Abstracts of Senior Theses at Williams College directed by Thomas Garrity

1990-1991

1. Andrew Beveridge, "Steps towards a Morse-Smale Algorithm"

   Our goal is to determine the rank of the homology groups of a real algebraic manifold M. In this exposition, we outline the steps needed in order to determine the homology of the manifold using a Morse-Smale flow. Our method employs a numerical solution to the Morse-Smale gradient flow of the height function. This reduces the problem to solving certain ordinary differential equations. One major assumption, found in section 4.2, must be made in order to solve these equations. We explicitly approximate the local coordinate system at a certain critical point given by the lemma of Morse. We also find an approximation for every orbit connecting a critical point p to a critical point q where the indices of these points satisfy $\lambda p - \lambda q = 1$. Further assumptions must be made in section 6.5 in order to use our approximations to determine the homology of the manifold.

2. Michael Donofrio,"Calculating the Intersection Number of Complementary-Dimensional Cycles"

   Given a manifold and two intersecting cycles of complementary dimension contained in the manifold, all defined by the complete intersections of polynomial equations, we provide an algorithm for calculating the intersection number of the cycles. The algorithm begins by finding as an orientation for the manifold a basis for its tangent space at any point on the manifold. We then perform a similar construction for the tangent spaces of the intersecting cycles at their intersection points. If the intersection point is transverse then we assign to it either a positive or negative orientation. If the point is not transverse, then the orientation of the cycles at that point is not well-defined. We present a method of perturbing one of the cycles so as to insure that the new cycle will intersect the unperturbed cycle transversely around the original non-transverse intersection, thus enabling us to assign an orientation to that intersection.

3. Erin Lagesen, "On the Projective Equivalency of Curves and Surfaces"

   This thesis constructs a theoretical method by which to determine (1) if two curves in the complex-projective plane are projectively equivalent and (2) if two generic smooth irreducible surfaces in complex-projective 3-space are projectively equivalent. We will accomplish this by looking for certain points on the curves, respectively surfaces, which possess distinguishing properties which remain invariant under projective change of coordinates. Knowing that such points on one curve, resp. surface, would necessarily map to the same such points of a projectively equivalent curve or surface, we will use these points to set up all of the projective linear transformations that would possibly map one to the other. We will then test the projective transformations on the curve or surfaces to determine whether they are projectively equivalent.

1991-1992

1. Elizabeth Gibbons,"On Computing the Orientability of Algebraic Manifolds"

   This thesis presents an algorithm which determines in simple exponential time if a given manifold embedded in Rn and described by rational polynomials is orientable, and assigns an orientation to the manifold if it is. For manifolds that are complete intersections, the problem is trivial. Given a manifold that is not, the algorithm finds pieces of the manifold that are complete intersections, which we shall call "chunks", which cover the manifold. Determining whether or not it is possible to consistently orient each of these chunks so that all orientations agree is equivalent to determining whether the manifold is orientable.

2. Amy Huston, "On Whitney Stratifications for Semi-Algebraic Sets"

   Sufficient background is given to understand a number of formulations of the Whitney Conditions. Several versions of the Whitney Conditions are presented and discussed. Time bounds are given to a known stratification algorithm. Also given are possible directions for future work in trying to create an efficient algorithm to Whitney stratify a set.

3. Cherie Macauley, "The Complexity Bounds of the Multivariate Sign Sequence Algorithm"

   We provide complexity bounds for John Canny's multivariate sign sequence algorithm. This is an algorithm for determining the signs of the polynomials

$$f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)$$

   at the isolated roots of the system of polynomial equations

$$P_1(x_1, \ldots, x_n), \ldots, P_k(x_1, ..., x_n).$$

   We find that the majority of the steps of the algorithm can be reduce to determinant calculations and hence, the complexity bounds of computing a determinant.

1994-1995

1. David Dela Cruz, "Analysis of Manifolds Using Morse-Smale Homololgy"

   A topological object can be defined as a set of points in a space. A particular beautiful topological object to examine is the manifold. Given two n-manifolds, it is interesting to know if they are topologically equivalent- that is, whether we can bend, twist and stretch one to get the other. The topological invariants of two manifolds must be the same if the two manifolds are the same under topology. One such invariant is the homology groups. Singular homology theory describes homology groups for manifolds in the abstract. Morse-Smale homology allows us to capture homology groups algebraically as well. This thesis is an exposition of singular homology theory, Morse theory, and Morse-Smale homology, followed by an algorithm that can be used to approximate Morse-Smale gradiant flows on algebraically-defined manifolds.

2. Christopher French, "Computing the Intersection Homology Groups of a Complex Algebraic Variety"

   An algorithm is given to compute the intersection homology groups for a complex algebraic variety. Two previously developed algorithms, the Collin's CAD Algorithm and Prill's Adjacency Algorithm, are presented and used.

3. Dimitry Korsunsky, "One Approach to Factoring Multivariate Rational Polynomials over the Complex Numbers"

   Factoring a given multivariate polynomial is an important task in symbolic computation. Potential uses for an efficient solution to this problem could be found in various branches of applied mathematics, such as computer-aided design and theorem proving. Several algorithms giving different methods for factoring multivariate polynomials had been created over the years (Noether 1922, Davenport and Trager 1981, Christov and Grigoryev 1983, etc.). The theoretical basis for the Bajaj, Canny, Garrity and Warren algorithm is topological in nature. There it is proved that the suggested approach when implemented in parallel will execute in shorter time as compared to earlier solutions. Although a sequential solution which was implemented achieves lower efficiency there is a significant advantage in that it can be used in a large variety of settings. The program had been written using the Mathematica software package.

1995-1996

1. Daniel Ebert, "Probabilistic Enumerative Geometry: How many inflection points are real"

   Classically, enumerative or counting questions have been answered in complex projective space, where the answers are precise. In this project, we ask these enumerative questions in real affine space; in real space the answers may no longer be precise. For instance, changing the coefficients of an nth degree polynomial gives a different number of roots. So, we must answer these enumerative questions in real projective space probabilistically, finding the expected, not exact, number. After examining Edeleman and Kostlans exposition and expansion on Kacs formula for the expected number of real roots of a random polynomial, the thesis focuses on trying to find the number of real inflection points on an nth degree polynomial. Interestingly enough, through Mathematica calculations, the expected number of real inflection points for a quartic and quintic is a little less than twice that of the expected number of real roots for an mth degree polynomial where m is the degree of the resultant of the Hessian curve with the original quartic or quintic polynomial curve.

2. Daniel Kim, "Examining G^k Continuity Along Tubular Neighborhoods"

   We use concepts of geometric continuity to develop a relationship between manifolds and tubular neighborhoods. Specifically, we examine a pair of C^k manifolds of dimension r whose intersection is along a C^k manifold of dimension r-1; if the r-manifolds meet with G^k continuity, then the boundaries of their corresponding tubular neighborhoods intersect with G^{k-1} continuity. We also discuss possible ways to extend the scope of this research.

3. Alexander Meadows, "The construction of the Moduli Space of Quadrilateral up to Similarity"

   A method for constructing a space of quadrilaterals modulo similarity is provided. We begin with several methods of constructing a space of triangles up to similarity. Then we construct the moduli space of quadrilaterals up to similarity using both algebraic and geometric methods.

1998-1999

1. Zachary Grossman, "Relations and Syzygies in Classical Invariant Theory for Vector-Valued Bilinear Forms"

   The goal of invariant theory is to describe the algebra of invariants for a vector space under a given group action. We examine the First Fundamental Theorems for the general linear group and for vector-valued linear forms, which describe the invariant rings for their respective vector space. We also examine the Second Fundamental Theorem for the general linear group, which describes the basis relations between the invariants of that group. The main theorem of the paper, the Second Fundamental Theorem for vector-valued bilinear forms, uses the Kunneth formula to describe the basis relations between invariants of vector-valued bilinear forms and to provide a method for finding the resolution of the entire syzygy chain.

1999-2000

1. Tegan Cheslack-Postava, "Questions of Uniqueness for Triangle Sequences in m Dimensions"

   The present paper focuses on the question of uniqueness-when does an m-dimensional triangle sequence correspond to a unique point in Rn. It is known in the two dimensional case that there are infinite sequences corresponding to entire line segments and that particular characteristics of the sequences will guarantee to identify unique points. We will extend both of these results to higher dimensions.

2. Adam Schuyler, "A Structural Analysis of the Triangle Iteration"

   Classically, it is known that the continued fraction sequence for a real number is eventually periodic if and only if the number is a quadratic irrational. In response to this, Hermite posed the general question which asks for ways of representing numbers that reflect special algebraic properties. Specifically, he was inquiring about possible generalizations of the continued fraction. In this paper we will study the triangle iteration, a two-dimensional analogue of the continued fraction. We will take a primarily geometric approach and look at the probabilities of the occurrences of certain sequences.

2002-2003

1. Michael Baiocchi, "Triangle Sequence Analogs to Pell's Equation"

   The solutions to $x^2 - dy^2 = 1$, the Pell equation, are intimately bound up with the periodicity of the continued fraction expansion of the square root of d. We present a higher dimensional analog to the Pell equation which is bound up the with the periodicity of the multi-dimensional continued fractions known as triangle sequences. We will show that this analog has many of the characteristics of the original Pell equation: infinite integral solutions determined by the expansion of a number, finitely generated group structure on the solutions sets, and a strong connection to the units of a given field.

2. Edvard Major, "Phase Transitions of a Generalized Farey Number Theoretical Chain"

   This thesis briefly reviews basic concepts of statistical mechanics. A detailed exposition of the Farey Number-Theoretical Chain (FTNC) model is provided. Critical phenomenon of this statistical-mechanical model is further discussed.

   The Knauf Number-Theoretical Chain (KNTC) model is revisited, and an elegant new proof of exact phase transition location is provided.

   A couple of new two-dimensional number sequence models that assume Knauf-like, denominator interactions are proposed. The first one is based on a sequence introduced by Von Rudolf Monkemeyer and D. Grabiner. The existence of the model's phase transition is verified.

   To construct the remaining models, a couple of new continued fraction R2 algorithm-generalizations are proposed, and their properties analyzed. The existence of respective phase transitions is proved.

3. Mark Rothlisberger, "Generalized Continued Fractions"

   Every real number has a continued fraction expansion which can be developed in several ways. Continued fractions are closely tied to distinguishing quadratic irrationals and determining properties of the algebraic number fields that they determine. The generalized continued fractions that we develop and investigate will follow the approach of Minkowski by using convex bodies in R3, namely parallelpipeds, to approximate certain vectors and planes. These methods will resemble the geometric development of continued fractions, and we will demonstrate that some of the results from continued fractions generalize as a result of this method. Two approaches will be given: the first is not original; the same approach is contained in Minkowski's A Criterion for Algebraic Numbers. The second generalization is original in the choice of parallelpiped, but employs methods from Minkowski's The Theory of Continued Fractions. We will examine the connection between the two generalizations.

2003-2004

1. Christopher Calfee, "Triangle Sequence Revisited: An In-Depth Look at Triangle Iterations"

   Purely periodic triangle sequences correspond to cubic irrationals alpha and beta. We will show a variety of methods for finding the irreducible cubic polynomials corresponding to both alpha and beta. Finally, we will explore some of the interesting polynomials which emerge from the sequences that are purely periodic of periodicity length one.

2. Andrew Marder, "The Farey-Bary Map Revisited"

   Two generalizations of the Minkowski ?(x) function are given. As ?(x) maps quadratic irrationals to rational numbers, it is shown that both generalizations send natural classes of pairs of cubic irrational numbers in the same cubic number field to pairs of rational numbers. It is also shown that these functions satisfy an analog to the fact that ?(x), while continuous and increasing, has derivative zero almost everywhere. Both extend earlier work of Beaver-Garrity on the Farey-Bary map.

2006-2007

1. Shea Daniel Chen, "Growth and Combinatorial Properties of the Triangle Sequence"

   Triangle sequences are a type of multi-dimensional continued fraction. We investigate growth rates of the denominators in triangle sequences, in analog to the growth rates of the denominators in continued fractions. In particular we look at the analog of the Euler totient function for triangle sequences, which gives us the number of points in the triangle sequence given a denominator. We also study the distribution of denominators for special sets of triangle sequences. Finally, we present a combinatorial representation for triangle sequences.

2007-2008

1. Son L. Ho, "Preliminary Work on Manifolds in Higher Codimension Embedded in $\mathbf{R}^{n+k}$"

   Our main goal is to build a machinery to work with manifolds of higher codimension in $\mathbf{R}^{n+k}$. We take a look at the map from the manifold M to a polynomial space, defined by using second fundamental form matrices. The multiplicity of roots of polynomials in (M) are preserved under various changes of coordinates. This leads us to believe that the map captures interesting geometric information of M. And in the last section we prove a result which indicates that it is the case for a certain type of codimension 2 manifolds.

2. Haydee Lindo, “On Computing the Intersection of Algebraic Hypersurfaces”

   A Groebner basis approach is developed for projecting the intersection of two algebraic hypersurfaces to a hypersurface. In terms of the Groebner basis, an algorithm is given for producing the rational map from the projected hypersurface back to the original intersection.

3. Amy Steele, “On Panti’s Generalization of the n-Dimensional Minkowski Question-Mark Function”

   A real number x is a quadratic irrational if and only if it has an eventually periodic continued fraction expansion. This property led Hermann Minkowski to construct a function that can be seen as the confrontation of regular continued fractions and the alternated dyadic system within [0,1]. The function has zero derivative almost everywhere, and is continuous and strictly increasing. In this, we discuss the n-dimensional analogue of Minkowski’s function as defined by Giovanni Panti.

4. Paul Alexander Woodard, “On Equivalence Relations on Sequence Spaces”

   Given a sequence space S, we can define an equivalence relation $\sim_X$ on S by setting

   $$(x_n) \sim_X (y_n)$$

   for sequences $(x_n), (y_n) \in S$ if the sequence $(y_n - x_n)$ is in X, where X is a subspace of S. Examples include spaces such as l_i, the space of absolutely summable sequences, and c_0, the space of sequences converging to 0. The quotient space $S/\sim_X$ is also a vector space, so we can study the linear functionals which act on it. To this end, we examine infinite matrices whose rows, as elements of the dual space of Xconverge weak* to (0).

2011-2012

1. Noah Goldberg, “Monkmeyer Map Analogues to Stern’s Diatomic Sequence”

   Stern’s Diatomic Sequence is a well-studied sequence of integers which stems from continued fractions. The Monkmeyer Map is a type of multidimensional continued fraction. We will examine an analogue of Stern’s Diatomic Sequence for the Monkmeyer Map.

2. Stephanie Jensen, “Ergodic Properties of TRIP Maps: A Family of Multidimensional Continued Fractions”

   We study the ergodic properties of several of the most relevant TRIP maps, as family of multidimensional continued fractions that encompasses many well-known algorithms. As a first step, we show these maps converge almost everywhere. From there, we are able to prove ergodicity.