The Expected Eigenvalue Distribution of a Large Regular Graph

Brendan D. McKay*
Department of Mathematics
University of Melbourne
Parkville, Victoria 3052, Australia

Submitted by Richard A. Brualdi

ABSTRACT

Let $X_1, X_2, \dots$ be a sequence of regular graphs with degree $v \geqslant 2$ such that $n(X_i) \to \infty$ and $c_k(X_i)/n(X_i) \to 0$ as $i \to \infty$ for each $k \geqslant 3$, where $n(X_i)$ is the order of X_i, and $c_k(X_i)$ is the number of k-cycles in X_i. We determine the limiting probability density $f(x)$ for the eigenvalues of X_i as $i \to \infty$. It turns out that

$$f(x)=\begin{cases} \dfrac{v\sqrt{4(v-1)-x^2}}{2\pi(v^2-x^2)} & \text{for} \quad |x| \leqslant 2\sqrt{v-1}, \\ 0 & \text{otherwise} \end{cases}.$$

It is further shown that $f(x)$ is the expected eigenvalue distribution for every large randomly chosen labeled regular graph with degree v.

1. INTRODUCTION

Let X be a regular graph with vertex set $\{1,2,\dots,n(X)\}$. As with all our graphs, X has no loops, multiple edges, or directed edges. The degree of X will be denoted by v, and the number of cycles of length k by $c_k(X)$, for each $k \geqslant 3$.

The adjacency matrix of X is the matrix $A(X)=(a_{ij})$ of order $n(X)$, where $a_{ij}=1$ if vertices i and j are adjacent and $a_{ij}=0$ otherwise. The eigenvalues of X are the eigenvalues of $A(X)$. Define the cumulative distribution function $F(X,x)$ to be the proportion of the eigenvalues of X which are less than or

*Present address: Computer Science Department, Vanderbilt University, Nashville, Tennessee 37235.

52 Vanderbilt Ave., New York, NY 10017 0024-3795/81/060203+14$02.50

equal to the real number x. Eigenvalues are counted according to their multiplicities. It is easy to see that the function $F(X,x)$ satisfies the following conditions:

(1) $F(X,x)=0$ if $x<-v$.
(2) $F(X,x)=1$ if $x\geqslant v$.
(3) $F(X,x)$ is monotonically nondecreasing and right-continuous for all x.

We can now state our main result.

THEOREM 1.1. *Let $X_1, X_2, \ldots$ be a sequence of regular graphs, each of degree $v\geqslant 2$, which satisfies the conditions*

(4) *$n(X_i)\to\infty$ as $i\to\infty$, and*
(5) *for each $k\geqslant 3$, $c_k(X_i)/n(X_i)\to 0$ as $i\to\infty$.*

Then for every x, $F(X_i,x)\to F(x)$ as $i\to\infty$, where $F(x)$ is the function defined as follows:

(6) $F(x)=0$ *if* $x\leqslant -2\sqrt{v-1}$.

(7) $$F(x)=\int_{-2\sqrt{v-1}}^{x}\frac{v\sqrt{4(v-1)-z^2}}{2\pi(v^2-z^2)}\,dz$$

$$=\frac{1}{2}+\frac{v}{2\pi}\left[\arcsin\frac{x}{2\sqrt{v-1}}-\frac{v-2}{v}\arctan\frac{(v-2)x}{v\sqrt{4(v-1)-x^2}}\right]$$

if $-2\sqrt{v-1}<x<2\sqrt{v-1}$.

(8) $F(x)=1$ *if* $x\geqslant 2\sqrt{v-1}$.

Conversely, if $F(X_i,x)$ does not converge to $F(x)$ for some x, then the condition (5) *fails for some k.* ■

For $v=2$, the theorem follows directly from the fact that the eigenvalues of the n-vertex polygon are

$$2\cos\frac{2k\pi}{n},\qquad k=0,1,2,\ldots,n-1.$$

Consequently, we will assume from now on that $v\geqslant 3$.

2. EXISTENCE AND UNIQUENESS OF $F(x)$

Let ν_0 be a vertex of a graph X. A *closed walk of length* $r \geq 0$ *starting at* ν_0 is a sequence $\nu_0, \nu_1, \nu_2, \dots, \nu_r$ of vertices of X such that $\nu_r = \nu_0$, and ν_{i-1} is adjacent to ν_i for $1 \leq i \leq r$.

LEMMA 2.1. *Suppose X is regular of degree v. Let ν_0 be a vertex of X, and suppose that the subgraph of X induced by the vertices at distance at most $r/2$ from ν_0 is acyclic. Then the number of closed walks of length r in X starting at ν_0 is $\theta(r)$, where $\theta(r)=0$ if r is odd, and*

$$\theta(2s) = \sum_{k=1}^{s} \binom{2s-k}{s} \frac{k}{2s-k} v^k (v-1)^{s-k} \qquad (s>0)$$

$$= v \sum_{k=0}^{s-1} \binom{2s}{k} \frac{s-k}{s} (v-1)^k \qquad (s>0)$$

$$= \sum_{k=1}^{s} \binom{2s}{k} \frac{2s-2k+1}{2s-k+1} (v-1)^k \qquad (s \geq 0).$$

Proof. Let $\boldsymbol{\nu} = \nu_0, \nu_1, \dots, \nu_r$ be a closed walk of length r. Corresponding to $\boldsymbol{\nu}$ we have a sequence of nonnegative integers $\boldsymbol{\delta} = \delta_0, \delta_1, \dots, \delta_r$, where δ_i is the distance in X of ν_i from ν_0. Thus $\boldsymbol{\delta}$ satisfies the conditions

(a) $\delta_0 = \delta_r = 0$,
(b) $\delta_i \geq 0$ for $0 \leq i \leq r$, and
(c) $|\delta_i - \delta_{i-1}| = 1$ for $1 \leq i \leq r$.

Conditions (a) and (c) together imply that r is even. Also, $\theta(0)=1$ obviously, so assume $r=2s$, where $s>0$.

It can be shown [1] that the number of sequences $\boldsymbol{\delta}$ which satisfy conditions (a), (b), and (c), and for which exactly k of the numbers $\delta_1, \delta_2, \dots, \delta_r$ are zero, is

$$\binom{2s-k}{s} \frac{k}{2s-k}.$$

Each such $\boldsymbol{\delta}$ corresponds to $v^k(v-1)^{s-k}$ closed walks $\boldsymbol{\nu}$. Summing over k gives the first expression for $\theta(2s)$. The other two can be derived from the first by algebraic manipulation. ■

LEMMA 2.2. *For $r \geqslant 0$, $i \geqslant 1$ let $\phi_r(X_i)$ denote the total number of closed walks of length r in X_i. Then for each r, $\phi_r(X_i)/n(X_i) \to \theta(r)$ as $i \to \infty$.*

Proof. Let $n_r(X_i)$ denote the number of vertices of X_i which satisfy the requirement of Lemma 2.1 for ν_0. By condition (5) above, $n_r(X_i)/n(X_i) \to 1$ as $i \to \infty$. By Lemma 2.1 the number of closed walks of length r starting at each such vertex is $\theta(r)$. For each of the remaining vertices the number of closed walks of length r is certainly less than v^r.

Therefore, for each r, there are numbers $\bar{\theta}_r(X_i)$ such that $0 \leqslant \bar{\theta}_r(X_i) \leqslant v^r$, and

$$\frac{\phi_r(X_i)}{n(X_i)} = \frac{n_r(X_i)\theta(r) + [n(X_i) - n_r(X_i)]\bar{\theta}_r(X_i)}{n(X_i)}$$

$$\to \theta(r) \qquad \text{as} \quad i \to \infty. \qquad \blacksquare$$

Lemma 2.2 can be written in terms of the functions $F(X_i, x)$ by means of the Lebesgue-Stieltjes integral.

LEMMA 2.3. *For each $r \geqslant 0$, $\int x^r \, dF(X_i, x) \to \theta(r)$ as $i \to \infty$.*

Proof. The number of closed walks of length r in X_i equals the trace of the rth power of $A(X_i)$, and so equals the sum of the rth powers of the eigenvalues of X_i. The lemma now follows from the definition of $F(X_i, x)$. $\blacksquare$

THEOREM 2.4. *There is a unique function $F(x)$ which is monotonically nondecreasing and right-continuous for all x such that*

$$\int x^r \, dF = \theta(r) \qquad \text{for each} \quad r \geqslant 0.$$

Furthermore $F(X_i, x) \to F(x)$ as $i \to \infty$, for every x at which $F(x)$ is continuous.

The theorem follows from Lemma 2.3 as a special case of Theorem C of [5]. However, it will be as easy to give a direct proof here as to prove that the requirements of the latter theorem are met. In fact we will prove a more general result, because the proof is the same.

Let $I = [\alpha, \beta]$ be a finite real interval. For each finite $M \geqslant 0$ define $\mathrm{RBV}(I, M)$ to be the set of all real functions $E(x)$ such that

(a) $E(x)=0$ if $x<\alpha$,
(b) $E(x)$ is constant if $x>\beta$,
(c) $E(x)$ is right-continuous for all x, and
(d) the total variation of E is at most M.

The next three lemmas are standard results.

LEMMA 2.5. *If $E\in \mathrm{RBV}(I,M)$ and $\int x^r dE=0$ for each $r\geq 0$, then $E(x)=0$ for all x.*

LEMMA 2.6 (Helley-Bray theorem). *Let $E_1, E_2,\dots$ be a sequence in $\mathrm{RBV}(I,M)$ which converges to some $E\in\mathrm{RBV}(I,M)$ at every x for which $E(x)$ is continuous. Then $\int x^r dE_i \to \int x^r dE$ as $i\to\infty$, for each $r\geq 0$.*

LEMMA 2.7 (Helley selection theorem). *Let $E_1, E_2,\dots$ be a sequence in $\mathrm{RBV}(I,M)$. Then there exists $E\in\mathrm{RBV}(I,M)$ and a subsequence of $E_1, E_2,\dots$ which converges to E at every x for which $E(x)$ is continuous.*

THEOREM 2.8. *Let $E_1, E_2,\dots$ be a sequence in $\mathrm{RBV}(I,M)$ such that, for each $r\geq 0$, $\int x^r dE_i\to\mu_r$ as $i\to\infty$, where $\mu_0,\mu_1,\dots$ are finite constants. Then there exists a unique function $E\in\mathrm{RBV}(I,M)$ such that $\int x^r dE=\mu_r$ for each $r\geq 0$. Furthermore $E_i(x)\to E(x)$ wherever $E(x)$ is continuous.*

Proof. By the preceding lemmas, there is a unique function $E\in\mathrm{RBV}(I,M)$ such that every infinite subsequence of $E_1, E_2,\dots$ contains a subsequence which converges to E wherever $E(x)$ is continuous. Now suppose that $E_1, E_2,\dots$ does not converge to E at some point x where $E(x)$ is continuous. Then there exists $\varepsilon>0$ and a subsequence $E_{i_1}, E_{i_2},\dots$ such that $|E_{i_j}(x)-E(x)|>\varepsilon$ for all j. Thus $E_{i_1}(x), E_{i_2}(x),\dots$ does not contain a subsequence which converges to $E(x)$, providing a contradiction. ■

Proof of Theorem 2.4. The functions $F(X_i, x)$ satisfy the requirements of Theorem 2.8 with $M=1$. Furthermore, they are all nondecreasing, which proves that the limit $F(x)$ is nondecreasing. ■

3. DERIVATIONS OF $F(x)$

Our first task will be to derive an asymptotic expression for $\theta(r)$.

LEMMA 3.1. $\theta(2s)\sim\dfrac{4^s v(v-1)^{s+1}}{s(v-2)^2\sqrt{\pi s}}$ *as $s\to\infty$.*

Proof. We need three elementary results:

(9) $1-(1-x)^m \leqslant mx$ if $m \geqslant 1$, $0 \leqslant x \leqslant 1$.

(10) $\displaystyle\sum_{i=1}^{m} iz^{i-1} = \frac{1-z^m(m+1-mz)}{(1-z)^2} \leqslant \frac{1}{(1-z)^2}$ if $0 \leqslant z < 1$.

(11) $\displaystyle\binom{2m}{m-1} \sim \frac{4^m}{\sqrt{\pi m}}$ as $m \to \infty$.

Assume $s \geqslant 1$ and define $z = 1/(v-1)$. The second formula of Lemma 2.1 can be written as

$$\begin{aligned}\theta(2s) &= \frac{v(v-1)^{s-1}}{s} \sum_{i=1}^{s} \binom{2s}{s-i} iz^{i-1} \\ &= \frac{v(v-1)^{s-1}}{s} \binom{2s}{s-1} \left\{ \sum_{i=1}^{s} iz^{i-1} - \varepsilon(s) \right\},\end{aligned}$$

where

$$\begin{aligned}\varepsilon(s) &= \sum_{i=1}^{s} \left[1 - \binom{2s}{s-i} \Big/ \binom{2s}{s-1} \right] iz^{i-1} \\ &= \sum_{i=1}^{s} \left[1 - \binom{s-1}{i-1} \Big/ \binom{s+i}{i-1} \right] iz^{i-1} \\ &= \sum_{i=1}^{k} \left[1 - \binom{s-1}{i-1} \Big/ \binom{s+i}{i-1} \right] iz^{i-1} + \sum_{i=k+1}^{s} \left[1 - \binom{s-1}{i-1} \Big/ \binom{s+i}{i-1} \right] iz^{i-1}\end{aligned}$$

for $1 \leqslant k < s$. Now

$$0 \leqslant 1 - \binom{s-1}{i-1} \Big/ \binom{s+i}{i-1} \leqslant 1 \qquad \text{for} \quad 1 \leqslant i \leqslant s,$$

and

$$\begin{aligned}1 - \binom{s-1}{i-1} \Big/ \binom{s+i}{i-1} &\leqslant 1 - \left[1 - \frac{2(k-1)}{s+k-1} \right]^{k-1} \qquad \text{if} \quad 1 \leqslant i \leqslant k \\ &\leqslant \frac{2(k-1)^2}{s+k-1}, \qquad \text{by (9).}\end{aligned}$$

Therefore

$$0 \leqslant \varepsilon(s) \leqslant \sum_{i=1}^{k} \frac{2(k-1)^2}{s+k-1} i z^{i-1} + \sum_{i=k+1}^{s} s z^{i-1}$$

$$\leqslant \frac{2(k-1)^2}{s+k-1} \cdot \frac{1}{(1-z)^2} + \frac{s z^k}{1-z}, \qquad \text{by (10).}$$

Now let $s \to \infty$ while holding $k = \lfloor s^{1/3} \rfloor$. Both terms on the right approach zero. Hence $\varepsilon(s) \to 0$ as $s \to \infty$. Therefore

$$\theta(2s) \sim \frac{v(v-1)^{s-1}}{s} \binom{2s}{s-1} \sum_{i=1}^{s} i z^{i-1}$$

$$\sim \frac{4^s v(v-1)^{s+1}}{s(v-2)^2 \sqrt{\pi s}}, \qquad \text{by (10) and (11).} \qquad \blacksquare$$

LEMMA 3.2. *Define* $\omega = \sup\{|x| \mid 0 < F(x) < 1\}$. *Then* $\omega = 2\sqrt{v-1}$.

Proof. For any $s \geqslant 0$, $\int x^{2s+2}\, dF \leqslant \omega^2 \int x^{2s}\, dF$, by the definition of ω. Therefore

$$\limsup_{s \to \infty} \frac{\int x^{2s+2}\, dF}{\int x^{2s}\, dF} \leqslant \omega^2.$$

Choose α and β such that $0 < \alpha < \beta < \omega$. Then

$$\int_{|x| \geqslant \alpha} x^{2s}\, dF \geqslant \beta^{2s}[1 - F(\beta) + F(-\beta)]$$

and

$$\int_{|x| \leqslant \alpha} x^{2s}\, dF \leqslant \alpha^{2s}[F(\alpha) - F(-\alpha)].$$

Therefore

$$\frac{\displaystyle\int_{|x|\leqslant\alpha} x^{2s}\,dF}{\displaystyle\int_{|x|\geqslant\alpha} x^{2s}\,dF}\to 0 \qquad \text{as} \quad s\to\infty,$$

since $1-F(\beta)+F(-\beta)>0$. Thus

$$\liminf_{s\to\infty}\frac{\displaystyle\int x^{2s+2}\,dF}{\displaystyle\int x^{2s}\,dF}\geqslant\beta^2.$$

Allowing β to approach ω, we find that $\omega^2=\lim_{s\to\infty}\theta(2s+2)/\theta(2s)$. Hence $\omega=2\sqrt{v-1}$ by Lemma 3.1. ■

LEMMA 3.3. *$F(x)$ is continuous at $x=\pm\omega$.*

Proof. Define $\delta=F(-\omega)-F(-\omega-0)+F(\omega)-F(\omega-0)$. Then $\theta(2s)\geqslant\omega^{2s}\delta=4^s(v-1)^s\delta$, by Lemma 3.2. Comparison with Lemma 3.1 shows that $\delta=0$. ■

Lemmas 3.2 and 3.3 together permit us to replace $\int x^r\,dF$ by $\int_{-\omega}^{\omega}x^r\,dF$. In fact it will be convenient to replace the interval $[-\omega,\omega]$ by $[-1,1]$. To accomplish this, define $G(x)=F(\omega x)$. Thus

$$\int_{-1}^{1}x^r\,dG=\frac{\theta(r)}{\omega^r}\qquad\text{for each}\quad r\geqslant 0. \tag{12}$$

We will now seek a solution of (12) such that the derivative $g(x)=G'(x)$ exists for $-1<x<1$. Under this assumption, (12) can be replaced by

$$\int_{-1}^{1}x^r g(x)\,dx=\frac{\theta(r)}{\omega^r}\qquad\text{for each}\quad r\geqslant 0. \tag{13}$$

The Tchebysheff polynomials $T_0(x),T_1(x),\ldots,$ are defined by $T_n(\cos\theta)=\cos n\theta$. The properties which we shall require are recalled in the next lemma.

LEMMA 3.4.

(a) *If* $n>0$ *then*

$$T_n(x)=\frac{n}{2}\sum_{k=0}^{[n/2]}(-1)^k\frac{(n-k-1)!}{k!(n-2k)!}(2x)^{n-2k}.$$

(b) $$\int_{-1}^{1}\frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}dx=\begin{cases}0 & \text{if } n\neq m,\\ \pi & \text{if } n=m=0,\\ \pi/2 & \text{if } n=m\neq 0.\end{cases}$$

We now define $h(x)=g(x)\sqrt{1-x^2}$ and suppose that for $-1<x<1$ we can expand

$$h(x)=\sum_{i=0}^{\infty}\alpha_i T_i(x).$$

Then Lemma 3.4(b) gives

$$\alpha_0=\frac{1}{\pi}\int_{-1}^{1}T_0(x)g(x)\,dx=\frac{1}{\pi},$$

$$\alpha_n=\frac{2}{\pi}\int_{-1}^{1}T_n(x)g(x)\,dx \qquad \text{if} \quad n>0.$$

If n is odd, $T_n(x)$ only contains odd powers of x, and so $\alpha_n=0$. Now suppose $n=2m>0$.

By Lemma 3.4(a),

$$\begin{aligned}\alpha_{2m}&=\frac{2m}{\pi}\sum_{k=0}^{m}(-1)^k\frac{(2m-k-1)!}{k!(2m-2k)!}2^{2m-2k}\int_{-1}^{1}x^{2m-2k}g(x)\,dx\\
&=\frac{2m}{\pi}\sum_{k=0}^{m}(-1)^k\frac{(2m-k-1)!}{k!(2m-2k)!}(v-1)^{k-m}\theta(2m-2k)\\
&=\frac{2m}{\pi}\sum_{k=0}^{m}(-1)^k\frac{(2m-k-1)!}{k!(2m-2k)!}(v-1)^{k-m}\\
&\quad\times\sum_{r=0}^{m-k}\binom{2m-2k}{r}\frac{2m-2k-2r+1}{2m-2k-r+1}(v-1)^r, \qquad \text{by Lemma 2.1,}\\
&=\frac{2}{\pi}\sum_{t=0}^{m}\beta_t(v-1)^{t-m},\end{aligned}$$

where

$$\beta_t = m \sum_{k=0}^{t} (-1)^k \frac{(2m-k-1)!}{k!(2m-2k)!} \binom{2m-2k}{t-k} \frac{2m-2t+1}{2m-k-t+1}.$$

By direct calculation, $\beta_0 = \frac{1}{2}$ and $\beta_1 = -\frac{1}{2}$. If $t \geqslant 2$,

$$\begin{aligned}\beta_t &= \frac{m(2m-2t+1)}{t(t-1)} \sum_{k=0}^{t} (-1)^k \binom{t}{k} \binom{2m-k-1}{t-2} \\ &= \frac{m(2m-2t+1)}{t(t-1)} \binom{2m-t-1}{2m-t+1}, \\ &\qquad\qquad \text{by the Vandermonde convolution [6]} \\ &= 0.\end{aligned}$$

Therefore

$$\begin{aligned}\alpha_{2m} &= \frac{2}{\pi} \left\{ \tfrac{1}{2}(v-1)^{-m} - \tfrac{1}{2}(v-1)^{1-m} \right\} \\ &= -\frac{v-2}{\pi(v-1)^m}.\end{aligned}$$

Thus we can expand

$$\begin{aligned}\pi h(x) &= 1 - (v-2) \sum_{m=1}^{\infty} \frac{T_{2m}(x)}{(v-1)^m} \\ &= 1 - (v-2) \left[\frac{(v-1)(2x^2-1)-1}{(v-1)^2 - 2(v-1)(2x^2-1)+1} \right],\end{aligned}$$

which gives

$$g(x) = \frac{2v(v-1)\sqrt{1-x^2}}{\pi\left[v^2 - 4(v-1)x^2\right]}.$$

Recalling that $g(x)=\omega F'(\omega x)$, we find that for $-\omega<x<\omega$,

$$F(x)=\int_{-\omega}^{x}\frac{v\sqrt{4(v-1)-z^2}}{2\pi(v^2-z^2)}dz$$

$$=\frac{1}{4}+\frac{v}{2\pi}\left[\arcsin\frac{x}{2\sqrt{v-1}}-\frac{v-2}{v}\arctan\frac{(v-2)x}{v\sqrt{4(v-1)-x^2}}\right].$$

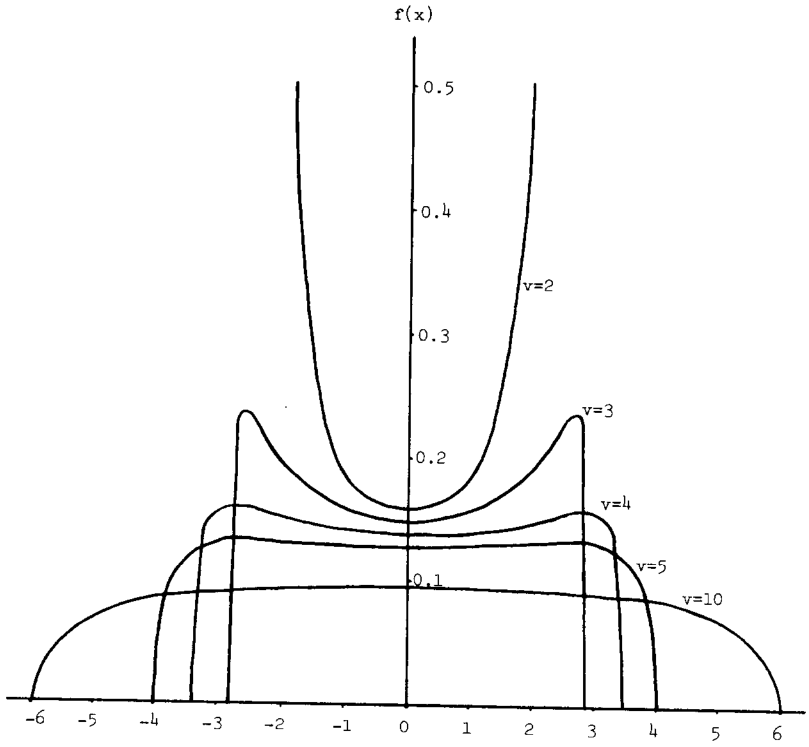

FIG. 1. Limiting eigenvalue densities for various v.

Proof of Theorem 1.1. The function $F(x)$ we have obtained satisfies the requirements that $G(x)$ be differentiable for $-1<x<1$ and that $g(x)$ have a convergent expansion in terms of Tchebysheff polynomials. Since it is also continuous, Theorem 1.1 follows from Theorem 2.4. ■

Figure 1 illustrates the shape of the probability density function $f(x)$ for various values of v. As v becomes very large, the shape approaches that of an ellipse.

4. RANDOM LABELED REGULAR GRAPHS

Fix $v \geqslant 2$. Let $n_1 < n_2 < n_3 < \cdots$ be the sequence of possible orders of regular graphs with degree v. For each i define R_i to be the set of all labeled regular graphs with degree v and order n_i. For each real x, define $F_i(x)$ to be the average value of $F(X, x)$, where the average is taken over all $X \in R_i$. We can think of $F_i(x)$ as giving the expected eigenvalue distribution of a random labelled regular graph with degree v and order n_i.

The following lemma appears in [7].

LEMMA 4.1. *For each $k \geqslant 3$ define $c_{k,i}$ to be the average number of k-cycles in the members of R_i. Then for each k, $c_{k,i} \to (v-1)^k/2k$ as $i \to \infty$.*

THEOREM 4.2. *For every x, $F_i(x) \to F(x)$ as $i \to \infty$, where $F(x)$ is the function defined in Theorem* 1.1.

Proof. Consider the graph Y_i consisting of the disjoint union of all members of R_i. Then $F_i(x) = F(Y_i, x)$. Consequently, the theorem follows from Lemma 4.1 and Theorem 1.1. ■

A stronger version of Lemma 4.1 [4] can be used to improve on theorem 4.2.

THEOREM 4.3. *For each i choose $X_i \in R_i$ at random. Then, with probability one, $F(X_i, x) \to F(x)$.*

Finally, we show that a precise bound can be given on the deviation of $F(X, x)$ from $F(x)$ in terms of the degree and the girth.

THEOREM 4.4. *Let X be a regular graph with degree v and girth r. Then, for every x,*

$$|F(X,x)-F(x)| < \frac{24e\sqrt{v-1}}{\pi^2(r+2)}.$$

Proof. From Equations 3.2, 3.6, and 3.12 of Chapter XVI of [2] we find that, for any $T>0$,

$$|F(X,x)-F(x)| \leqslant \frac{1}{\pi}\int_{-T}^{T}\left|\frac{\phi(z)-\gamma(z)}{z}\right|\left(1-\frac{|z|}{T}\right)dz+\frac{24m}{\pi T},$$

where $\phi(z)$ and $\gamma(z)$ are the characteristic functions of $F(x)$ and $F(X,x)$ respectively, and m is an upper bound on $|F'(x)|$ for all x. From Theorem 1.1, we can take $m=\omega/2\pi v$. Furthermore, the power-series expansions of $\phi(z)$ and $\gamma(z)$ agree up to the terms involving z^{r-1}. Therefore,

$$|\phi(z)-\gamma(z)| \leqslant \frac{(\omega^r+v^r)|z|^r}{r!}.$$

Performing the integration, we obtain

$$|F(X,x)-F(x)| \leqslant \frac{2(v^r+\omega^r)T^r}{\pi r(r+1)!}+\frac{12\omega}{\pi^2 vT},$$

and inserting the value of T for which this bound is smallest, we find

$$\begin{aligned}|F(X,x)-F(x)| &\leqslant \frac{12\omega(r+1)}{\pi^2 r}\left[\frac{\pi(1+\omega^r/v^r)}{6\omega(r+1)!}\right]^{1/(r+1)}\\ &< \frac{24e\sqrt{v-1}}{\pi^2(r+2)},\end{aligned}$$

since $v\geqslant 3$ and $r\geqslant 3$. ∎

The problem was introduced to me by B. E. Eichinger and J. E. Martin, who also suggested the use of moments for its solution. An error in an early

version of Lemma 2.1 was pointed out by R. G. Cowell. Finally, C. D. Godsil has suggested that Theorem 1.1 might be proved more easily using the existing statistical theory of random walks. See, for example, pp. 108–109 of [3].

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