The Probability Lifesaver:
Change of Variable Theorem

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Contents

Greetings again!

In this supplemental chapter we state and sketch the proof of the Change of Variable Theorem, as well as cover some of the most important examples.

Chapter 1

Change of Variable Formula

In this chapter we review the Change of Variable formula. For a first course in probability, it's usually sufficient to just know it in the big three special cases: polar coordinates, cylindrical coordinates, and spherical coordinates. We state the results in a bit more generality and try to motivate the result by giving many of the details of the proof.

While the more linear algebra you know, the easier time you'll have, for the most part all we need are the concepts of a square matrix and its determinant. For a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

the determinant, denoted $|A|$ or $\det(A)$, is

$$\det(A) = ad - bc.$$

There's a very nice geometric interpretation to $\det(A)$ here; it's the area of the parallelogram spanned by the vectors (a, b) and (c, d) (or by the vectors (a, c) and (b, d)). The formula is more involved in higher dimensions, but a similar interpretation holds (in three dimensions the determinant is the volume of the parallelpiped spanned by the rows *or* by the columns).

1.1 Statement

We begin by looking at what happens when we have a map from the plane to itself. Thus we have two variables, say x and y, and we change to new variables u and v. We first need to recall some notation from multivariable calculus.

A region $D \subset \mathbb{R}^2$ is x-**simple** if there are continuous functions ψ_1 and ψ_2 defined on $[c, d]$ such that

$$\psi_1(y) \ \leq \ \psi_2(y)$$

and

$$D \ = \ \{(x, y) : \psi_1(y) \leq x \leq \psi_2(y) \ \text{ and } \ c \leq y \leq d\};$$

similarly, D is y**-simple** if there are continuous functions $\phi_1(x)$ and $\phi_2(x)$ such that

$$\phi_1(x) \le \phi_2(x)$$

and

$$D = \{(x,y) : \phi_1(x) \le y \le \phi_2(x) \text{ and } a \le x \le b\}.$$

If D is both x-simple and y-simple then we say D is **simple**. A region that is either x-simple, y-simple or simple is frequently called an **elementary region**. **ADD PICTURES!**

Let's look at the definition for a y-simple region. What it's saying is that we have two boundary curves, $y = \phi_1(x)$ and $y = \phi_2(x)$, such that the first curve is the 'bottom' of our region and the second is the 'top'. In other words, for each x we go up and first enter the region at $\phi_1(x)$, we're in our region for all y from $\phi_1(x)$ to $\phi_2(x)$, and then once we leave the region at $\phi_2(x)$ we never return (for this x).

We can now state the Change of Variables Formula (in the plane).

Theorem 1.1.1 (Change of Variables Formula in the Plane) *Let S be an elementary region in the xy-plane (such as a disk or parallelogram for example). Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be an invertible and differentiable mapping, and let $T(S)$ be the image of S under T. Then*

$$\int\int_S 1 \cdot dxdy = \int\int_{T(S)} 1 \cdot \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right| dudv,$$

or more generally

$$\int\int_S f(x,y) \cdot dxdy = \int\int_{T(S)} f\left(T^{-1}(u,v)\right) \cdot \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right| dudv.$$

Some notes on the above.

1. We assume our map T has an inverse function, denoted T^{-1}. Thus $T(x,y) = (u,v)$ and $T^{-1}(u,v) = (x,y)$.

2. As T is invertible, for each $(x,y) \in S$ there's one and only one (u,v) that it's mapped to, and conversely each (u,v) is mapped to one and only one (x,y).

3. The derivative of $T^{-1}(u,v) = (x(u,v), y(u,v))$ is the matrix

$$(DT^{-1})(u,v) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix},$$

and the absolute value of the determinant of the derivative (often called the **Jacobean** and denoted JT) is

$$\left|\det\left(DT^{-1})(u,v)\right)\right| = \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right|,$$

which implies the area element transforms as

$$dxdy = \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right| dudv.$$

4. Note that f takes as input x and y, but when we change variables our new inputs are u and v. The map T^{-1} takes u and v and gives x and y, and thus we need to evaluate f at $T^{-1}(u, v)$. Remember that we're now integrating over u and v, and thus the integrand must be a function of u and v.

5. Note that the formula requires an absolute value of the determinant. The reason is that the determinant can be negative, and we want to see how a small area element transforms. Area is supposed to be positively counted. Note in one-variable calculus that $\int_a^b f(x)dx = -\int_b^a f(x)dx$; we need the absolute value to take care of issues such as this.

6. While we stated T is a differentiable mapping, our assumptions imply T^{-1} is differentiable as well.

We occasionally need a more general version, such as when we study the chi-square distribution in Chapter **??**.

Theorem 1.1.2 (Change of Variables) *Let V and W be bounded open sets in $\mathbb{R}^n$. Let $h : V \to W$ be a 1-1 and onto map, given by*

$$h(u_1, \dots, u_n) \;=\; (h_1(u_1, \dots, u_n), \dots, h_n(u_1, \dots, u_n)).$$

Let $f : W \to \mathbb{R}$ be a continuous, bounded function. Then

$$\begin{aligned}\int \cdots \int_W & f(x_1, \dots, x_n) dx_1 \cdots dx_n \\ &= \int \cdots \int_V f\left(h(u_1, \dots, u_n)\right) J(u_1, \dots, u_v) du_1 \cdots du_n,\end{aligned}$$

where J is the ***Jacobian***

$$J \;=\; \begin{vmatrix} \frac{\partial h_1}{\partial u_1} & \cdots & \frac{\partial h_1}{\partial u_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_n}{\partial u_1} & \cdots & \frac{\partial h_n}{\partial u_n} \end{vmatrix}.$$

1.2 Sketch of Proof

The Change of Variable Theorem (or Formula) is one of the most important results of multivariable calculus. The reason is that numerous problems have a natural coordinate system where, if we look at it from the right perspective, the analysis greatly simplifies. It's very important to be able to convert from one coordinate system to another and be able to exploit the advantages of each.

Our first example is mapping the unit square to a rectangle (see Figure 1.1). Note the original square, S, has area 1 and the region it maps to, $T(S)$, has area 6. Thus

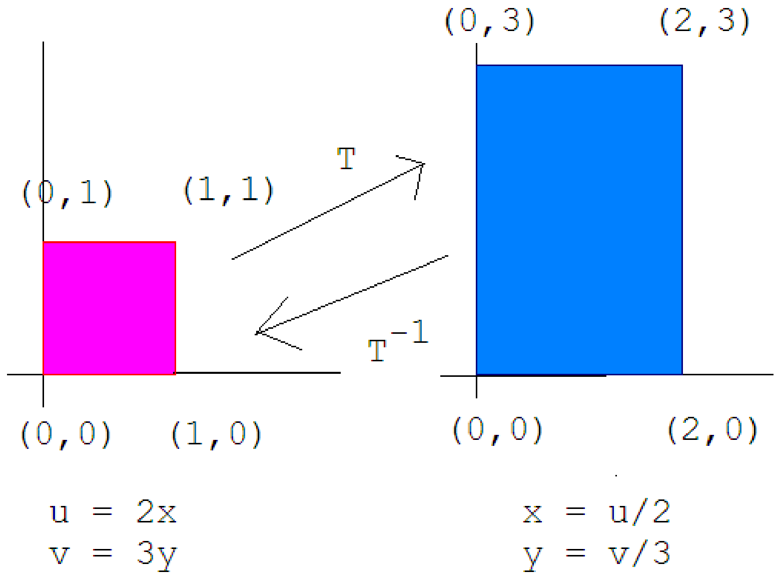

Figure 1.1: Mapping the unit square via $u = 2x$ and $v = 3y$, so $T(x, y) = (2x, 3y)$.

$dxdy$ corresponds to $\frac{1}{6}dudv$. If we compute the derivative matrix associated to T^{-1}, since

$$x(u, v) \;=\; u/2 \quad \text{and} \quad y(u, v) \;=\; v/3,$$

we find

$$\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix},$$

which has a determinant of $1/6$. This verifies the formula's prediction, namely that the conversion factor to go from area in xy-space to area in uv-space is $1/6$. In other words,

$$\int\int_S 1 \cdot dxdy \;=\; \int\int_{T(S)} \frac{1}{6} \cdot dudv;$$

clearly we don't expect $\int\int_S 1 \cdot dxdy$ to equal $\int\int_{T(S)} 1 \cdot dudv$. I think a great way to view the absolute value of the determinant of the derivative matrix is that it gives the exchange rate in converting from area in one space to the other.

We now sketch the proof of the formula. It'll involve several of the major concepts in multivariable calculus, from the cross product to determinants and areas to the definition of the derivative being that the tangent plane is a great approximation. *The proof below may safely be skipped; we've included it because the Change of Variable Theorem is so important, and because the proof reviews a lot of concepts from multivariable calculus.*

Recall we have

$$T^{-1}(u, v) \;=\; (x(u, v), y(u, v)).$$

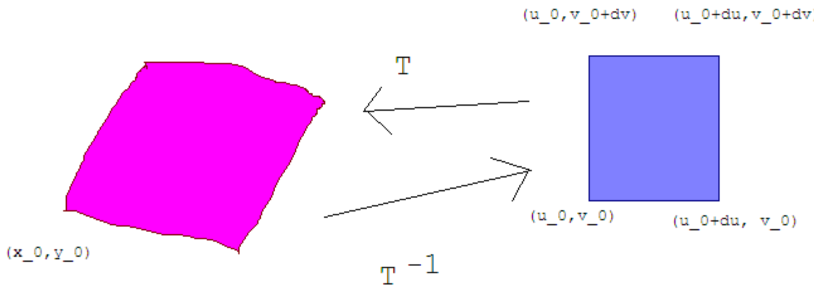

Figure 1.2: Mapping of the general case: $x(u, v)$ and $y(u, v)$. Note: to save time I've written $u_0 + du$ for $u_0 + \Delta u$, and similarly for the v's, above. **FIX IMAGE AND TEXT.**

We want to see what a small rectangle in uv-space corresponds to in xy-space; see Figure 1.2. Let's look at where the four corners of the rectangle in the uv-plane are mapped. Recall that if we have a function $f(u, v)$, then

$$f(u, v) \;=\; f(u_0, v_0) + (\nabla f)(u_0, v_0) \cdot (u - u_0, v - v_0) + \text{small}$$

if (u, v) is close to (u_0, v_0). There are many ways to look at this. It's a Taylor expansion, it's a definition of the derivative, it's taking a directional derivative.

As T^{-1} is differentiable, we can write

$$\begin{aligned} x(u, v) &= x(u_0, v_0) + (\nabla x)(u_0, v_0) \cdot (u - u_0, v - v_0) + \text{small} \\ y(u, v) &= y(u_0, v_0) + (\nabla y)(u_0, v_0) \cdot (u - u_0, v - v_0) + \text{small}, \end{aligned}$$

so long as (u, v) is close to (u_0, v_0). This is simply the definition of the derivative in several variables, namely the statement that we can approximate a complicated function locally by a plane. What makes it a little confusing is that x is now our function name, not a coordinate (this is why we considered the example with a function f above). Thus, while the square with lengths Δu and Δv in uv-space doesn't map to exactly a square, rectangle or parallelogram in xy-space, it maps to almost a parallelogram.

Let's see where the four corners of the rectangle map to. Expanding the gradient we find

$$\begin{aligned} x(u, v) &= x_0 + \frac{\partial x}{\partial u}(u - u_0) + \frac{\partial x}{\partial v}(v - v_0) + \text{small} \\ y(u, v) &= y_0 + \frac{\partial y}{\partial u}(u - u_0) + \frac{\partial y}{\partial v}(v - v_0) + \text{small}. \end{aligned}$$

Note that $x(u_0, v_0)$ is what we're calling x_0 and $y(u_0, v_0)$ is what we're calling y_0, the base point of the square. For definiteness we're assuming the four corners' orientations are preserved under the mapping (we had to choose how to draw / discuss things). We have

$$\begin{aligned}(x(u_0+\Delta u, v_0), y(u_0+\Delta u, v_0)) &= \left(x_0 + \frac{\partial x}{\partial u}\Delta u, y_0 + \frac{\partial y}{\partial u}\Delta u\right)\\ (x(u_0, v_0+\Delta v), y(u_0, v_0+\Delta v)) &= \left(x_0 + \frac{\partial x}{\partial v}\Delta v, y_0 + \frac{\partial y}{\partial v}\Delta v\right).\end{aligned}$$

The original rectangle in uv-space had sides given by the vectors $(u_0+\Delta u, v_0)-(u_0, v_0)$ and $(u_0, v_0+\Delta v)-(u_0, v_0)$. Thus the area is equivalent to that of a rectangle given by the vectors $(\Delta u, 0)$ and $(0, \Delta v)$, for an area of $\Delta u \Delta v$.

What about the region it's mapped to? It's essentially a parallelogram; this is the content of the function T^{-1} being differentiable. The side $(u_0+\Delta u, v_0) - (u_0, v_0)$ which was equivalent to the vector $(\Delta u, 0)$ corresponds to

$$(x(u_0+\Delta u, v_0) - (x_0, y_0)),$$

which is approximately

$$\left(\frac{\partial x}{\partial u}\Delta u, \frac{\partial y}{\partial u}\Delta u\right);$$

similarly the other side is basically

$$\left(\frac{\partial x}{\partial v}\Delta v, \frac{\partial y}{\partial v}\Delta v\right).$$

To find the area of a parallelogram with sides $\overrightarrow{w}_1$ and $\overrightarrow{w}_2$ we need only take the cross product. We must be careful, though. The cross product takes as input two vectors with three components and outputs a vector with three components. We can consider our vectors as living in three-dimensional space by appending a zero as the third component, and then the area of the parallelogram is the length of the cross product. We must compute

$$\left(\frac{\partial x}{\partial u}\Delta u, \frac{\partial y}{\partial u}\Delta u, 0\right) \times \left(\frac{\partial x}{\partial v}\Delta v, \frac{\partial y}{\partial v}\Delta v, 0\right).$$

Recall this is given by

$$\left|\begin{array}{ccc}\overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k}\\ \frac{\partial x}{\partial u}\Delta u & \frac{\partial y}{\partial u}\Delta u & 0\\ \frac{\partial x}{\partial v}\Delta v & \frac{\partial y}{\partial v}\Delta v & 0\end{array}\right| = \left(0, 0, \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right)\Delta u \Delta v\right),$$

and the length is clearly just

$$\left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right| \Delta u \Delta v,$$

or equivalently

$$dxdy \sim \left|\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right| dudv.$$

The outline above highlights the key ideas in the proof. One of course needs to perform a careful analysis of the error terms, but you've seen the main ideas. The key observation is that locally any differentiable map is linear (and takes rectangles to parallelograms), and then we piece the contributions over the entire region together. The absolute value of the determinant of the derivative map gives us the exchange rate between the two different areas.

1.3 Special Cases

We'll first state the big three special cases, and then do a few examples. Briefly, change of variables allows us to convert integrals over complicated regions to integrals over simpler regions, at a cost of a more complicated integrand. Often this is a good exchange; in other words, in many problems it's better to have to integrate a harder function over a simpler area than a simpler function over a harder area. We won't prove the theorems below, which are essentially consequences of the Change of Variable Formula in 2 and 3 dimensions. We have to say essentially as there's one minor, annoying technicality. Our map is supposed to be singly valued, but for the three maps below, multiple points (r, θ) are mapped to $(0, 0)$. Fortunately $(0, 0)$ is the only point where this breaks down, and a more involved proof shows that the claim still holds in this case.

Theorem 1.3.1 (Change of Variables Theorem: Polar Coordinates) *Let*

$$x \;=\; r\cos\theta, \quad y \;=\; r\sin\theta$$

with $r \geq 0$ and $\theta \in [0, 2\pi)$; note the inverse functions are

$$r \;=\; \sqrt{x^2+y^2}, \quad \theta \;=\; \arctan(y/x).$$

Let D be an elementary region in the xy-plane, and let D^* be the corresponding region in the $r\theta$-plane. Then

$$\int\int_D f(x,y)dxdy \;=\; \int\int_{D^*} f(r\cos\theta, r\sin\theta)rdrd\theta.$$

For example, if D is the region $x^2+y^2 \leq 1$ in the xy-plane then D^* is the rectangle $[0,1]\times[0,2\pi]$ in the $r\theta$-plane. A little thought shows why polar coordinates are so useful. They allow us to convert a hard shape, a circle, to a simple shape, a rectangle. A rectangle is the easiest shape to integrate over; much easier than a circle! Think back to how the integral was defined, in one and in several dimensions. We had Riemann sums, breaking the regions up into rectangles which fit together nicely. The hope is that, if we're integrating over a rectangular region, the integration can split into two one-dimensional integrals, where neither depends on the other. This reduces one two-dimensional problem to two one-dimensional problems, and often it's better to do more simple integrals than fewer complicated ones. If we were to write down the bounds of integration for the unit circle, we would have x goes from -1 to 1 and y goes from $-\sqrt{1-x^2}$ to $\sqrt{1-x^2}$. Notice how one variable arises in the bounds for the other. In polar coordinates, we have the significantly simpler r goes from 0 to 1 and θ goes from 0 to 2π.

Theorem 1.3.2 (Change of Variables Theorem: Cylindrical Coordinates) *Let*

$$x \;=\; r\cos\theta, \quad y \;=\; r\sin\theta, \quad z \;=\; z$$

with $r \geq 0$, $\theta \in [0, 2\pi)$ *and* z *arbitrary; note the inverse functions are*

$$r \;=\; \sqrt{x^2+y^2}, \quad \theta \;=\; \arctan(y/x), \quad z \;=\; z.$$

Let D *be an elementary region in* xyz*-space, and let* D^* *be the corresponding region in* $r\theta z$*-space. Then*

$$\int\int\int_D f(x,y,z)dxdydz \;=\; \int\int\int_{D^*} f(r\cos\theta, r\sin\theta, z)rdrd\theta dz.$$

Similar to how polar coordinates convert a circle to a rectangle, cylindrical coordinates convert a cylinder to a box. **PICTURES!**

Theorem 1.3.3 (Change of Variables Theorem: Spherical Coordinates) *Let*

$$x \;=\; \rho\sin\phi\cos\theta, \quad y \;=\; \rho\sin\phi\sin\theta, \quad z \;=\; \rho\cos\phi$$

with $\rho \geq 0$, $\theta \in [0, 2\pi]$ *and* $\phi \in [0, \pi)$. *Note that the angle* ϕ *is the angle made with the* z*-axis; many books (such as physics texts) interchange the role of* ϕ *and* θ. *Let* D *be an elementary region in* xyz*-space, and let* D^* *be the corresponding region in* $\rho\theta\phi$*-space. Then*

$$\begin{aligned} & \int\int\int_D f(x,y,z)dxdydz \\ = & \int\int\int_{D^*} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi d\rho d\theta d\phi. \end{aligned}$$

We get another box; this time we replace a sphere with a box. It's worth noting the most common mistake here is to have incorrect bounds of integration, typically confusing which angle goes to 2π and which goes to π. Another common mistake is to have the volume element wrong, either forgetting the $\sin^2\phi$ or having $\sin^2\theta$. Sadly, there are two different notations used for spherical coordinates. In our notation, ϕ is the angle made coming down from the z-axis; lots of other books use θ for this angle. We chose to follow the convention that calls this ϕ as then θ is reserved for the angle in the xy-plane, and we can then 'see' spherical coordinates as an extension of polar coordinates. In fact, taking projections we see the component of our point (x, y, z) in the xy-plane has length $\rho\sin\phi$ (it's component along the z-axis is $\rho\cos\phi$), and the expressions for x, y and z come from now applying polar coordinates to a vector of length $\rho\sin\phi$ with angle θ around the xy-plane.

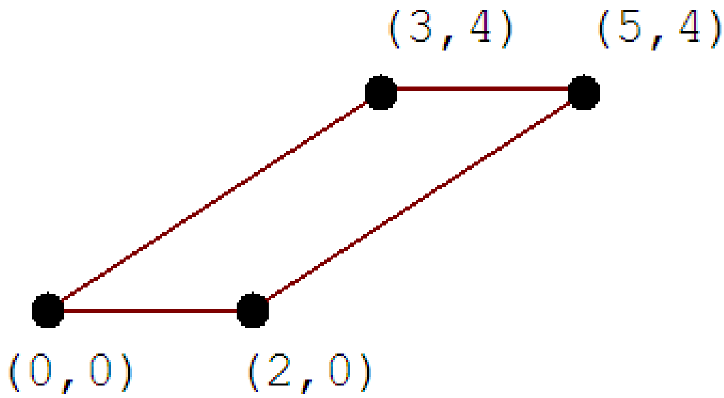

Figure 1.3: Region the unit square is mapped to under $T(x, y) = (2x + 3y, 4y)$.

Consider the change of variables $u = 2x + 3y$ and $v = 4y$. We'll show that this map takes the unit square $[0, 1] \times [0, 1]$ (i.e., the set of points (x, y) with $0 \leq x, y \leq 1$) to a parallelogram, and then use the Change of Variable formula to find the area.

The unit square is mapped to the parallelogram shown in Figure 1.3. To see this, look and see where each vertex of the unit square is sent. We have $(0, 0)$ goes to $(0, 0)$, we have $(1, 0)$ goes to $(2, 0)$, $(0, 1)$ goes to $(3, 4)$ and finally $(1, 1)$ goes to $(5, 4)$. More generally, if we take a point of the form $(x, 0)$ it's mapped to the point $(2x, 0)$, so we see the interval $[0, 1]$ on the x-axis is mapped to the interval $[0, 2]$ on the u-axis. A similar analysis shows all the other lines of the unit square are mapped to lines in the uv-plane. For example, consider the line $(x, 1)$ with $0 \leq x \leq 1$. This is mapped to the line $(2x + 3, 4)$ in the uv-plane, or equivalently the line from $(3, 4)$ (corresponding to $x = 0$) to the point $(5, 4)$ (corresponding to $x = 1$).

We need the inverse transformation T^{-1}, which gives us the x and y corresponding to a choice of u and v. We have to invert the relations

$$u \ = \ 2x + 3y, \quad v \ = \ 4y.$$

The second is the easiest; we clearly need to have $y = v/4$. Knowing this, we then find

$$u \ = \ 2x + \frac{3v}{4},$$

or

$$x \ = \ \frac{u}{2} - \frac{3v}{8}.$$

In other words, we have

$$T^{-1}(u, v) \ = \ (x(u, v), y(u, v)) \ = \ \left(\frac{u}{2} - \frac{3v}{8}, \frac{v}{4}\right).$$

We now find the determinant of the derivative. First we compute

$$(DT^{-1})(u, v) \ = \ \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \ = \ \begin{pmatrix} \frac{1}{2} & -\frac{3}{8} \\ 0 & \frac{1}{4} \end{pmatrix}.$$

The determinant is

$$\det((DT^{-1})(u,v)) \;=\; \frac{1}{2}\cdot\frac{1}{4}+\frac{3}{8}\cdot 0 \;=\; \frac{1}{8},$$

and thus the absolute value of the determinant is

$$\left|\det((DT^{-1})(u,v))\right| \;=\; \frac{1}{8},$$

which means

$$dxdy \quad\longrightarrow\quad \left|\det((DT^{-1})(u,v))\right| dudv \;=\; \frac{1}{8}dudv.$$

By the Change of Variables formula, if S is the original unit square in xy-space and $P = T(S)$ is the parallelogram in uv-space, we have

$$\int\int_S 1dxdy \;=\; \int\int_{T(S)} 1\left|\det((DT^{-1})(u,v))\right| dudv \;=\; \int\int_{T(S)} 1\cdot\frac{1}{8}dudv.$$

As 1/8 is constant, we can pull it out of the integral and find

$$\int\int_S 1dxdy \;=\; \frac{1}{8}\int\int_{T(S)} 1dudv;$$

the left double integral is the area of the unit square, while the right double integral is the area of our parallelogram. We thus find

$$\text{Area}(S) \;=\; \frac{1}{8}\text{Area}(T(S)) \;=\; \frac{1}{8}\text{Area}(P),$$

or equivalently that the area of the parallelogram is 8:

$$\text{Area}(P) \;=\; 8\text{Area}(S) \;=\; 8\cdot 1 \;=\; 8.$$

We could consider more general maps from squares to parallelograms, but this illustrates the principle and proves a nice, known result: the area of a parallelogram is its base times height. For our parallelogram, it has length 2 and height 4, which do multiply to give an area of 8.

Notice that we're able to deduce the formula for the parallelogram's area by knowing the area of the square *because* the absolute value of the determinant of the derivative matrix is constant (i.e., independent of u and v). This allows us to pull out that common factor of 1/8 and leaves us with the integral of 1 over the parallelogram, which is thus its area. Whenever we have a change of variables where the determinant is constant, these calculations can often allow us to deduce the area of one region from knowing another. For example, you can use this to find the area of an ellipse knowing the area of a circle.

For the ellipse

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2 \;\le\; 1,$$

consider the change of variables $u = x/a$ and $v = y/b$, so

$$T(x,y) \;=\; (u(x,y), v(x,y)) \;=\; (x/a, y/b).$$

Equivalently the inverse map T^{-1} is

$$T^{-1}(u,v) \;=\; (x(u,v), y(u,v)) \;=\; (au, bv).$$

Note this maps the ellipse to the unit disk

$$u^2 + v^2 \;\leq\; 1,$$

and we know the area of the unit disk is just $\pi 1^2 = \pi$! After a little bit of algebra, you'll find the area of the ellipse is πab.

Whenever we have an answer, a great way to check our work is to see if it agrees with previous results for special cases. For example, if we take $a = b = r$ in the ellipse problem above, it reduces to finding the area of a circle of radius r. We know that this is just πr^2, and as expected our formula πab equals this when $a = b = r$. While this isn't a proof that our formula for the ellipse's area is correct, it's suggestive that we're correct.

For our next problem, let D be the unit disk $x^2 + y^2 \leq 1$, and let's evaluate the integral

$$\int\int_D \cos(x^2 + y^2)dxdy.$$

This problem screams polar coordinates, as we're integrating over a circle and the integrand only depends on r, the distance from the center of the circle. We have $dxdy$ goes to $rdrd\theta$, and the unit disk becomes $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$. We replace $f(x,y)$ with $f(r\cos\theta, r\sin\theta)$, and thus find

$$\begin{aligned}
\int\int_D \cos(x^2+y^2)dxdy \;&=\; \int_{\theta=0}^{2\pi}\int_{r=0}^{1} \cos(r^2\cos^2\theta + r^2\sin^2\theta)rdrd\theta \\
&=\; \int_{\theta=0}^{2\pi}\int_{r=0}^{1} \cos(r^2)rdrd\theta \\
&=\; \int_{\theta=0}^{2\pi} \frac{1}{2}\left[\int_{r=0}^{1} \cos(r^2)2rdr\right] d\theta \\
&=\; \int_{\theta=0}^{2\pi} \frac{1}{2}\left[\sin(r^2)\Big|_0^1\right] d\theta \\
&=\; \int_{\theta=0}^{2\pi} \frac{\sin 1}{2} d\theta \\
&=\; \frac{\sin 1}{2}\cdot 2\pi \;=\; \pi\sin 1.
\end{aligned}$$

Consider the cylinder C given by $x^2 + y^2 \leq 9$ and $-1 \leq z \leq 2$. Evaluate

$$\int\int\int_C f(x,y,z)dxdydz$$

where

$$f(x,y,z) \;=\; z\sqrt{x^2+y^2}.$$

If we wanted to write down the integral explicitly in Cartesian coordinates, we would have

$$\int_{z=-1}^{2}\int_{y=-3}^{3}\int_{x=-\sqrt{9-y^2}}^{\sqrt{9-y^2}} z\sqrt{x^2+y^2}dxdydz.$$

To see this, note that on the boundary $x^2+y^2 = 9$, so if we have chosen a value of y then x ranges from $-\sqrt{9-y^2}$ to $\sqrt{9-y^2}$; these are not integrals we desire to evaluate! For cylindrical coordinates, we have

$$dxdydz \quad\longrightarrow\quad rdrd\theta dz,$$

and

$$x \;=\; r\cos\theta, \quad y \;=\; r\sin\theta, \quad z \;=\; z.$$

Our function $f(x, y, z)$ becomes $f(r\cos\theta, r\sin\theta, z)$, or in our case

$$z\sqrt{x^2+y^2} \quad\longrightarrow\quad z\sqrt{r^2\cos^2\theta + r^2\sin^2\theta} \;=\; zr.$$

The bounds of integration are z ranges from -1 to 2, θ ranges from 0 to 2π, and r ranges from 0 to 3. We thus have

$$\begin{aligned}
\int\int\int_C f(x,y,z)dxdydz \quad &= \quad \int_{z=-1}^{2}\int_{\theta=0}^{2\pi}\int_{r=0}^{3} f(r\cos\theta, r\sin\theta, z)rdrd\theta dz \\
&= \quad \int_{z=-1}^{2}\int_{\theta=0}^{2\pi}\int_{r=0}^{3} z\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}rdrd\theta dz \\
&= \quad \int_{z=-1}^{2}\int_{\theta=0}^{2\pi}\int_{r=0}^{3} zr\cdot rdrd\theta dz \\
&= \quad \int_{z=-1}^{2} z\int_{\theta=0}^{2\pi}\left[\int_{r=0}^{3} r^2dr\right]d\theta dz \\
&= \quad \int_{z=-1}^{2} z\int_{\theta=0}^{2\pi}\left[\left.\frac{r^3}{3}\right|_0^3\right]d\theta dz \\
&= \quad \int_{z=-1}^{2} z\int_{\theta=0}^{2\pi}\frac{27}{3}d\theta dz \\
&= \quad 9\int_{z=-1}^{2} z\left[\int_{\theta=0}^{2\pi} d\theta\right]dz \\
&= \quad 9\int_{z=-1}^{2} z2\pi dz \\
&= \quad 18\pi\int_{z=-1}^{2} zdz \\
&= \quad 18\pi\left.\frac{z^2}{2}\right|_{-1}^{2} \\
&= \quad 18\pi\left[\frac{4}{2}-\frac{1}{2}\right] \\
&= \quad 18\pi\cdot\frac{3}{2} \\
&= \quad 27\pi.
\end{aligned}$$

Finally, consider the unit sphere S given by $x^2 + y^2 + z^2 \leq 1$. Let's find

$$\int\int\int_S f(x, y, z) dx dy dz$$

for

$$f(x, y, z) \;=\; \frac{1}{(x^2 + y^2 + z^2)}.$$

Writing the integral out explicitly in Cartesian coordinates gives

$$\int_{z=-1}^{1} \int_{y=-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{x=-\sqrt{1-y^2-z^2}}^{\sqrt{1-y^2-z^2}} f(x, y, z) dx dy dz,$$

and these bounds of integration are horrible! We now convert to spherical coordinates:

$$x \;=\; \rho \sin\phi \cos\theta, \quad y \;=\; \rho \sin\phi \sin\theta, \quad z \;=\; \rho \cos\phi,$$

with

$$0 \;\leq\; \rho \;\leq\; 1, \quad 0 \;\leq\; \theta \;\leq\; 2\pi, \quad 0 \;\leq\; \phi \;\leq\; \pi.$$

Our function $f(x, y, z)$ becomes

$$f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi) \;=\; \frac{1}{\rho^2}$$

after some simple algebra. Finally,

$$dx dy dz \quad \longrightarrow \quad \rho^2 \sin\phi d\rho d\theta d\phi.$$

Note: remember, other textbooks change the role of θ and ϕ, especially physics books. We thus have

$$
\begin{aligned}
& \int\int\int_S f(x,y,z)dxdydz \\
= & \int_{\phi=0}^{\pi}\int_{\theta=0}^{2\pi}\int_{\rho=0}^{1} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi d\rho d\theta d\phi \\
= & \int_{\phi=0}^{\pi}\int_{\theta=0}^{2\pi}\int_{\rho=0}^{1} \frac{1}{\rho^2}\rho^2\sin\phi d\rho d\theta d\phi \\
= & \int_{\phi=0}^{\pi}\int_{\theta=0}^{2\pi}\int_{\rho=0}^{1} \sin\phi d\rho d\theta d\phi \\
= & \int_{\phi=0}^{\pi}\int_{\theta=0}^{2\pi} \sin\phi\left[\int_{\rho=0}^{1} d\rho\right] d\theta d\phi \\
= & \int_{\phi=0}^{\pi} \sin\phi\left[\int_{\theta=0}^{2\pi} d\theta\right] d\phi \\
= & \ 2\pi\int_{\phi=0}^{\pi} \sin\phi d\phi \\
= & \ 2\pi\left[-\cos\phi\Big|_0^{\pi}\right] \\
= & \ 2\pi\left[(-\cos\pi)-(-\cos 0)\right] \\
= & \ 2\pi\left(1+1\right) \\
= & \ 4\pi.
\end{aligned}
$$