = \sigma + it$, then one has a significant contribution to the integral only for $t \asymp \sqrt{qX}$ in which case

$$\text{(C.5)} \qquad B(s) = (1-i)\sqrt{\frac{2\pi}{t}}\phi\left(\frac{t^2}{4\pi^2 qX}\right)\left(\frac{t}{2\pi e\sqrt{qX}}\right)^{2s} + \dots .$$

Here and later the dots indicate that the remaining terms are very small or are insignificant for our analysis. Now

$$\int_{(\sigma)} |B(s)ds| \ll (qX)^{1/4}X^{\sigma}$$

whence taking σ close to $\frac{1}{2}$ yields

$$\text{(C.6)} \qquad S_q(X) \ll q^{1/4}X^{3/4+\varepsilon} .$$

(The pole at $s = 1$ does not contribute significantly to S_q.)

The basic question is: can one improve on the exponent $\frac{3}{4}$? One can try to shift the contour to the left of $\sigma = \frac{1}{2}$. This requires a modified form of (C.4), and can be done if A(s) is an L-function with one gamma factor (i.e. Dirichlet L-function) but it just fails if A(s) is an L-function with two gamma factors (i.e. a $GL_2/\mathbb{Q}$ L-function). The simplest example of the first case is

$$\text{(I)} \qquad a_n = 1, \quad A(s) = \zeta(s) .$$

For examples of the latter case we take

(II) $a_n = \tau(n),\ A(s) = \zeta^2(s)$

and

(III) $a_n = \lambda(n),\ A(s) = L(s, f)$.

Here $\tau(n)$ is the divisor function and $\lambda(n)$ are the normalized Fourier coefficients of a fixed cusp form f of weight $k \geqslant 12$ for the modular group $SL(2, \mathbf{Z})$

$$f(z) = \sum_{n=1}^{\infty} \lambda(n) n^{k-1/2} e(nz) .$$

We will treat these examples additively by Fourier rather than Mellin transforms.

The Mellin transform is nevertheless quite revealing for the above sequences when restricted to primes. That is for

(PI) $a_n = \Lambda(n),\ A(s) = \frac{-\zeta'}{\zeta}(s)$

(PII) $a_n = \tau(n)\Lambda(n),\ A(s) = \frac{-2\zeta'}{\zeta}(s) + \ldots$

(PIII) $a_n = \lambda(n)\Lambda(n),\ A(s) = \frac{-L'}{L}(s, f) + \ldots$.

In these cases assuming the Riemann Hypothesis for the corresponding zeta functions one shows (C.4) for $0 < \sigma < 1, \sigma \neq \frac{1}{2}$. Passing the integration through the poles on $\mathrm{Re}(s) = \frac{1}{2}$ we obtain

$$S_q(X) = \frac{1-i}{e} \sum_{\gamma} \left(\frac{2\pi q}{\gamma}\right)^{1/2} \phi\left(\frac{\gamma^2}{4\pi^2 qX}\right) \left(\frac{\gamma}{2\pi e\sqrt{q}}\right)^{2i\gamma} + \ldots . \tag{C.7}$$

Here the effective range of summation is those zeros of height $\gamma \asymp (qX)^{1/2}$.

Estimating (C.7) trivially recovers (C.6). Therefore an improvement of the exponent $\frac{3}{4}$ (for q fixed) amounts to getting cancellations in the sums

$$\sum_{\gamma \asymp \sqrt{qX}} \left(\frac{\gamma}{2\pi e\sqrt{q}}\right)^{2i\gamma} . \tag{C.8}$$

For later we note that this is a Weyl sum pertaining to the equidistribution of ($q = 1$)

$$2\frac{\gamma \log(\gamma/2\pi e)}{2\pi} \pmod 1 \tag{C.8'}$$

(the rest of the Weyl sums related to this sequence can also be given in terms of related sums over primes). Based on the analysis below we will see that there is cancellation in (C.8) (i.e. power savings) for (PI) and (PII), but not for (PIII).

Returning to (I), (II) and (III) we establish approximate formulae (a'la Poisson) for the sums $S_q(X)$.

Case I. — Applying Poisson summation and evaluating the resulting Fourier integrals by stationary phase, we obtain

$$\sum_n e(-2\sqrt{nq})\phi\left(\frac{n}{X}\right) = (1+i)q^{1/2}\sum_h e\left(\frac{-q}{h}\right)h^{-3/2}\phi\left(\frac{q}{h^2X}\right) + \ldots \tag{C.9}$$

This is a special case of Van der Corput's method (see [Tit]). For $q \leqslant X^{1-\varepsilon}$ the right hand side of (C.9) is negligible, otherwise estimating it trivially yields

$$S_q(X) \ll (qX)^{1/4}. \tag{C.10}$$

Cases II and III are similar, we deal with III only.

Case III. — We appeal to the Voronoi type summation formula

$$\sum_{m=1}^{\infty}\lambda(m)F(m) = \sum_{n=1}^{\infty}\lambda(n)G(n) \tag{C.11}$$

where $G(y)$ is a Hankel type transform of $F(x)$ given by

$$G(y) = 2\pi i^k\int_0^\infty F(x)J_{k-1}(4\pi\sqrt{xy})\,dx. \tag{C.12}$$

In particular if $F(x) = e(-2\sqrt{xq})\phi\left(\frac{x}{X}\right)$ and we apply the asymptotic expansion of the Bessel function $J_{k-1}(z)$ as $z \to \infty$, we get

$$G(y) = \frac{X^{3/4}}{y^{1/4}}\widehat{B}\left(-2\sqrt{X}(\sqrt{y}-\sqrt{q})\right) + \ldots \tag{C.13}$$

where

$$B(x) = i^k(1-i)x^{1/2}\phi(x^2) \tag{C.14}$$

and $\widehat{B} \in \mathscr{S}(\mathbf{R})$ is its Fourier transform.

Clearly $G(y)$ is small unless y is in the short interval

$$y - q \ll \sqrt{q/X}. \tag{C.15}$$

Combining the above gives the approximate formula

$$\sum_m \lambda(m)e(-2\sqrt{mq})\phi\left(\frac{m}{X}\right) = X^{3/4}\sum_n \lambda(n)\,\widehat{B}\left(-2\sqrt{X}\left(\sqrt{n}-\sqrt{q}\right)\right)n^{-1/4} + \ldots \tag{C.16}$$

where B is given in (C.14).

For q a positive integer, $n = q$ occurs in (C.16) and applying the trivial bound $G(n) \ll n^{-1/4}X^{3/4}$ for the remaining terms we arrive at

$$\sum_m \lambda(m) e\left(-2\sqrt{mq}\right) \phi\left(\frac{m}{X}\right) = \frac{\lambda(q)}{q^{1/4}} X^{3/4} \widehat{B}(O) + O\left((qX)^{1/4+\varepsilon}\right) . \tag{C.17}$$

Here

$$\widehat{B}(0) = i^k \frac{1-i}{2} \int_0^\infty \phi(x) x^{-1/4} dx . \tag{C.18}$$

Hence it is clear that we cannot improve the exponent $\frac{3}{4}$ in (C.6) for any fixed q with $\lambda(q) \neq 0$ and ϕ with $\widehat{B}(0) \neq 0$. In the sequel we normalize the test function ϕ so that

$$\widehat{B}(0) = 1 . \tag{C.19}$$

We are ready to consider the sums (PI), (PII) and (PIII). As we will see the automatic assumption that oscillatory sums over primes cancel to order square root of the number of terms can be false. We find it much safer to rely on the randomness in the Möbius function $\mu(m)$. We therefore derive estimates for exponential sums over primes from the more reliable estimates for similar sums against $\mu(m)$. We need the following:

Hypothesis A. — For any positive integer $\ell \leqslant X$ we have

$$\sum_{m \leqslant X} \mu(m) e(-2\sqrt{\ell m}) \ll_\varepsilon X^{1/2+\varepsilon} \tag{C.20}$$

with any $\varepsilon > 0$.

Hypothesis B. — Fix a cusp form f as above, then for any positive integer $\ell \leqslant X$

$$\sum_{m \leqslant X} \mu(m) \lambda(\ell m) e(-2\sqrt{\ell m}) \ll_\varepsilon X^{1/2+\varepsilon} \tag{C.21}$$

with any $\varepsilon > 0$.

We write the von Mangolt function as follows

$$\Lambda(n) = -\sum_{m|n} \mu(m) \log m = \Lambda^\flat(n) + \Lambda^\#(n) . \tag{C.22}$$

where $\Lambda^\flat(n)$ denotes the partial convolution restricted by $m \leqslant M$ and $\Lambda^\#(n)$ is the complementary part. Here M is a parameter which we choose later.

Consider first the sums (PI). According to (C.22) the sum

(C.23) $$S(X) = \sum_n \Lambda(n) e(-2\sqrt{n}) \phi\left(\frac{n}{X}\right)$$

splits into $S(X) = S^\flat(X) + S^\#(X)$. Now

$$\begin{aligned} S^\flat(X) &= \sum_n \Lambda^\flat(n) e(-2\sqrt{n}) \phi\left(\frac{n}{X}\right) \\ &= -\sum_{m \leqslant M} \mu(m)(\log m) \sum_\ell e(-2\sqrt{\ell n}) \phi\left(\frac{\ell m}{X}\right) \\ &= -(1+i) \sum_h h^{-3/2} \sum_{m \leqslant M} \mu(m)(\log m) m^{1/2} e\left(\frac{-m}{h}\right) \phi\left(\frac{m^2}{h^2 X}\right) + \dots \end{aligned}$$

by (C.9). In this sum, $h \ll MX^{-1/2}$ and $m \asymp hX^{1/2}$. By GRH for $L(s, \chi)$ with $\chi \pmod h$ the inner sum is $O(hX^{1/4+\varepsilon})$. Hence

(C.24) $$S^\flat(X) \ll MX^\varepsilon .$$

In order to estimate $S^\#(X)$ we reverse the order of summation and apply Hypothesis A getting:

$$\begin{aligned} S^\#(X) &= \sum_\ell \sum_{m > M} \mu(m)(\log m) e(-2\sqrt{\ell m}) \phi\left(\frac{\ell m}{X}\right) \\ &\ll \sum_{\ell \ll X/M} \ell^{-1/2} X^{1/2+\varepsilon} \ll M^{-1/2} X^{1+\varepsilon} . \end{aligned}$$

Adding this to (C.24) and taking $M = X^{2/3}$ we obtain the estimate for (PI) (on Hypothesis A),

(C.25) $$\sum_n \Lambda(n) e(-2\sqrt{n}) \phi\left(\frac{n}{X}\right) \ll X^{2/3+\varepsilon} .$$

Thus though Hypothesis A does not yield (at least along these lines) our expected exponent $\frac{1}{2}$ in (C.25), it does go below the critical exponent of $\frac{3}{4}$ (which is what is needed in the applications).

We skip (PII) and go to (PIII) which yields a somewhat surprising result. Again we split the sum

(C.26) $$S(X) = \sum_n \lambda(n) \Lambda(n) e(-2\sqrt{n}) \phi\left(\frac{n}{X}\right)$$

into $S^\flat(X) + S^\#(X)$. We arrange $S^\flat(X)$ as

$$S^\flat(X) = \sum_{m\leqslant M} \mu(m)(\log m) \sum_{\ell} \lambda(\ell m) e(-2\sqrt{m})\phi\left(\frac{\ell m}{X}\right).$$

Using the multiplicativity of the Hecke eigenvalues,

$$\lambda(\ell m) = \sum_{d|(\ell, m)} \mu(d)\lambda\left(\frac{\ell}{d}\right)\lambda\left(\frac{m}{d}\right)$$

we can write the inner sum as

$$\sum_{\ell} = \sum_{d|m} \mu(d)\lambda\left(\frac{m}{d}\right) \sum_{a} \lambda(a) e(-2\sqrt{adm})\phi\left(\frac{adm}{X}\right).$$

Now for the inner sum if we apply (C.17) we get

$$\textbf{(C.27)} \qquad \sum_{a} = \frac{\lambda(dm)}{dm} X^{3/4} + O(X^{1/4+\varepsilon})$$

and this gives

$$\sum_{\ell} = X^{3/4} \sum_{d|m} \frac{\mu(d)}{dm} \lambda\left(\frac{m}{d}\right)\lambda(dm) + O(X^{1/3+\varepsilon}).$$

Hence

$$\textbf{(C.28)} \qquad S^\flat(X) = X^{3/4} Z(M) + O(MX^{1/4+\varepsilon})$$

where

$$\textbf{(C.29)} \qquad Z(M) = \sum_{m\leqslant M} \frac{\mu(m)\log m}{m} \left(\sum_{d|m} \lambda\left(\frac{m}{d}\right)\lambda(dm)\right).$$

If instead of applying (C.17) for the a sum in (C.27) one applies (C.16) with $H(n)$ given in (C.14), one can obtain a better estimate for the remainder if we assume the following hypothesis.

Hypothesis C. — For $0 < h_1 \leqslant X$, and $0 < |h_1| \leqslant X$

$$\sum_{m\leqslant X} \mu(m)\lambda(m)\lambda(h_1 m + h_2) \ll X^{1/2+\varepsilon}.$$

Remarks. — Note that this is a more standard (as compared with A and B) arithmetical sum. Also in order to estimate the remainder in $S^\flat(X)$ successfully, we only need Hypothesis C with the exponent $\frac{1}{2}$ replaced by $1-\delta$, for $\delta > 0$. This would lead to (C.23) with error term $O(x^{3/4-\delta/6})$.

The corresponding improvement in (C.28) is

$$\text{(C.28')} \qquad S^\flat(X) = X^{3/4} Z(M) + O(M^{1/2} X^{1/4+\varepsilon}) .$$

Again to estimate $S^\#(X)$ we reverse the order of summation and apply Hypothesis B getting

$$\begin{aligned} S^\#(X) &= -\sum_{\ell}\sum_{m>M} \mu(m)(\log m)\lambda(\ell m) e(-2\sqrt{\ell m})\phi\left(\frac{\ell m}{X}\right) \\ &\ll \sum_{\ell \ll X/M} \ell^{-1/2} X^{1/2+\varepsilon} \ll M^{-1/2} X^{1+\varepsilon} . \end{aligned}$$

Adding this bound to (C.28′) and taking $M = X^{3/4}$ we conclude that (under Hypothesis B and Hypothesis C)

$$\text{(C.30)} \qquad S(X) = X^{3/4} Z(M) + O(X^{5/8+\varepsilon}) .$$

It remains to evaluate $Z(M)$. To this end consider the series

$$\text{(C.31)} \qquad L(s) = \sum_{m=1}^{\infty} \frac{\mu(m)}{m^s} \left(\sum_{d|m} \frac{\mu(d)}{d} \lambda\left(\frac{m}{d}\right) \lambda(dm) \right) .$$

We have

$$\begin{aligned} L(s) &= \sum_{d} \frac{\mu^2(d)}{d^{s+1}} \lambda(d^2) \sum_{(m,d)=1} \frac{\mu(m)}{m^s} \lambda^2(m) \\ &= \prod_p \left(1 - \frac{\lambda^2(p)}{p^s} + \frac{\lambda(p^2)}{p^{s+1}}\right) = \frac{Q(s,f)}{\zeta(s) L(s, \operatorname{sym}^2 f)} \end{aligned}$$

where $Q(s,f)$ is given by an absolutely convergent Euler product in $\operatorname{Re}(s) = \sigma > \frac{1}{2}$. Hence shifting the contour and using the Riemann Hypothesis for $L(s, \operatorname{sym}^2 f)$ we get that

$$Z(M) = \frac{1}{2\pi i} \int_{(1)} L'(s+1) \frac{M^s}{s} ds = Z + O(M^{\varepsilon - 1/2})$$

where

$$\text{(C.32)} \qquad Z = \frac{Q(1,f)}{L(1, \operatorname{sym}^2 f)} = \prod_p \left(1 - \frac{\lambda(p^2)}{p}\right) \neq 0 .$$

Inserting this in (C.30) yields our main analysis of (PIII):

$$\textbf{(C.33)} \qquad \sum_n \lambda(n)\Lambda(n)e(-2\sqrt{n})\phi\left(\frac{n}{X}\right) = ZX^{3/4} + O(X^{5/8+\varepsilon})\,.$$

Thus according to (C.25) and (C.33) there is cancellation beyond the exponent $\frac{3}{4}$ for (PI), but not for (PIII). This is quite surprising at first glance and indicates how delicate such sums over primes can be. Note that we can also view the sum (C.25) as (C.33) with $\lambda(n)$ replaced by $\tau(n)$ because $\tau(p) = 2$. Moreover $\tau(n)$ are Fourier coefficients of a modular form; precisely of $\frac{\partial}{\partial_s}E(z, s)$ at $s = \frac{1}{2}$, where $E(z, s)$ is the Eisenstein series for $SL_2(\mathbf{Z})$. Since $\tau(n)$ satisfies a summation formula similar to (C.11) and obeys the same multiplicativity law as $\lambda(n)$, one also has (C.30) with $\lambda(n)$ replaced by $\tau(n)$. The corresponding series

$$L(s) = \prod_p \left(1 - \frac{4}{p^s} + \frac{3}{p^{s+1}}\right)$$

has a zero of order four at $s = 1$ (rather than order 1 as in the cuspidal case) and hence $Z = 0$. So these different analyses of the equivalent sums PI and PII yield consistent results.

Further confirmation of the above analysis has been provided by numerical experiments by Rubinstein [R]. Rather than examining PI and PIII he experiments with the equivalent dual sums (C.8) and (C.8′) for $\rho = \frac{1}{2} + i\gamma$ the zeros of $\zeta(s)$ and $L(s, \Delta)$, Δ the weight 12 cusp form for $SL_2(\mathbf{Z})$. He finds that for $\zeta(s)$, the numbers $2\frac{\gamma}{2\pi}\log(\frac{\gamma}{2\pi e})$ are very well distributed mod1, while for $L(s, \Delta)$ the same sequence is badly distributed mod 1.

Indeed it appears, and an analysis with the Möbius function as in this Appendix confirms, that if $\rho = \frac{1}{2} + i\gamma$ are the zeros of $L(s,f)$, f a cusp form on $GL_n/\mathbf{Q}$, then

$$A \cdot \frac{n\gamma}{2\pi}\log\left(\frac{\gamma}{2\pi e}\right)$$

is well distributed mod 1 iff A is not the reciprocal of a positive integer.

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REFERENCES

[AL] A. Atkin and J. Lehner, Hecke operators on $\Gamma_0(m)$, *Math. Ann.* **185** (1970), 134-160.

[AU] A. Abbes and E. Ullmo, Comparison des métriques d'Arakelov et de Poincaré sur $X_0(N)$, *Duke Math. J.* **80** **(2)** (1995), 295-307.

[CI] B. Conrey and H. Iwaniec, The cubic moment of central values of automorphic L-functions, *Ann. of Math.*, **151** (2000), 1175-1216.

[Del] P. Deligne, La conjecture de Weil. I, II, *Publ. Math. IHES*, **48** (1974), 273-308; **52** (1981), 313-428.

[DFI] W. Duke, J. Friedlander and H. Iwaniec, Class group L-functions, *Duke Math. J.* **79** (1995), 1-56.

[Fuj] A. Fujii, Some observations concerning the distribution of zeros of zeta functions II, *Comment. Math. Univ. St. Pauli* **40** (1991), 125-231.

[Gal] P. X. Gallagher, Pair correlation of zeros of the zeta function, *J. Reine Angew. Math.* **362** (1985), 72-86.

[GJ] S. Gelbart and H. Jacquet, A relation between automorphic representations of GL(2) and GL(3), *Ann. Sci. Ecole Norm. Sup.* **11** (1978), 411-542.

[GR] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, New York, Academic Press, 1965.

[Hej] D. Hejhal, *On the triple correlation of the zeros of the zeta-function*, IMRN (1994), 293-302.

[HL] J. Hoffstein and P. Lockhart, Coefficients of Maass forms and the Siegel zero; Appendix by D. Goldfeld, J. Hoffstein and D. Lieman, An effective zero-free region, *Ann. of Math.* **140** (1994), 161-176-181.

[Iwa] H. Iwaniec, Topics in classical automorphic forms, *Grad. Studies in Math.*, **(17)** AMS (1997).

[IS] H. Iwaniec and P. Sarnak, The non-vanishing of central values of automorphic L-functions and Landau-Siegel zeros, *Israel Math. J.*, to appear.

[JP-SS] H. Jacquet, I. Piatetski-Shapiro, J. Shalika, Rankin-Selberg convolutions, *Amer. J. Math.* **105** (1983), 367-464.

[KS1] N. Katz and P. Sarnak, Random Matrices, Frobenius Eigenvalues and Monodromy, *AMS Colloq. Publ.* **45** (1999).

[KS2] N. Katz and P. Sarnak, Zeros of zeta functions and symmetry, *Bull. AMS* **36** (1999), 1-26.

[KM1] E. Kowalski and P. Michel, The analytic rank of $J_0(q)$ and zeros of automorphic L-functions, *Duke Math. J.* **100** (1999), 503-547.

[KM2] E. Kowalski and P. Michel, A lower bound for the rank of $J_0(q)$, *Acta Arith.*, **94** (2000), 303-343.

[KMV1] E. Kowalski, P. Michel and J. Vanderkam, Non-vanishing of high derivatives of automorphic L-functions at the center of the critical strip, *J. Reine Angew. Math.*, **526** (2000), 1-34.

[KMV2] E. Kowalski, P. Michel and J. Vanderkam, Mollification of the fourth moment of automorphic L-functions and arithmetic applications, *Invent. Math.* **142** (2000), 95-151.

[KMV3] E. Kowalski, P. Michel and J. Vanderkam, *Rankin-Selberg* L-*functions in the level aspect*, Preprint (2000).

[LW] S. Lang and A. Weil, Number of points of varieties in finite fields, *Amer. J. Math.* **76** (1954), 819-827.

[Li] W. Li, Newforms and functional equations, *Math. Ann.* **212** (1975), 285-315.

[Miy] T. Miyake, *Modular Forms*, Springer-Verlag, 1989.

[Mon] H. L. Montgomery, The pair correlation of zeros of the zeta-function, *Proc. Symp. Pure Math. AMS* **24** (1973), 181-193.

[P1] H. Petersson, Zur analytischen Theorie der Grenzkreisgruppen. I-V, *Math. Ann.* **115** (1938), 23-61; 175-204; 518-512; 670-709; *Math. Z.* **44** (1938), 127-155.

[P2] H. Petersson, Über die Entwicklungskoefficienten der automorphen formen, *Acta Math.* **58** (1932), 169-215.

[R] M. Rubinstein, private communication, 1998.

[RS] Z. Rudnick and P. Sarnak, Principal L-functions and random matrix theory, *Duke Math. J.* **81** **(2)** (1996), 269-322.

[S1] G. Shimura, On the holomorphy of certain Dirichlet series, *Proc. London Math. Soc.* **31** **(3)** (1975), 79-98.

[S2] G. Shimura, *Introduction to the Arithmetic Theory of Automorphic Functions*, Princeton, NJ, Princeton University Press, 1971.

[Tit] E. C. Titchmarsh, *The Theory of the Riemann Zeta-function*, Oxford (second edition), Clarendon Press, 1986.

[Van] J. Vanderkam, The rank of quotients of $J_0(N)$, *Duke Math. J.* **97** **(3)** (1999), 545-577.

[Vin] I. M. Vinogradov, Estimate of a prime-number trigonometric sum (in Russian), *Izv. Akad. Nauk SSSR Ser. Mat.* **23** (1959), 157-164; *Special Variants of the Method of Trigonometric Sums*, Selected Works, Springer-Verlag, 1984.

[Wat] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, 1944.

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