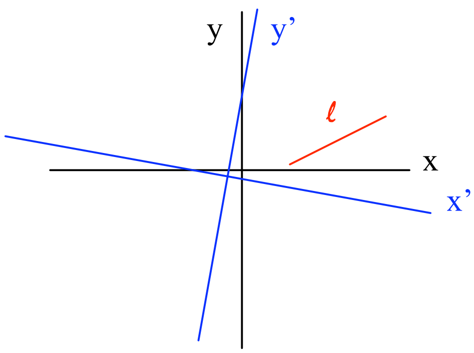
Metrics and Geometry

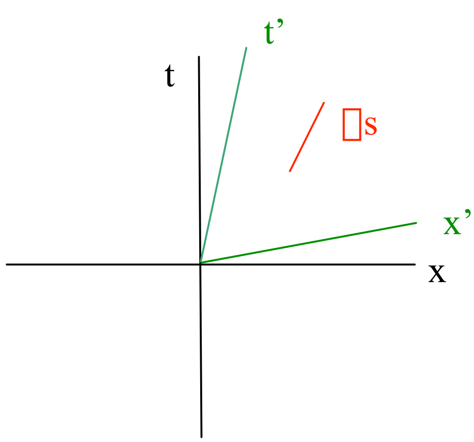
What's a **metric**? A metric is no more than a rule that describes how to measure lengths. Just as you might have been surprised to discover that you have been speaking prose all your life, you have also been using the Euclidean metric every time you did a calculation with the Pythagorean theorem! Here's an example in the familiar x-y plane of Euclidean geometry. The example below shows that the length of a line segment, ℓ, is *invariant*, or in other words, independent of the orientation or origin of the coordinate grid in which it is measured.



$$\ell^2 = \Box x^2 + \Box y^2 = \Box x'^2 + \Box y'^2$$

The length of ℓ is *INVARIANT*.

Here is the **Minkowski spacetime metric,** valid for a flat universe (k=0), shown in only one spatial dimension for simplicity. Remember that in special relativity, for a moving frame the coordinate axes representing space and time tilt toward the 45º light-traveling line. The graph below illustrates that the length of the spacetime interval, $\Box s$, is *invariant*, or in other words, independent of the particular inertial frame in which it is measured.



$$\Box s^2 = (c\Box t)^2 - (\Box x^2)$$
$$= (c\Box t')^2 - (\Box x'^2)$$

The length of $\Box s$ is *INVARIANT*.

Now, if we allow the universe to expand with time, and if we let x be a *co-moving coordinate*, (like an address that doesn't change), then we can use the **scale factor, R(t),** to describe the expansion**:**

$$\Delta s^2 = (c\Delta t)^2 - \mathbf{R^2(t)}(\Delta x^2)$$

Finally, here is the most general form of the metric that is compatible with general relativity in a homogeneous and isotropic universe, shown in spherical coordinates, which happen to be the most convenient. The most important thing to see in this equation is the introduction of the **curvature constant, k.**

$$\Delta s^2 = (c\Delta t)^2 - R^2(t)\left[\frac{\Delta r^2}{1-\mathbf{k}r^2} + \Delta\theta^2 + \sin^2\theta\,\Delta\phi^2\right]$$

The Robertson-Walker Metric

What are the consequences of different values for k?

k = 0 → no curvature – a flat universe

k = +1 → positive curvature – analogous to the surface of a sphere

k = -1 → negative curvature – analogous to the surface of a saddle, or hyperboloid

For diagrams and more description, see http://scholar.uwinnipeg.ca/courses/38/4500.6-001/Cosmology/Properties-of-Space.htm