LOW-LYING ZEROS OF DIRICHLET L-FUNCTIONS

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Abstract. We study the distribution of the zeros near the central point for weighted and unweighted families of Dirichlet L-functions. As the conductors tend to infinity, the main term of the 1-level densities agrees with the scaling limit of unitary matrices for even C^2 test functions whose Fourier transforms are supported in $(-2, 2)$, supporting the Katz-Sarnak conjecture. The lower order terms agree with the prediction from the L-function Ratios Conjecture in the regime where both can be computed, though we are able to compute beyond square-root cancelation, thus going further than what the Ratios Conjecture can predict. We also investigate the consequences of conjectures about the modulus dependence in the error terms in the distribution of primes in residue classes. We show how some natural conjectures imply that the 1-level densities agree with unitary matrices for arbitrary support, while some weaker conjectures still give an improvement over $(-2, 2)$, allowing support up to $(-4, 4)$.

Contents

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1. Introduction

1.1. **Previous Results.** Assuming GRH, the non-trivial zeros of any nice L-function lie on its critical line, and therefore it is possible to investigate the statistics of its normalized zeros. The spacing statistics of these zeros are fundamental in many problems, ranging from the distribution of primes in congruence classes to the class number [CI, Go, GZ, RubSa]. Numerical and theoretical evidence [Hej, Mon2, Od1, Od2, RS] support a universality in behavior of zeros of an individual automorphic L-function high above the central point, specifically that they are well-modeled by ensembles of random matrices (see [FM, Ha] for histories of the emergence of random matrix theory in number theory). The story is different near the central point. In this case, an individual L-function no longer provides enough zeros to average over, and one instead studies families of L-functions. The Katz and Sarnak Density Conjecture states that the behavior of zeros near the central point, in the limit as the conductors tend to infinity, agrees with the scaling limit of eigenvalues near 1 of a classical compact group.

A convenient way to study these low-lying zeros is via the 1-level density, which we now describe. Let ϕ be an even Schwartz class test function on $\mathbb{R}$ whose Fourier transform

$$\widehat{\phi}(y) \ = \ \int_{-\infty}^{\infty} \phi(x) e^{-2\pi i x y} dx \tag{1.1}$$

has compact support. Let $\mathcal{F}_N$ be a (finite) family of L-functions satisfying GRH.[1] The 1-level density associated to $\mathcal{F}_N$ is defined by

$$D_{1;\mathcal{F}_N}(\phi) \ = \ \frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} \sum_j \phi\left(\frac{\log c_f}{2\pi} \gamma_f^{(j)}\right), \tag{1.2}$$

where $\frac{1}{2} + i\gamma_f^{(j)}$ runs through the non-trivial zeros of $L(s,f)$. Here c_f is the "analytic conductor" of f, and gives the natural scale for the low zeros. As ϕ is Schwartz, only low-lying zeros (i.e., zeros within a distance $1/\log c_f$ of the central point $s = 1/2$) contribute significantly. Thus the 1-level density can help identify the symmetry type of the family. To evaluate (1.2), one applies the explicit formula, converting sums over zeros to sums over primes.

Based in part on the function-field analysis where $G(\mathcal{F})$ is the monodromy group associated to the family $\mathcal{F}$, Katz and Sarnak conjectured that for each reasonable irreducible family of L-functions there is an associated symmetry group $G(\mathcal{F})$ (one of the following five: unitary U, symplectic USp, orthogonal O, SO(even), SO(odd)), and that the distribution of critical zeros near 1/2 mirrors the distribution of eigenvalues near 1. The five groups have distinguishable 1-level densities. To date, for suitably restricted test functions the main terms from number theory have been shown to agree with random matrix theory for many families, including Dirichlet L-functions, elliptic curves, cuspidal newforms, Maass forms, number field L-functions, and symmetric powers of GL_2 automorphic representations [AILMZ, DM1, FI, Gao, Gü, HM, HR, ILS, KaSa1, KaSa2, Mil1, MilPe, RR, Ro, Rub, Ya, Yo2], to name a few, as well as non-simple families formed by Rankin-Selberg convolution [DM2].

While Random Matrix Theory [Con, KaSa1, KaSa2, KeSn1, KeSn2, KeSn3] has done an excellent job predicting the main term, it suffers from a serious defect in that it cannot see the

[1] We often do not need GRH for the analysis, but only to interpret the results. If the GRH is true, the zeros lie on the critical line and can be ordered, which suggests the possibility of a spectral interpretation.

arithmetic in families, which surfaces as lower order terms. These lower order terms can be isolated in many families [HKS, HMM, Mil2, Yo1], and thus another theory is needed which is capable of making more detailed predictions. Recently the L-function Ratios Conjecture [CFZ1, CFZ2] has had great success in determining lower order terms.[2] To study the 1-level density, it suffices to obtain good estimates for

$$R_{\mathcal{F}_N}(\alpha,\gamma) \ := \ \frac{1}{|\mathcal{F}_N|}\sum_{f\in\mathcal{F}_N}\frac{L(1/2+\alpha,f)}{L(1/2+\gamma,f)}. \tag{1.3}$$

The conjectured formulas are believed to hold up to errors of size $O(|\mathcal{F}_N|^{-1/2+\epsilon})$. We quote the recipe from [GJMMNPP] (see also [CS1] for an accessible overview of the Ratios Conjecture).

(1) Use the approximate functional equation to expand the numerator into two sums plus a remainder, ignoring the remainder term.
(2) Expand the denominator by using the generalized Mobius function.
(3) Execute the sum over $\mathcal{F}(q)$, keeping only main (diagonal) terms; however, before executing these sums replace any product over epsilon factors (arising from the signs of the functional equations) with the average value of the sign of the functional equation in the family.[3]
(4) Extend the m and n sums to infinity (i.e., complete the products).
(5) Differentiate with respect to the parameters, and note that the size of the error term does not significantly change upon differentiating.[4]
(6) A contour integral involving $\frac{\partial}{\partial\alpha}R_{\mathcal{F}(q)}(\alpha,\gamma)|_{\alpha=\gamma=s}$ yields the 1-level density.

While steps (1), (3) and (4) provably can involve adding or discarding terms as large as the main term, all the errors seem to cancel and the resulting prediction is believed to be correct to almost square-root cancelation in the family size $|\mathcal{F}_N|$, significantly beyond the first order correction terms of size $1/\log c_f$ (where c_f is the conductor). For suitably restricted test functions, these predictions have been confirmed for a variety of families [CS1, CS2, GJMMNPP, HKS, HMM, Mil3, Mil4, MilMo], sometimes down to $|\mathcal{F}_N|^{\epsilon-1/2}$, sometimes only to a power savings.

A major goal of this work was to verify these detailed predictions for our family; interesting, we are not only able to verify these predictions, but we are able to determine the 1-level density *beyond* what even the Ratios Conjecture predicts!

1.2. **Results (With and Without GRH).** We study families of Dirichlet L-functions below. Özlük and C. Snyder [OS1, OS2] and Hughes and Rudnick [HR] show that for certain families of primitive characters, the 1-level density agrees with unitary matrices for test functions ϕ with $\text{supp}(\widehat{\phi}) \in [-2,2]$ (see also [BFHR, FiMa, RubSa] for more on zeros of Dirichlet L-functions, including applications to Chebyshev's bias, where more than GRH is often assumed). Our goal is to show how reasonable conjectures allow us to increase the

[2]Another promising approach is through hybrid models [GHK], which combine the Euler product of an L-function (which encodes the arithmetic) with the Hadamard product of its zeros (which is thought to behave similarly as random matrix ensembles).

[3]One may weaken the Ratios Conjecture by not discarding these terms; this is done in [Mil4, MilMo], where as predicted it is found that these terms do not contribute.

[4]There is no error in this step, which can be justified by elementary complex analysis because all terms under consideration are analytic. See Remark 2.2 of [Mil4] for details.

support, as well as to compute the lower order terms as far as possible, thus discovering the arithmetic dependence. In this regard our work is similar to [ILS], where they show that if a classical exponential sum over primes has some cancelation, then the 1-level density of weight k level 1 cusp forms (split by sign) agrees with the corresponding orthogonal group for $\operatorname{supp}(\widehat{\phi}) \subset (-22/9, 22/9)$. For us, the corresponding quantities involve the modulus dependence in the error terms in primes in residue classes, and relates how natural conjectures on the distributions of primes can be used to provide further support for the density conjectures.

Specifically, we study the 1-level density of non-principal Dirichlet L-functions with modulus q. We adopt a slightly more general setting than usual, that is we let f be an even C^2 function with compact support (which will play the role of $\hat{\phi}$). Denoting by $\rho_\chi = \frac{1}{2} + i\gamma_\chi$ the non-trivial zeros of $L(s,\chi)$ (i.e., $0 < \Re(\rho_\chi) < 1$) and choosing Q a scaling parameter close to q, the 1-level density is[5]

$$D_{1;q}(\widehat{f}) \ := \ \frac{1}{\phi(q)} \sum_{\chi \bmod q} \sum_{\gamma_\chi} \widehat{f}\left(\gamma_\chi \frac{\log Q}{2\pi}\right); \tag{1.4}$$

throughout this paper a sum over $\chi \bmod q$ *always means a sum over all characters, including the principal character.* If we assume GRH, then the γ_χ are real. As $\widehat{f}(y)$ is defined for complex values of y, it makes sense to consider (1.4) for complex γ_χ, in case GRH is false (in other words, GRH is only needed to interpret the 1-level density as a spacing statistic arising from an ordered sequence of real numbers; see Footnote 1). We also study the average of (1.4) over the moduli $Q/2 < q \le Q$, which is easier to understand in general:

$$D_{1;Q/2,Q}(\widehat{f}) \ := \ \frac{1}{Q/2} \sum_{Q/2<q\le Q} D_{1;q}(\widehat{f}). \tag{1.5}$$

Note that we are normalizing all of the zeros by the same factor, $\frac{\log Q}{2\pi}$. This corresponds to the global rescaling, and facilitates performing averaging over the family as we can move the family sum past the test function to the arithmetic coefficients.[6]

Remark 1.1. *In many papers in the literature, the roles of f and $\widehat{f}$ are reversed, with the sum in* (1.4) *being the test function evaluated at the scaled zeros, and not the Fourier transform. As most of our computations are on the prime side, we prefer to evaluate the scaled zeros at the Fourier transform of the test function, and have the test function on the arithmetic side.*

Our results depend on the support of f. The larger $\sigma := \sup(\operatorname{supp} f)$ is, the stronger the needed hypotheses. We also exclude the trivial case by assuming that $\sigma > 0$.

By restricting the support of f to $[-1,1]$, we are able to obtain unconditional results. Interestingly, we are able to compute the lower-order terms incredibly well, going beyond what the Ratios Conjecture can predict. These lower-order terms will be studied separately in §3.

[5]Since f is C^2, we have that $\hat{f}(\xi) \ll \xi^{-2}$ for large ξ, hence the sum over the zeros is absolutely convergent.

[6]It is possible to consider the local rescaling, where for a character of conductor q we instead use $(\log q)/2\pi$. If we were studying the 2-level density, this would be necessary. See [Mil1] for more on local versus global rescaling.

Theorem 1.2. *Suppose that f is supported on the interval $[-1,1]$, so $\sigma \le 1$. There exists a positive constant c such that the 1-level density $D_{1;q}(\widehat{f})$ (from (1.4) with scaling parameter $Q = q$) equals*

$$f(0)\left(1 - \frac{\log(8\pi e^{\gamma})}{\log q} - \frac{\sum_{p|q}\frac{\log p}{p-1}}{\log q}\right) + \int_0^{\infty} \frac{f(0)-f(t)}{q^{t/2}-q^{-t/2}}dt$$
$$- \frac{2}{\log q}\sum_{\substack{p^{\nu}\|q \\ p^e\equiv 1 \bmod q/p^{\nu} \\ e,\nu\ge 1}} \frac{\log p}{\phi(p^{\nu})p^{e/2}} f\left(\frac{\log p^e}{\log q}\right) - \frac{2}{\phi(q)}\int_0^1 q^{u/2}\left(\frac{f(u)}{2} - \frac{f'(u)}{\log q}\right)du$$
$$+ O\left(\frac{q^{\frac{\sigma}{2}-1}}{e^{c\sqrt{\sigma\log q}}}\right). \tag{1.6}$$

(One could replace $q^{t/2} - q^{-t/2}$ by $2\sinh((t/2)\log q)$ in the first integral above.)

Remark 1.3. *As the size of the family is $\phi(q) \gg q/\log\log q$, the second integral in (1.6) is $O(q^{\sigma/2-1}\log\log q/\log q)$, and is thus smaller than the square-root cancelation predictable by the Ratios Conjecture.*

This should be compared to the main result of Goes, Jackson, Miller, Montague, Ninsuwan, Peckner and Pham [GJMMNPP], where they show one can extend the support of f to $[-2,2]$ and still get the main term, as well as the lower order terms down to a power savings.

Theorem 1.4 (Goes, Jackson, Miller, Montague, Ninsuwan, Peckner, Pham)**.** *If $1 < \sigma \le 2$, then the 1-level density $D_{1;q}(\widehat{f})$ (from (1.4) with scaling parameter $Q = q$) equals*

$$f(0)\left(1 - \frac{\log(8\pi e^{\gamma})}{\log q} - \frac{\sum_{p|q}\frac{\log p}{p-1}}{\log q}\right) + \int_0^{\infty} \frac{f(0)-f(t)}{q^{t/2}-q^{-t/2}}dt + o\left(q^{\frac{\sigma}{2}-1}\right), \tag{1.7}$$

and this agrees with the Ratios Conjecture's prediction up to an error of size $O(q^{-1/2+\epsilon} + q^{\sigma/2-1+\epsilon})$.

Remark 1.5. *Goes et al.* [GJMMNPP] *actually proved* (1.7) *for any $\sigma \le 2$, with the additional error term $O(q^{-1/2+\epsilon})$. We prefered not to include the case $\sigma \le 1$, as Theorem 1.2 is more precise in this range.*

Remark 1.6. *Note that in* (1.7) *we get the asymptotic for any $\sigma \le 2$ (even $\sigma = 2$).*

Our next results are conditional on GRH. While assuming GRH alone is not sufficient to understand the asymptotic behavior of the 1-level density for f with unrestricted support, it does extend the support to $\sigma > 1$. We again find interesting lower-order terms, beyond what the Ratios Conjecture can predict.

Theorem 1.7. *Assume GRH.*

(1) *If f is supported in $(-2,-a] \cup [-1,1] \cup [a,2)$ for some $1 \le a < 2$ (if $a = 1$, then we have the full interval $(-2,2)$), then the average 1-level density, $D_{1;Q/2,Q}(\widehat{f})$, equals*

$$\frac{f(0)}{\log Q}\left(\log Q - 1 - \gamma - \log(4\pi) - \sum_p \frac{\log p}{p(p-1)}\right) + \int_0^\infty \frac{f(0)-f(t)}{Q^{t/2}-Q^{-t/2}}dt$$
$$- \frac{4\log 2}{Q}\frac{\zeta(2)\zeta(3)}{\zeta(6)}\int_0^1 Q^{u/2}\left(\frac{f(u)}{2} - \frac{f'(u)}{\log Q}\right)du + O(Q^{-\frac{a}{2}} + Q^{\sigma-2}\log Q). \quad (1.8)$$

Unless $a > 1$ and $\sigma < \frac{3}{2}$, the third term of (1.8) goes in the error term.

(2) *If f is supported in $(-\frac{3}{2}, -1-\kappa] \cup [-1,1] \cup [1+\kappa, \frac{3}{2})$ for some $\kappa > 0$, then for any $\epsilon > 0$ we have that $D_{1;Q/2,Q}(\widehat{f})$ equals*

$$\frac{f(0)}{\log Q}\left(\log Q - 1 - \gamma - \log(4\pi) - \sum_p \frac{\log p}{p(p-1)}\right) + \int_0^\infty \frac{f(0)-f(t)}{Q^{t/2}-Q^{-t/2}}dt$$
$$- \frac{4\log 2}{Q}\frac{\zeta(2)\zeta(3)}{\zeta(6)}\int_0^1 Q^{u/2}\left(\frac{f(u)}{2} - \frac{f'(u)}{\log Q}\right)du$$
$$- \int_{1+\kappa}^{4/3}((u-1)\log Q + C_6)Q^{-u/2}\left(\frac{f(u)}{2} - \frac{f'(u)}{\log Q}\right)du$$
$$+ O_\epsilon(Q^{-\frac{1}{2}-\kappa+\epsilon} + Q^{-\frac{2}{3}}\log Q + Q^{\sigma-2}\log Q), \quad (1.9)$$

with $C_6 := \log(\pi/2) + 1 + \gamma + \sum_p \frac{\log p}{p(p-1)}$.
Note that for $\sigma \ge \frac{4}{3}$, unless $f(x)$ has some mass near $x = \lambda$ for some $1 < \lambda < 4-2\sigma$, the fourth term in (1.9) goes in the error term (and hence (1.9) reduces to (1.8)). However, if $1 < \sigma < \frac{4}{3}$, it is always a genuine lower-order term.

(3) *Finally, if f is supported in $(-\frac{3}{2}, \frac{3}{2})$, then $D_{1;Q/2,Q}(\widehat{f})$ equals*

$$\frac{f(0)}{\log Q}\left(\log Q - 1 - \gamma - \log(4\pi) - \sum_p \frac{\log p}{p(p-1)}\right) + \int_0^\infty \frac{f(0)-f(t)}{Q^{t/2}-Q^{-t/2}}dt + \frac{Q^{-1/2}}{\log Q}S_f(Q), \quad (1.10)$$

where

$$S_f(Q) = (2-\sqrt{2})Q^{-1/2}\zeta\left(\frac{1}{2}\right)\prod_p\left(1 + \frac{1}{(p-1)p^{1/2}}\right) \times$$
$$\left(f(1) + \left(\frac{\sqrt{2}+4}{3} - \left(\frac{\zeta'}{\zeta}\left(\frac{1}{2}\right) - \sum_p \frac{\log p}{(p-1)p^{1/2}+1}\right)\right)\frac{f'(1)}{\log Q}\right) + O\left(\frac{(\log\log Q)^2}{(\log Q)^2}\right). \quad (1.11)$$

Remark 1.8. *All the action is near the point $u = 1$, as this is where the primes in arithmetic progressions congruent to 1 modulo q first appear (see (2.1) for an explicit statement of how these primes enter the computations).*

Remark 1.9. *If f is $K+1$ times continuously differentiable, then we can evaluate $S_f(Q)$ more precisely:*

$$S_f(Q) \;=\; \sum_{i=0}^{K} \frac{a_i(f)}{(\log Q)^i} + O_{\epsilon,K}\left(\frac{Q^{-1/2}}{(\log Q)^{K+1-\epsilon}}\right), \tag{1.12}$$

where the $a_i(f)$ are constants depending (linearly) on the Taylor coefficients of $f(t)$ at $t=1$. In fact, $S_f(Q)$ is a truncated linear functional supported on $\{1\}$ (in the sense of distributions).

Remark 1.10. *Parts (1) and (2) of Theorem 1.7 depend heavily on the support of f, since for example in part (2), the last term is of order $Q^{-\sigma/2-o(1)}$. In part (3), however, $S_f(Q)$ depends only on the Taylor coefficients of $f(x)$ at $x=1$, and hence (3) is in a sense much more natural. For this reason, we believe that (3) actually uncovers the terms of order $Q^{-1/2}/(\log Q)^k$, whatever the support of f is, and suggests the following conjecture.*

Conjecture 1.11. *Theorem 1.7 (3) holds for f with arbitrarily large finite support σ.*

1.3. **Results (Beyond GRH).** As the GRH is insufficient to compute the 1-level density for test functions supported beyond $[-2,2]$, we explore the consequences of other standard conjectures in number theory involving the distribution of primes among residue classes. Before stating these conjectures, we first set the notation. Let

$$\psi(x) \;:=\; \sum_{n\le x}\Lambda(n), \qquad \psi(x,q,a) \;:=\; \sum_{\substack{n\le x\\ n\equiv a \bmod q}} \Lambda(n), \tag{1.13}$$

$$E(x,q,a) \;:=\; \psi(x,q,a) - \frac{\psi(x)}{\phi(q)}. \tag{1.14}$$

If we assume GRH, we have that

$$\begin{aligned} \psi(x) \;&=\; x + O\left(x^{1/2}(\log x)^2\right) \\ E(x,q,a) \;&=\; O\left(x^{1/2}(\log x)^2\right). \end{aligned} \tag{1.15}$$

Our first result uses GRH and the following de-averaging hypothesis, which depends on a parameter $\eta\in[0,1]$.

Hypothesis 1.11$_\eta$. *We have*

$$\sum_{Q/2<q\le Q}\left|\psi(x;q,1)-\frac{\psi(x)}{\phi(q)}\right|^2 \;\ll\; Q^{\eta-1}\sum_{Q/2<q\le Q}\;\sum_{\substack{1\le a\le q:\\ (a,q)=1}}\left|\psi(x;q,a)-\frac{\psi(x)}{\phi(q)}\right|^2. \tag{1.16}$$

Hypothesis 1.11$_\eta$ is trivially true for $\eta=1$ (which we will see recovers our previous result of support in $(-2,2)$), and while it is unlikely to be true for $\eta=0$, it is reasonable to expect it to hold for any $\eta>0$. What we need is some control over biases of primes congruent to 1 mod q. For the residue class a mod q, $\left|\psi(x;q,a)-\frac{\psi(x)}{\phi(q)}\right|^2$ is the variance; the above conjecture can be interpreted as bounding $\left|\psi(x;q,1)-\frac{\psi(x)}{\phi(q)}\right|^2$ in terms of the average variance.[7]

Under these hypotheses, we show how to extend the support to a wider but still limited range.

[7]Note that we only need this de-averaging hypothesis for the special residue class $a=1$.

Theorem 1.13. *Assume GRH and Hypothesis 1.11$_\eta$ for some $\eta \in (0,1)$. The average 1-level density $D_{1;Q/2,Q}(\widehat{f})$ equals*

$$\frac{f(0)}{\log Q}\left(\log Q - 1 - \gamma - \log(4\pi) - \sum_p \frac{\log p}{p(p-1)}\right) + \int_0^\infty \frac{f(0)-f(t)}{Q^{t/2}-Q^{-t/2}}dt$$
$$+ O(Q^{\frac{\eta-1}{2}}(\log Q)^{\frac{3}{2}} + Q^{\frac{\sigma+2\eta}{4}-1}(\log Q)^{\frac{1}{3}}), \quad (1.17)$$

which is asymptotic to $f(0)$ provided the support of f is contained in $(-4+2\eta, 4-2\eta)$.

The proof of Theorem 1.13 is given in §5. It uses a result of Goldston and Vaughan [GV], which is an improvement of the Barban-Davenport-Halberstam-Montgomery-Hooley Theorem.

Remark 1.14. *A similar result holds if we replace*

$$D_{1;Q/2,Q}(\widehat{f}) \ := \ \sum_{Q/2<q\le Q} \frac{1}{\phi(q)} \sum_{\chi \bmod q} \sum_{\gamma_\chi} \widehat{f}\left(\gamma_\chi \frac{\log Q}{2\pi}\right) \quad (1.18)$$

with

$$D^{\text{unweighted}}_{1;Q/2,Q}(\widehat{f}) \ := \ \frac{1}{\frac{9}{\pi^2}(Q/2)^2} \sum_{Q/2<q\le Q} \sum_{\chi \bmod q} \sum_{\gamma_\chi} \widehat{f}\left(\gamma_\chi \frac{\log Q}{2\pi}\right), \quad (1.19)$$

that is, if we study the unweighted 1-level density for Dirichlet L-functions in the family of principal characters $\{\chi \bmod q, Q/2 < q \le Q\}$. It is technically more convenient to use the weighted 1-level density; this is similar to many other families of L-functions, such as cuspidal newforms [ILS, MilMo] *and Maass forms* [AILMZ]*, where the introduction of weights (arising from the Petersson and Kuznetsov trace formulas) facilitates evaluating the arithmetical terms.*

Finally, we show how we can determine the 1-level density for arbitrary finite support, under a hypothesis of Montgomery [Mon1].

Hypothesis 1.15 (Montgomery)**.** *For any a, q such that $(a,q) = 1$ and $q \le x$, we have*

$$\psi(x;q,a) - \frac{\psi(x)}{\phi(q)} \ \ll_\epsilon \ x^\epsilon \left(\frac{x}{q}\right)^{1/2}. \quad (1.20)$$

It is by gaining some savings in q in the error $E(x,q,a)$ that we can increase the support for families of Dirichlet L-functions. The following weaker version of Montgomery's Conjecture, which depends on a parameter $\theta \in (0,1/2]$, also suffices to increase the support beyond $[-2,2]$.

Hypothesis 1.15$_\theta$. *We have*

$$\psi(x;q,1) - \frac{\psi(x)}{\phi(q)} \ \ll_\epsilon \ \frac{x^{\frac{1}{2}+\epsilon}}{q^\theta}. \quad (1.21)$$

In other words, any power savings in the modulus suffices.

The last hypothesis is the weakest, but is still sufficient to allow us to take any test function with compact support.

Hypothesis 1.15*. *Fix $\epsilon > 0$. We have for $x^\epsilon \le q \le \sqrt{x}$ that*

$$\sum_{\substack{n \le x \\ n \equiv 1 \bmod q}} \Lambda(n)\left(1 - \frac{n}{x}\right) - \frac{1}{\phi(q)}\sum_{n\le x}\Lambda(n)\left(1-\frac{n}{x}\right) \;=\; o(x^{1/2}). \tag{1.22}$$

Note that this is a weighted version of $\psi(x;q,1) - \frac{\psi(x)}{\phi(q)}$; that is, we added the weight $\left(1 - \frac{n}{x}\right)$. The reason for this is that it makes the count smoother, and this makes it easier to understand in general since the Mellin transform of $g(y) := 1 - y^{-1}$ in the interval $[1,\infty)$ is decaying much faster in vertical strips than that of $g(y) \equiv 1$.

We obtain

Theorem 1.16. *Let f be a test function with arbitrarily large (but finite) support.*

1. *If we assume Hypothesis 1.15*, then the 1-level density $D_{1;q}(\widehat{f})$ equals $f(0) + o(1)$, agreeing with the scaling limit of unitary matrices.*
2. *If we assume Hypothesis 1.15$_\theta$ for some $0 < \theta \le \frac{1}{2}$, then $D_{1;q}(\widehat{f})$ equals*

$$f(0)\left(1 - \frac{\log(8\pi e^\gamma)}{\log q} - \frac{\sum_{p|q}\frac{\log p}{p-1}}{\log q}\right) + \int_0^\infty \frac{f(0)-f(t)}{q^{t/2}-q^{-t/2}}dt + O_\epsilon(q^{-\theta+\epsilon}). \tag{1.23}$$

Remark 1.17. *Under GRH, the left hand side of* (1.22) *is $O(x^{1/2})$. Therefore, if we win by any amount over GRH, that is if we can replace the big-Oh with a little-oh, then we have the expected asymptotic for the 1-level density for any f of arbitrarily large finite support.*

Interestingly, if we assume Montgomery's original conjecture times x^ϵ (Hypothesis 1.15), then we can take $\theta = 1/2$ in (1.23)*, and doing so we end up precisely with the Ratios Conjecture's prediction.*

We derive the explicit formula for the families of Dirichlet characters in §2, as well as some useful estimates for some of the resulting sums. We give the unconditional results in §3, Theorems 1.2 and 1.4. The proof of Theorem 1.7 is conditional on GRH, and uses results of [FG2] and [Fi1]; we give it in §4. We conclude with an analysis of the consequences of the hypotheses on the distribution of primes in residue classes, using the de-averaging hypothesis to prove Theorem 1.13 in §5 and Montgomery's hypothesis to prove Theorem 1.16 in §6.

2. The Explicit Formula and Needed Sums

The starting point to investigating the behavior of low-lying zeros is the explicit formula, which relates sums over zeros to sums over primes. We follow the derivation in [MonVa2] (see also [ILS, RS], and [Da, IK] for all needed results about Dirichlet L-functions). We first derive the expansion for Dirichlet characters with fixed conductor q, and then extend to $q \in (Q/2, Q]$. We conclude with some technical estimates that will be of use in proving Theorem

2.1. The Explicit Formula for fixed q.

Proposition 2.1 (Explicit Formula for the Family of Dirichlet Characters Modulo q)**.** *Let f be an even, twice differentiable test function with compact support. Denote the non-trivial* *zeros of $L(s,\chi)$ by $\rho_\chi = 1/2 + i\gamma_\chi$. Then the 1-level density $D_{1,q}(\widehat{f})$ equals*

$$\begin{aligned}\frac{1}{\phi(q)}\sum_{\chi \bmod q}\sum_{\gamma_\chi}\widehat{f}\left(\gamma_\chi\frac{\log Q}{2\pi}\right) &= \frac{f(0)}{\log Q}\left(\log q - \log(8\pi e^\gamma) - \sum_{p|q}\frac{\log p}{p-1}\right.\\ &\quad + \int_0^\infty \frac{f(0)-f(t)}{Q^{t/2}-Q^{-t/2}}dt - \frac{2}{\log Q}\sum_{\substack{p^\nu \| q\\ p^e\equiv 1 \bmod q/p^\nu\\ e,\nu\ge 1}}\frac{\log p}{\phi(p^\nu)p^{e/2}}f\left(\frac{\log p^e}{\log Q}\right)\\ &\quad - \frac{2}{\log Q}\left(\sum_{n\equiv 1 \bmod q} - \frac{1}{\phi(q)}\sum_n\right)\frac{\Lambda(n)}{n^{1/2}}f\left(\frac{\log n}{\log Q}\right) + O\left(\frac{1}{\phi(q)}\right).\end{aligned} \tag{2.1}$$

Proof. We start with Weil's explicit formula for $L(s,\chi)$, with χ mod q a non-principal character (we add the contribution from the principal character later). We can replace $L(s,\chi)$ by $L(s,\chi^*)$ (where χ^* is the primitive character of conductor q^* inducing χ), since these have the same non-trivial zeros. Taking $F(x) := \frac{2\pi}{\log Q}f\left(\frac{2\pi x}{\log Q}\right)$ in Theorem 12.13 of [MonVa2] (whose conditions are satisfied by our restrictions on f), we find $\Phi(s) = \widehat{f}\left(\frac{\log Q}{2\pi}\frac{(s-\frac{1}{2})}{i}\right)$, and

$$\begin{aligned}\sum_{\rho_\chi}\widehat{f}\left(\frac{\log Q}{2\pi}\gamma_\chi\right) &= \frac{f(0)}{\log Q}\left(\log(q^*/\pi) + \frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{a(\chi)}{2}\right)\right)\\ -\frac{2}{\log Q}\sum_{n=1}^\infty\frac{\Lambda(n)\Re(\chi^*(n))}{n^{1/2}}f\left(\frac{\log n}{\log Q}\right) &+ \frac{4\pi}{\log Q}\int_0^\infty\frac{e^{-(1+2a(\chi))\pi x}}{1-e^{-4\pi x}}\left(f(0)-f\left(\frac{2\pi x}{\log Q}\right)\right)dx,\end{aligned} \tag{2.2}$$

where $a(\chi)=0$ for the half of the characters with $\chi(-1)=1$ and 1 for the half with $\chi(-1)=-1$. Making the substitution $t=\frac{2\pi x}{\log Q}$ in the integral and summing over $\chi\neq\chi_0$, we find

$$\begin{aligned}\sum_{\chi\neq\chi_0}\sum_{\gamma_\chi}\widehat{f}\left(\gamma_\chi\frac{\log Q}{2\pi}\right) &= \frac{f(0)}{\log Q}\left(\sum_{\chi\neq\chi_0}\log(q^*/\pi) + \frac{\phi(q)}{2}\frac{\Gamma'}{\Gamma}\left(\frac{3}{4}\right) + \frac{\phi(q)}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}\right)\right)\\ &\quad + \phi(q)\int_0^\infty\frac{Q^{-3t/2}+Q^{-t/2}}{1-Q^{-2t}}(f(0)-f(t))dt\\ &\quad - \frac{2}{\log Q}\left(\phi(q)\sum_{n\equiv 1 \bmod q} - \sum_n\right)\frac{\Lambda(n)}{n^{1/2}}f\left(\frac{\log n}{\log Q}\right)\\ &\quad - \frac{2}{\log Q}\sum_{\chi\neq\chi_0}\sum_n\frac{\Lambda(n)\Re(\chi^*(n)-\chi(n))}{n^{1/2}}f\left(\frac{\log n}{\log Q}\right) + O(1).\end{aligned} \tag{2.3}$$

To get (2.3) from (2.2) we added zero by writing $\chi^*(n)$ as $(\chi^*(n)-\chi(n))+\chi(n)$. Summing $\chi(n)$ over all χ mod q gives $\phi(q)$ if $n\equiv 1 \bmod q$ and 0 otherwise; as our sum omits the principal character, the sum of $\chi(n)$ over the non-principal characters yields the sum on the third line above. We also replaced $(\phi(q)-1)/2$ by $\phi(q)/2$ in the first term, hence the $O(1)$.

We use Proposition 3.3 of [FiMa] for the first term (which involves the sum over the conductor of the inducing character). We then use the duplication formula of the digamma function $\psi(z) = \Gamma'(z)/\Gamma(z)$ to simplify the next two terms, namely $\psi(1/4) + \psi(3/4)$. As $\psi(1/2) = -\gamma - 2\ln 2$ (equation 6.3.3 of [AS]) and $\psi(2z) = \frac{1}{2}\psi(z) + \frac{1}{2}\psi(z+\frac{1}{2}) + \ln 2$ (equation 6.3.8 of [AS]), setting $z = 1/4$ yields $\psi(1/4) + \psi(3/4) = -2\gamma - 6\ln 2$. We keep the next two terms as they are, and then apply Proposition 3.4 of [FiMa] for the last term, obtaining that it equals

$$-\frac{2}{\log Q} \sum_n \frac{\Lambda(n)}{n^{1/2}} f\left(\frac{\log n}{\log Q}\right) \mathrm{Re}\left(\sum_{\chi \neq \chi_0} \left(\chi^*(n) - \chi(n)\right)\chi(1)\right). \tag{2.4}$$

Writing $n = p^e$, this term is zero unless $p \mid q$. If $p \mid q$, then it is zero unless $p^e \equiv 1 \bmod q/p^\nu$, where $\nu \geq 1$ is the largest ν such that $p^\nu \mid q$. Therefore this term equals

$$-\frac{2}{\log Q} \sum_p \sum_{\substack{e,\nu \geq 1 \\ p^\nu \| q, p^e \equiv 1 \bmod q/p^\nu}} \frac{\Lambda(p^e)}{\phi(p^\nu) p^{e/2}} f\left(\frac{\log p^e}{\log Q}\right). \tag{2.5}$$

Combining the above and some elementary algebra yields

$$\begin{aligned}
\frac{1}{\phi(q)} \sum_{\chi \neq \chi_0} \sum_{\gamma_\chi} \widehat{f}\left(\gamma_\chi \frac{\log Q}{2\pi}\right) \ &= \ \frac{f(0)}{\log Q}\left(\log q - \log(8\pi e^\gamma) - \sum_{p \mid q} \frac{\log p}{p-1}\right) \\
&+ \int_0^\infty \frac{f(0) - f(t)}{Q^{t/2} - Q^{-t/2}} dt - \frac{2}{\log Q}\left(\sum_{n \equiv 1 \bmod q} - \frac{1}{\phi(q)} \sum_n\right) \frac{\Lambda(n)}{n^{1/2}} f\left(\frac{\log n}{\log Q}\right) \\
&- \frac{2}{\log Q} \sum_{\substack{p^\nu \| q \\ p^e \equiv 1 \bmod q/p^\nu \\ e,\nu \geq 1}} \frac{\log p}{\phi(p^\nu) p^{e/2}} f\left(\frac{\log p^e}{\log Q}\right) + O\left(\frac{1}{\phi(q)}\right).
\end{aligned} \tag{2.6}$$

Finally, since the non-trivial zeros of $L(s, \chi_0)$ coincide with those of $\zeta(s)$, the difference between the left hand side of (2.1) and that of (2.6) is

$$\frac{1}{\phi(q)} \sum_{\gamma_\zeta} \widehat{f}\left(\gamma_\zeta \frac{\log Q}{2\pi}\right) \ \ll \ \frac{1}{\phi(q)} \tag{2.7}$$

(since f is twice continuously differentiable, $\widehat{f}(y) \ll 1/y^2$), completing the proof.[8] □

2.2. **The Averaged Explicit Formula for $q \in (Q/2, Q]$.** We now average the explicit formula for $D_{1;q}(\widehat{f})$ (Proposition 2.1) over $q \in (Q/2, Q]$. We concentrate on deriving useful expansions, which we then analyze in later sections when we determine the allowable support.

[8]While the explicit formula for $\zeta(s)$ has a term arising from its pole at $s = 1$, that term does not matter here as it is insignificant upon division by the family's size.

Proposition 2.2 (Explicit Formula for the Averaged Family of Dirichlet Characters Modulo q)**.** *The averaged 1-level density, $D_{1;Q/2,Q}(\widehat{f})$, equals*

$$\begin{aligned} D_{1;Q/2,Q}(\widehat{f}) &= \frac{1}{Q/2}\sum_{Q/2<q\leq Q} D_{1;q}(\widehat{f}) \\ &= \frac{f(0)}{\log Q}\left(\log Q - 1 - \gamma - \log(4\pi) - \sum_p \frac{\log p}{p(p-1)}\right) + \int_0^\infty \frac{f(0)-f(t)}{Q^{t/2}-Q^{-t/2}}dt \\ &\quad + \frac{2}{Q/2}\sum_{Q/2<q\leq Q}\int_0^\infty \left(\frac{f(u)}{2} - \frac{f'(u)}{\log Q}\right)\frac{\psi(Q^u;q,1) - \frac{\psi(Q^u)}{\phi(q)}}{Q^{u/2}}\,du + O\left(\frac{1}{Q}\right). \end{aligned} \tag{2.8}$$

Setting

$$\psi_2(x;q,a) := \sum_{\substack{n\leq x\\ n\equiv a \bmod q}} \Lambda(n)\left(1-\frac{n}{x}\right), \qquad \psi_2(x) := \sum_{n\leq x}\Lambda(n)\left(1-\frac{n}{x}\right), \tag{2.9}$$

the last integral in (2.8) *may be replaced with*

$$-2\int_0^\infty \left(\frac{3f(u)}{4} - \frac{2f'(u)}{\log Q} + \frac{f''(u)}{(\log Q)^2}\right)\frac{\psi_2(Q^u;q,1) - \frac{\psi_2(Q^u)}{\phi(q)}}{Q^{u/2}}\,du. \tag{2.10}$$

Proof. The main term in the expansion of $D_{1;q}(\widehat{f})$ from Proposition 2.1 is

$$T_1(q) := \frac{f(0)}{\log Q}\left(\log q - \log(8\pi e^\gamma) - \sum_{p|q}\frac{\log p}{p-1}\right). \tag{2.11}$$

Using the anti-derivative of $\log x$ is $x\log x - x$, one easily finds its average over $Q/2 < q \leq Q$ is

$$\frac{1}{Q/2}\sum_{Q<q\leq 2Q} T_1(q) = \frac{f(0)}{\log Q}\left(\log Q - 1 - \gamma - \log(4\pi) - \sum_p \frac{\log p}{p(p-1)}\right) + O\left(\frac{1}{Q}\right). \tag{2.12}$$

We now turn to the lower-order term

$$T_2(q) := -\frac{2}{\log Q}\sum_{\substack{p^\nu \| q\\ p^e\equiv 1 \bmod q/p^\nu\\ e,\nu\geq 1}} \frac{\log p}{\phi(p^\nu)p^{e/2}} f\left(\frac{\log p^e}{\log Q}\right). \tag{2.13}$$

Before determining its average behavior, we note that its size can vary greatly with q. It is very small for prime q (so $\nu = 1$ and $p = q$ in the sum), since

$$T_2(q) \ll \frac{1}{\log Q}\sum_{e\geq 1}\frac{\log q}{\phi(q)q^{e/2}} \ll \frac{1}{(q-1)(q^{\frac{1}{2}}-1)}; \tag{2.14}$$

however, it can be as large as $\frac{C}{\sqrt{q}\log Q}$ for other values of q (such as $q = 2(2^e - 1)$). This is more or less as large as it can get, since for general q we have

$$T_2(q) \ \ll\ \frac{1}{\log Q} \sum_{\substack{p^\nu \| q \\ e,\nu \geq 1 \\ p^e \leq Q^\sigma}} \frac{\log p}{\phi(p^\nu)(q/p^\nu)^{1/2}} \ \ll\ \frac{(\log q)^{\frac{1}{2}}}{q^{\frac{1}{2}} \log\log q}. \tag{2.15}$$

On average however, $T_2(q)$ is very small:

$$\begin{aligned}
\frac{1}{Q/2} \sum_{Q/2 < q \leq Q} T_2(q) \ &\ll\ \frac{1}{Q} \sum_{Q/2 < q \leq Q} \sum_{\substack{p^\nu \| q \\ p^e \equiv 1 \bmod q/p^\nu \\ e,\nu \geq 1}} \frac{\log p}{p^{\nu + e/2}} \ \ll\ \frac{1}{Q} \sum_{\substack{p^\nu \\ \nu, e \geq 1}} \frac{\log p}{p^{\nu+e/2}} \sum_{\substack{q \leq Q \\ p^\nu | q \\ \frac{q}{p^\nu} | p^e - 1}} 1 \\
&\ll\ \frac{1}{Q} \sum_{\substack{p^\nu \\ \nu, e \geq 1}} \frac{\log p}{p^{\nu + e/2}} \tau(p^e - 1) \ \ll_\epsilon\ \frac{1}{Q} \sum_{\substack{p^\nu \\ \nu, e \geq 1}} \frac{\log p}{p^{\nu + (1-\epsilon)\frac{e}{2}}} \\
&\ll\ \frac{1}{Q} \sum_p \frac{\log p}{p^{\frac{3}{2} - \frac{\epsilon}{2}}} \ \ll\ \frac{1}{Q}.
\end{aligned} \tag{2.16}$$

While we will not re-write the next lower order term, it is instructive to determine its size. Set

$$T_3(q) \ :=\ \int_0^\infty \frac{f(0) - f(t)}{Q^{t/2} - Q^{-t/2}} dt. \tag{2.17}$$

Letting $t = 2\pi x / \log Q$, we find

$$T_3(q) \ =\ \frac{2\pi}{\log Q} \int_0^\infty \frac{f(0) - f\left(\frac{2\pi x}{\log Q}\right)}{2\sinh(\pi x)} dx. \tag{2.18}$$

Since f is twice differentiable with compact support, $|f(0) - f(x)| \ll |x|$, thus

$$T_3(q) \ \ll\ \frac{2\pi}{\log Q} \int_0^\infty \frac{x}{2\sinh(\pi x)}\ dx \ =\ \frac{\pi}{4\log Q}. \tag{2.19}$$

As

$$\int_0^\infty \frac{x^k dx}{\sinh(\pi x)} \ =\ \frac{2^{k+1} - 1}{2^k \pi^{k+1}} \Gamma(k+1)\zeta(k+1), \tag{2.20}$$

if f has a Taylor series of order $K + 1$ we have

$$T_3(q) \ =\ \sum_{k=1}^{K} \frac{(2^{k+1} - 1)\zeta(k+1) f^{(k)}(0)}{\log^{k+1} Q} + O\left(\frac{1}{\log^{K+1} Q}\right). \tag{2.21}$$

If the Taylor coefficients of f decay very fast, we can even make our bounds uniform and get an error term smaller than a negative power of Q.

The remaining term from Proposition 2.1 is the most important, and controls the allowable support. The arithmetic lives here, as this term involves primes in arithmetic progressions.

It is

$$T_4(q) \;:=\; -\frac{2}{\log Q}\left(\sum_{n\equiv 1 \bmod q} - \frac{1}{\phi(q)}\sum_n\right)\frac{\Lambda(n)}{n^{1/2}} f\left(\frac{\log n}{\log Q}\right)$$
$$= \; -\frac{2}{\log Q}\int_1^\infty t^{-\frac12} f\left(\frac{\log t}{\log Q}\right) d\left(\psi(t;q,1) - \frac{\psi(t)}{\phi(q)}\right)$$
$$= \; \frac{2}{\log Q}\int_1^\infty \frac{\frac12 f\left(\frac{\log t}{\log Q}\right) - \frac{1}{\log Q} f'\left(\frac{\log t}{\log Q}\right)}{t^{\frac32}}\left(\psi(t;q,1) - \frac{\psi(t)}{\phi(q)}\right) dt. \tag{2.22}$$

The claim in the proposition follows by changing variables by setting $t = Q^u$; specifically, the final integral is

$$T_4(q) \;=\; 2\int_0^\infty \left(\frac{f(u)}{2} - \frac{f'(u)}{\log Q}\right)\frac{\psi(Q^u;q,1) - \frac{\psi(Q^u)}{\phi(q)}}{Q^{u/2}}\, du. \tag{2.23}$$

We give an alternative expansion for the final integral. This expansion involves a smoothed sum of $\Lambda(n)$, which will be technically easier to analyze when we turn to determining the allowable support under Montgomery's hypothesis (Theorem 1.16(1)). Recall

$$\psi_2(x;q,a) \;:=\; \sum_{\substack{n\le x\\ n\equiv a \bmod q}} \Lambda(n)\left(1-\frac{n}{x}\right), \qquad \psi_2(x) \;:=\; \sum_{n\le x}\Lambda(n)\left(1-\frac{n}{x}\right). \tag{2.24}$$

We integrate by parts in (2.22). Since

$$\int_1^x \left(\psi(t;q,1) - \frac{\psi(t)}{\phi(q)}\right) dt \;=\; \int_1^x \left(\sum_{\substack{n\le t\\ n\equiv 1 \bmod q}} \Lambda(n) - \frac{1}{\phi(q)}\sum_{n\le t}\Lambda(n)\right) dt$$
$$= \sum_{\substack{n\le x\\ n\equiv 1 \bmod q}} \Lambda(n)\int_n^x dt - \frac{1}{\phi(q)}\sum_{n\le x}\Lambda(n)\int_n^x dt$$
$$= x\left(\sum_{\substack{n\le x\\ n\equiv 1 \bmod q}} \Lambda(n)\left(1-\frac{n}{x}\right) - \frac{1}{\phi(q)}\sum_{n\le x}\Lambda(n)\left(1-\frac{n}{x}\right)\right), \tag{2.25}$$

we find

$$T_4(q) \;=\; -2\int_0^\infty \left(\frac{3f(u)}{4} - \frac{2f'(u)}{\log Q} + \frac{f''(u)}{(\log Q)^2}\right)\frac{\psi_2(Q^u;q,1) - \frac{\psi_2(Q^u)}{\phi(q)}}{Q^{u/2}}\, du, \tag{2.26}$$

completing the proof. □

Remark 2.1. *It will be convenient later that in the averaged case ψ and ψ_2 are both evaluated at $(Q^u;q,1)$ and not $(q^u;q,1)$; this is because we are rescaling all L-function zeros by the same quantity (a global rescaling instead of a local rescaling).*

2.3. **Technical Estimates.** In the proof of Theorem 1.7, we need the following estimation of a weighted sum of the reciprocal of the totient function.

Lemma 2.3. *Let ϕ be Euler's totient function. We have*

$$\sum_{r\le R}\frac{1}{\phi(r)}\left(R^{1/2}+\frac{r}{R^{1/2}}-2r^{1/2}\right)=D_1R^{1/2}\log R+D_2R^{1/2}+D_3+O\left(\frac{\log R}{R^{1/2}}\right), \tag{2.27}$$

where

$$D_1 := \frac{\zeta(2)\zeta(3)}{\zeta(6)}, \qquad D_2 := D_1\left(\gamma-3-\sum_p\frac{\log p}{p^2-p+1}\right),$$
$$D_3 := -2\zeta\left(\frac{1}{2}\right)\prod_p\left(1+\frac{1}{(p-1)p^{1/2}}\right). \tag{2.28}$$

More generally, if $P(u):=\sum_{i=0}^d a_iu^i$ is a polynomial of degree d and of norm

$$\|P\| := \max_i|a_i|, \tag{2.29}$$

then

$$\sum_{r\le R}\frac{1}{\phi(r)}\int_{\frac{\log r}{\log R}}^1 P(u)\left(R^{\frac{u}{2}}-\frac{r}{R^{\frac{u}{2}}}\right)du = E_1\log R\int_{-\infty}^1 R^{\frac{u}{2}}uP(u)du$$
$$+E_2\int_{-\infty}^1 R^{\frac{u}{2}}P(u)du+\sum_{j=1}^{d+1}\frac{F_j(P)}{(\log R)^j}+O_d(R^{-\frac{1}{2}}\|P\|) \tag{2.30}$$

where

$$E_1 := \frac{\zeta(2)\zeta(3)}{\zeta(6)}, \qquad E_2 := E_1\left(\gamma-1-\sum_p\frac{\log p}{p^2-p+1}\right), \tag{2.31}$$

and the $F_j(P)$ are constants depending on P which can be computed explicitly. For example,

$$F_1(P) = -4\zeta\left(\frac{1}{2}\right)\prod_p\left(1+\frac{1}{(p-1)p^{1/2}}\right)\sum_{i=0}^d(-1)^iP^{(i)}(1)$$
$$F_2(P) = -4\zeta\left(\frac{1}{2}\right)\prod_p\left(1+\frac{1}{(p-1)p^{1/2}}\right)\left(\frac{\zeta'}{\zeta}\left(\frac{1}{2}\right)-\sum_p\frac{\log p}{(p-1)p^{1/2}+1}\right)\sum_{i=1}^d(-1)^iP^{(i)}(1). \tag{2.32}$$

Finally,

$$\sum_{r\le R}\frac{1}{\phi(r)}\int_{\frac{\log(r/2)}{\log(R/2)}}^1 P(u)\left((R/2)^{\frac{u}{2}}-\frac{r}{2(R/2)^{\frac{u}{2}}}\right)du$$
$$= E_1\log(R/2)\int_{-\infty}^1(R/2)^{\frac{u}{2}}uP(u)du+(E_2+E_1\log 2)\int_{-\infty}^1(R/2)^{\frac{u}{2}}P(u)du$$
$$+\sum_{j=1}^{d+1}\frac{F_j^{(2)}(P)}{(\log(R/2))^j}+O_d(R^{-\frac{1}{2}}\|P\|), \tag{2.33}$$

where the first two constants are given by

$$F_1^{(2)}(P) \;\; := \;\; \frac{F_1(P)}{\sqrt{2}}$$

$$F_2^{(2)}(P) \;\; := \;\; -2\sqrt{2}\zeta\left(\frac{1}{2}\right)\prod_p\left(1+\frac{1}{(p-1)p^{1/2}}\right) \times \left(\frac{\zeta'}{\zeta}\left(\frac{1}{2}\right)-\sum_p\frac{\log p}{(p-1)p^{1/2}+1}+\log 2\right)\sum_{i=1}^{d}(-1)^iP^{(i)}(1). \quad (2.34)$$

Remark 2.4. *It is possible to improve the estimates in* (2.27), (2.30) *and* (2.33) *to ones with an error term of* $O_{\epsilon,d}(R^{-5/4+\epsilon}\|P\|)$*; however, this is not needed for our purposes.*

Proof. By Mellin inversion, for $c\geq 2$ the left hand side of (2.27) equals

$$\frac{1}{2\pi i}\int_{\Re(s)=c}Z(s)\left(\frac{R^{s+\frac{1}{2}}}{s}+\frac{R^{s+\frac{1}{2}}}{s+1}-2\frac{R^{s+\frac{1}{2}}}{s+\frac{1}{2}}\right)ds \;=\; \frac{1}{2\pi i}\int_{\Re(s)=c}Z(s)\frac{R^{s+\frac{1}{2}}}{2s(s+\frac{1}{2})(s+1)}ds, \quad (2.35)$$

where

$$Z(s) \;:=\; \sum_{n\geq 1}\frac{1}{n^s\phi(n)}. \quad (2.36)$$

Taking Euler products,

$$Z(s) \;=\; \zeta(s+1)\zeta(s+2)Z_2(s), \quad (2.37)$$

where

$$Z_2(s) \;:=\; \prod_p\left(1+\frac{1}{p(p-1)}\left(\frac{1}{p^{s+1}}-\frac{1}{p^{2s+2}}\right)\right), \quad (2.38)$$

which converges for $\Re(s)>-\frac{3}{2}$. We shift the contour of integration to the left to the line $\Re(s)=-\frac{3}{2}+\epsilon$. By a standard residue calculation, we get that (2.35) equals

$$D_1R^{1/2}\log R+D_2R^{1/2}+D_3+D_4\frac{\log R}{R^{1/2}}+\frac{D_5}{R^{1/2}}+\frac{1}{2\pi i}\int_{\Re(s)=-\frac{3}{2}+\epsilon}Z(s)\frac{R^{s+\frac{1}{2}}}{2s(s+\frac{1}{2})(s+1)}ds \quad (2.39)$$

for some constants D_4 and D_5. The proof now follows from standard bounds on the zeta function, which show that this integral is $\ll_\epsilon R^{-1+\epsilon}$. See the proof of Lemma 6.9 of [Fi1] for more details.

We now move to (2.30). The Mellin transform in this case is (for $\Re(s)>0$)

$$\begin{aligned}\alpha(s) \;&:=\; \int_0^R r^{s-1}\int_{\frac{\log r}{\log R}}^1 P(u)\left(R^{\frac{u}{2}}-\frac{r}{R^{\frac{u}{2}}}\right)dudr\\ &=\; \int_{-\infty}^1 P(u)\int_0^{R^u} r^{s-1}\left(R^{\frac{u}{2}}-\frac{r}{R^{\frac{u}{2}}}\right)drdu\\ &=\; \int_{-\infty}^1 P(u)\frac{R^{u(s+\frac{1}{2})}}{s(s+1)}\,du, \end{aligned} \quad (2.40)$$

which is now defined for $\Re(s) > -1/2$. To meromorphically extend $\alpha(s)$ to the whole complex plane, we integrate by parts n times:

$$\alpha(s) = \frac{R^{s+\frac{1}{2}}}{s(s+1)} \sum_{i=0}^{n} \frac{(-1)^i P^{(i)}(1)}{(s+\frac{1}{2})^{i+1}(\log R)^{i+1}}, \tag{2.41}$$

which is a meromorphic function with poles at the points $s = 0, -1/2, -1$. The integral we need to compute is

$$\frac{1}{2\pi i} \int_{\Re(s)=1} Z(s)\alpha(s)ds. \tag{2.42}$$

We remark that

$$\alpha(-3/2+\epsilon+it) \ll_{\epsilon,d} \frac{R^{-1+\epsilon}}{t^3}\|P\|, \tag{2.43}$$

hence the proof is similar as in the previous case, since by shifting the contour of integration to the left, we have

$$\frac{1}{2\pi i} \int_{\Re(s)=1} Z(s)\alpha(s)ds = A + O_{\epsilon,d}(R^{-1+\epsilon}\|P\|), \tag{2.44}$$

where A is the sum of the residues of $Z(s)\alpha(s)$ for $-3/2+\epsilon \le \Re(s) \le 2$. Note that if $\beta(s) := s(s+1)\alpha(s)$, then

$$\beta(0) = \int_{-\infty}^{1} R^{\frac{u}{2}} P(u)du, \qquad \beta'(0) = \log R \int_{-\infty}^{1} R^{\frac{u}{2}} uP(u)du, \tag{2.45}$$

so the residue at $s = 0$ equals

$$\frac{\zeta(2)\zeta(3)}{\zeta(6)}\beta(0)\left(\frac{\beta'}{\beta}(0) + \gamma - 1 - \sum_p \frac{\log p}{p^2 - p + 1}\right). \tag{2.46}$$

For the pole at $s = -1/2$, we need to use the analytic continuation of $\alpha(s)$ provided in (2.41), which shows that this residue equals

$$\sum_{j=1}^{n+1} \frac{F_j(P)}{(\log R)^j}, \tag{2.47}$$

where the $F_j(P)$ are constants depending on P which can be computed explicitly. For example,

$$\begin{aligned} F_1(P) &= -4\zeta\left(\frac{1}{2}\right)\prod_p\left(1+\frac{1}{(p-1)p^{1/2}}\right)\sum_{i=0}^{d}(-1)^i P^{(i)}(1) \\ F_2(P) &= -4\zeta\left(\frac{1}{2}\right)\prod_p\left(1+\frac{1}{(p-1)p^{1/2}}\right)\left(\frac{\zeta'}{\zeta}\left(\frac{1}{2}\right) - \sum_p \frac{\log p}{(p-1)p^{1/2}+1}\right)\sum_{i=1}^{d}(-1)^i P^{(i)}(1). \end{aligned} \tag{2.48}$$

Moreover, $F_i(P) \ll_d \|P\|$ for all i.

At $s=-1$, we have a double pole with residue

$$R^{-\frac{1}{2}}\sum_{j=0}^{n+1}\frac{G_j(P)}{(\log R)^j}, \tag{2.49}$$

for some constants $G_j(P)\ll_d \|P\|$, hence the the proof of (2.30) is complete.

For the proof of (2.33), we proceed in the same way, noting that the Mellin transform is

$$\alpha_2(s) \;=\; \frac{2^s}{s(s+1)}\int_{-\infty}^{1} P(u)(R/2)^{u(s+\frac{1}{2})}du. \tag{2.50}$$

□

3. Unconditional Results (Theorems 1.2 and 1.4)

Using the expansion for the 1-level density $D_{1,q}(\widehat{f})$ and the averaged 1-level density $D_{1;Q/2,Q}(\widehat{f})$ from Propositions 2.1 and 2.2, we prove our unconditional results.

Proof of Theorem 1.2. We start from Proposition 2.1. The only term of (2.1) we need to understand is the last one (the "prime sum"), which is given by

$$T_4(q) \;:=\; 2\int_0^1\left(\frac{f(u)}{2}-\frac{f'(u)}{\log q}\right)\frac{\psi(q^u;q,1)-\frac{\psi(q^u)}{\phi(q)}}{q^{u/2}}\,du. \tag{3.1}$$

(We used that the support of f is contained in $[-1,1]$ and we made the substitution $t=q^u$.) However, since there are no integers congruent to 1 mod q in the interval $[2,q^u]$ when $u\le 1$ (this is also true when q^u is replaced by Q^u, with $Q/2<q\le Q$), the $\psi(q^u;q,1)$ term equals zero. By the Prime Number Theorem there is a $c>0$ such that

$$\begin{aligned} T_4(q) \;&=\; -2\int_0^1\left(\frac{f(u)}{2}-\frac{f'(u)}{\log q}\right)\frac{\psi(q^u)}{q^{u/2}\phi(q)}\,du \\ &=\; -\frac{2}{\phi(q)}\int_0^1 q^{u/2}\left(\frac{f(u)}{2}-\frac{f'(u)}{\log q}\right)du+O\left(\frac{1}{\phi(q)}\int_0^{\sigma}\frac{q^{u/2}}{e^{c\sqrt{u\log q}}}\,du\right), \end{aligned} \tag{3.2}$$

and the error term is

$$\ll\; \frac{q^{\sigma/4}}{\phi(q)}\int_0^{\sigma/2}e^{-c\sqrt{u\log q}}du+\frac{e^{-c\sqrt{\frac{\sigma}{2}\log q}}}{\phi(q)}\int_{\sigma/2}^{\sigma}q^{u/2}du \;\ll\; \frac{q^{\sigma/2-1}}{e^{c'\sqrt{\sigma\log q}}} \tag{3.3}$$

for q large enough (in terms of σ), completing the proof. □

Proof of Theorem 1.4. Starting again from Proposition 2.1, we have that

$$-\frac{2}{\log Q}\sum_{\substack{p^\nu\|q\\ p^e\equiv 1 \bmod q/p^\nu\\ e,\nu\ge 1}}\frac{\log p}{\phi(p^\nu)p^{e/2}}f\left(\frac{\log p^e}{\log Q}\right)\;\ll\;\frac{(\log q)^{\frac{1}{2}}}{q^{\frac{1}{2}}\log\log q} \tag{3.4}$$

(see (2.15)), hence this goes in the error term and the only term we need to worry about is the last one.

As our support exceeds $[-1,1]$, the $\psi(q^u;q,1)$ no longer trivially vanishes, and the last term is

$$T_4(q) \;=\; 2\int_0^2 \left(\frac{f(u)}{2} - \frac{f'(u)}{\log q}\right) \frac{\psi(q^u;q,1) - \frac{\psi(q^u)}{\phi(q)}}{q^{u/2}}\, du. \tag{3.5}$$

In the proof of Theorem 1.2 above we showed that the contribution from the integral where $0 \le u \le 1$ is $O(q^{-1/2})$.

For any fixed $\epsilon > 0$, trivial bounds for the region $1 \le u \le 1+\epsilon$ yield a contribution that is

$$\ll \int_1^{1+\epsilon} (u \log q) q^{\frac{u}{2}-1} du \;\ll\; q^{-\frac{1}{2}+\epsilon}. \tag{3.6}$$

We use the Brun-Titchmarsh Theorem (see [MonVa1]) for the region where $1+\epsilon \le u \le 2$, which asserts that for $q < x$,

$$\pi(x;q,a) \;\le\; \frac{2x}{\phi(q)\log(x/q)}. \tag{3.7}$$

It is straightforward to bound the contribution from prime powers, as

$$\begin{aligned}
\sum_{e \ge 2} \sum_{\substack{p \le x^{1/e} \\ p^e \equiv 1 \bmod q}} \log p \;&\ll\; \sum_{2 \le e \le \frac{2}{\epsilon}} e^{\omega(q)} \max_{b \bmod q} \left(\sum_{\substack{p \le x^{1/e} \\ p \equiv b \bmod q}} \log p \right) + \sum_{\frac{2}{\epsilon} \le e \le 2\log x} \sum_{p \le x^{1/e}} \log p \\
&\ll\; \sum_{2 \le e \le \frac{2}{\epsilon}} e^{\omega(q)} \left(1 + \frac{x^{1/e}}{q}\right) \log x + \sum_{\frac{2}{\epsilon} \le e \le 2\log x} x^{1/e} \\
&\ll\; \left(\frac{2}{\epsilon}\right)^{\omega(q)+1} \left(1 + \frac{x^{1/2}}{q}\right) \log x + x^{\epsilon/2} \log x \\
&\ll_\epsilon\; x^\epsilon \left(1 + \frac{x^{1/2}}{q}\right),
\end{aligned} \tag{3.8}$$

provided q is large enough in terms of ϵ.

Thus, for $1+\epsilon \le u \le 2$, we have

$$\psi(q^u;q,1) \;\ll_\epsilon\; \frac{q^{u-1}\log(q^u)\log\log q}{(u-1)\log q} + q^\epsilon + q^{\frac{u}{2}-1+\epsilon} \;\ll_\epsilon\; q^{u-1}\log\log q, \tag{3.9}$$

which bounds the integral from $1+\epsilon$ to σ by

$$\ll \int_{1+\epsilon}^{\sigma} q^{\frac{u}{2}-1} \log\log q du \;\ll\; \frac{\log\log q}{\log q} q^{\frac{\sigma}{2}-1}, \tag{3.10}$$

completing the proof. □

4. Results Under GRH (Theorem 1.7)

In this section we assume GRH (but none of the stronger results about the distribution of primes among residue classes) and prove Theorem 1.7. The theorem follows from the results of [Fi1] (see also [Fi2], where a dyadic interval version is done). The following is the needed conditional version.

Theorem 4.1. *Assume GRH. Fix an integer $a \neq 0$ and $\epsilon > 0$. We have for $M = M(x) \leq x^{\frac{1}{4}}$ that*

$$\sum_{\substack{\frac{x}{2M} < q \leq \frac{x}{M} \\ (q,a)=1}} \left(\psi(x; q, a) - \Lambda(a) - \frac{\psi(x)}{\phi(q)} \right) \;=\; \frac{\phi(a)}{a} \frac{x}{2M} \mu_0(a, M) + O_{a,\epsilon} \left(\frac{x}{M^{\frac{3}{2}-\epsilon}} + \sqrt{x} M (\log x)^2 \right), \tag{4.1}$$

where

$$\mu_0(a, M) \;:=\; \begin{cases} -\frac{1}{2} \log M - \frac{C_6}{2} & \text{if } a = \pm 1 \\ -\frac{1}{2} \log p & \text{if } a = \pm p^e \\ 0 & \text{otherwise,} \end{cases} \tag{4.2}$$

with

$$C_6 \;:=\; \log \pi + 1 + \gamma + \sum_p \frac{\log p}{p(p-1)}. \tag{4.3}$$

Proof. See Remark 1.5 of [Fi1]. Note that the restriction $M = o(x^{\frac{1}{4}} / \log x)$ is required for the error term to be negligible compared to the main term, but it can be changed to $M \leq x^{\frac{1}{4}}$. □

Proof of Theorem 1.7. By the averaged 1-level density (Proposition 2.2), the proof is completed by analyzing the average of $T_4(q)$:

$$\frac{1}{Q/2} \sum_{Q/2 < q \leq Q} T_4(q) \;=\; 2 \int_0^\sigma \left(\frac{f(u)}{2} - \frac{f'(u)}{\log Q} \right) \frac{1}{Q/2} \sum_{Q/2 < q \leq Q} \frac{\psi(Q^u; q, 1) - \frac{\psi(Q^u)}{\phi(q)}}{Q^{u/2}} \, du. \tag{4.4}$$

We now break the integral into regions and bound each separately. Going through the proof of Theorem 1.2 and applying GRH, we see that the contribution to the integral from $u \in [0, 1]$ equals

$$-\frac{4 \log 2}{Q} \frac{\zeta(2)\zeta(3)}{\zeta(6)} \int_0^1 Q^{u/2} \left(\frac{f(u)}{2} - \frac{f'(u)}{\log Q} \right) du + O\left(\frac{\log^2 Q}{Q} \right). \tag{4.5}$$

We now analyze the three cases of the theorem, corresponding to different support restrictions for our test function.

(1) *Proof of Theorem 1.7(1).* To prove (1.8), we need to understand the part of the integral in (4.4) with $a \leq u \leq 2$. Arguing as in [FG2] and using Proposition 3.4 of [Fi1], we have that for $x^{1/2} \leq Q \leq x$,

$$\sum_{Q/2 < q \leq Q} \left(\psi(x; q, 1) - \frac{\psi(x)}{\phi(q)} \right) \;\ll\; Q \left(\log(x/Q) + 1 \right) + \frac{x^{3/2} (\log x)^2}{Q}. \tag{4.6}$$

Using this, the part of the integral in (4.4) with $a \leq u \leq 2$ is

$$\ll \; \int_a^\sigma \left(Q^{-u/2} (\log(Q^{u-1}) + 1) + Q^{u-2} (\log(Q^u))^2 \right) du \;\ll\; Q^{-\frac{a}{2}} + Q^{\sigma-2} \log Q. \tag{4.7}$$

(2) *Proof of Theorem 1.7(2).* We now prove (1.9).

We need to study the part of the integral in (4.4) with $1+\kappa \le u \le \frac{3}{2}$. We first see that by (4.7), the part of the integral with $\frac{4}{3} \le u \le \frac{3}{2}$ is

$$\ll \; Q^{-\frac{2}{3}} + Q^{\sigma-2}\log Q. \tag{4.8}$$

We turn to the part of the integral with $1+\kappa \le u \le \frac{4}{3}$. We have by Theorem 4.1 (setting $x := Q^u$ and $M := Q^{u-1}$) that it is

$$\begin{aligned} &= 2\int_{1+\kappa}^{\frac{4}{3}} \left(\frac{f(u)}{2} - \frac{f'(u)}{\log Q}\right) Q^{-u/2}\Big(-\frac{1}{2}\log(Q^{u-1}) - \frac{C_6}{2} \\ &\qquad\qquad + O_\epsilon\left(Q^{\frac{1-u}{2}(1-\epsilon)} + Q^{\frac{3}{2}u-2}(\log Q^u)^2\right)\Big)du \\ &= -\int_{1+\kappa}^{\frac{4}{3}} ((u-1)\log Q + C_6)Q^{-u/2}\left(\frac{f(u)}{2} - \frac{f'(u)}{\log Q}\right)du + O_\epsilon\left(\frac{Q^{-\frac{1}{2}-\kappa(1-\epsilon)}}{\log Q} + Q^{-\frac{2}{3}}\log Q\right), \end{aligned} \tag{4.9}$$

hence (1.9) holds.

(3) *Proof of Theorem 1.7(3).* We now turn to (1.10), with f supported in $(-\frac{3}{2}, \frac{3}{2})$. Set $\kappa := \frac{A\log\log Q}{\log Q}$ with $A \ge 1$ a constant. As the big-Oh constant in (4.9) is independent of κ, we may use (4.9) to estimate the contribution to (4.4) from $u \in [1+\kappa, \frac{4}{3}]$. This part of the integral contributes

$$\begin{aligned} -\int_{1+\kappa}^{4/3} ((u-1)\log Q + C_6)Q^{-u/2}\left(\frac{f(u)}{2} - \frac{f'(u)}{\log Q}\right)du + O_\epsilon\left(\frac{Q^{-1/2}}{(\log Q)^{A(1-\epsilon)+1}}\right) \\ \ll \frac{Q^{-1/2}}{(\log Q)^{A/2}}. \end{aligned} \tag{4.10}$$

The part of the integral with $\frac{4}{3} \le u \le \frac{3}{2}$ was already shown to be $\ll Q^{-\frac{2}{3}} + Q^{\sigma-2}\log Q$, and hence is absorbed into the error term since $\sigma < 3/2$.

We now come to the heart of the argument, the part of the integral where $1 \le u \le 1+\kappa$. Since $f \in C^2(\mathbb{R})$, we have that in our range of u, $g(u) := \frac{f(u)}{2} - \frac{f'(u)}{\log Q}$ satisfies

$$\begin{aligned} g(u) &= \frac{f(1)}{2} + \frac{f'(1)}{2}(u-1) + O((u-1)^2) - \frac{f'(1)}{\log Q} + O\left(\frac{u-1}{\log Q}\right) \\ &= P(u-1) + O\left(\frac{(\log\log Q)^2}{(\log Q)^2}\right), \end{aligned} \tag{4.11}$$

where $P(u) := \frac{f(1)}{2} - \frac{f'(1)}{\log Q} + \frac{f'(1)}{2}u$. At this point, if f were $C^K(\mathbb{R})$, we could take its Taylor expansion and get an error of $O_{\epsilon,A}\left(\frac{(\log\log Q)^K}{(\log Q)^K}\right)$.

We cannot apply Theorem 4.1 directly since the error term is not got enough for moderate values of M. Instead, we argue as in the proof of Proposition 6.1 of [Fi1].

Slightly modifying the proof and using GRH, we get that

$$\sum_{Q/2<q\le Q}\left(\psi(x;q,a)-\frac{\psi(x)}{\phi(q)}\right)$$
$$= x\Bigg(-C_1-\sum_{r<\frac{x-1}{Q}}\frac{1}{\phi(r)}\left(1-\frac{r}{x/Q}\right)+\sum_{r<\frac{x-1}{Q/2}}\frac{1}{\phi(r)}\left(1-\frac{r}{2x/Q}\right)\Bigg)+O_\epsilon\left(\frac{x^{3/2+\frac{\epsilon}{2}}}{Q}\right), \tag{4.12}$$

with

$$C_1 := \frac{\zeta(2)\zeta(3)}{\zeta(6)}\log 2. \tag{4.13}$$

(We used that $\sum_{Q/2<q\le Q}=\sum_{Q/2<q\le x}-\sum_{Q<q\le x}$, as in the proof of Theorem 4.1* of [Fi2].) The contribution of the error term in (4.12) to the part of the integral in (4.4) with $1\le u\le 1+\kappa$ is (remember $\kappa\log Q = A\log\log Q$)

$$\ll \int_1^{1+\kappa}\frac{1}{Q/2}\frac{Q^{\frac{3u}{2}+\frac{\epsilon u}{2}}/Q}{Q^{u/2}}\,du \ \ll_\epsilon\ Q^{-1+\epsilon}. \tag{4.14}$$

Therefore, all that remains to complete the proof of Theorem 1.7(3) it to estimate the contribution to (4.4) from $u\in[1,1+\kappa]$. Using Lemma 5.9 of [Fi1] to bound the error in replacing $g(u)$ with $P(u-1)$, we find

$$\begin{aligned}
&2\int_1^{1+\kappa} g(u)\frac{Q^{\frac{u}{2}}}{Q/2}\Bigg(-C_1-\sum_{r<\frac{Q^u-1}{Q}}\frac{1}{\phi(r)}\left(1-\frac{r}{Q^{u-1}}\right)\\
&\qquad\qquad+\sum_{r<\frac{Q^u-1}{Q/2}}\frac{1}{\phi(r)}\left(1-\frac{r}{2Q^{u-1}}\right)\Bigg)du\\
&= 4\int_1^{1+\kappa}P(u-1)Q^{\frac{u}{2}-1}\Bigg(-C_1-\sum_{r\le\frac{Q^u-1}{Q}}\frac{1}{\phi(r)}\left(1-\frac{r}{Q^{u-1}}\right)\\
&\qquad+\sum_{r\le\frac{Q^u-1}{Q/2}}\frac{1}{\phi(r)}\left(1-\frac{r}{2Q^{u-1}}\right)\Bigg)du+O\left(\frac{Q^{-1/2}(\log\log Q)^2}{(\log Q)^3}\right);
\end{aligned} \tag{4.15}$$

we changed $r<\cdots$ to $r\le\cdots$ in the sums above, which gives a negligible error term.

Setting $R := Q^\kappa - \frac{1}{Q}$, we compute that

$$\begin{aligned}
&\int_1^{1+\kappa} P(u-1)Q^{\frac{u}{2}-1} \sum_{r\le \frac{Q^u-1}{Q}} \frac{1}{\phi(r)}\left(1-\frac{r}{Q^{u-1}}\right)du \\
&= \frac{1}{Q}\sum_{r\le Q^\kappa-\frac{1}{Q}} \frac{1}{\phi(r)}\int_{1+\frac{\log(r+Q^{-1})}{\log Q}}^{1+\kappa} P(u-1)\left(Q^{u/2}-\frac{r}{Q^{\frac{u}{2}-1}}\right)du \\
&= \frac{1}{Q}\sum_{r\le R} \frac{1}{\phi(r)}\int_{1+\kappa\frac{\log r}{\log R}}^{1+\kappa} P(u-1)\left(Q^{u/2}-\frac{r}{Q^{\frac{u}{2}-1}}\right)du + O_\epsilon(Q^{-\frac{3}{2}+\epsilon}) \\
&= Q^{-1/2}\sum_{r\le R} \frac{1}{\phi(r)}\int_{\kappa\frac{\log r}{\log R}}^{\kappa} P(u)\left(Q^{u/2}-\frac{r}{Q^{u/2}}\right)du + O_\epsilon(Q^{-\frac{3}{2}+\epsilon}) \\
&= Q^{-1/2}\sum_{r\le R} \frac{1}{\phi(r)}\int_{\frac{\log r}{\log R}}^{1} \kappa P(\kappa v)\left(R^{\frac{v}{2}}-\frac{r}{R^{\frac{v}{2}}}\right)dv + O_\epsilon(Q^{-\frac{3}{2}+\epsilon}).
\end{aligned} \tag{4.16}$$

Let

$$F_1 := -4\zeta\left(\frac{1}{2}\right)\prod_p\left(1+\frac{1}{(p-1)p^{1/2}}\right) \text{ and } F_2 := F_1\left(\frac{\zeta'}{\zeta}\left(\frac{1}{2}\right)-\sum_p\frac{\log p}{(p-1)p^{\frac{1}{2}}+1}\right). \tag{4.17}$$

By Lemma 2.3, we find (4.16) equals

$$\begin{aligned}
&\frac{\kappa}{Q^{1/2}}\Bigg(E_1\log R\int_{-\infty}^1 R^{u/2}vP(\kappa v)dv + E_2\int_{-\infty}^1 R^{\frac{v}{2}}P(\kappa v)dv \\
&\qquad F_1\frac{P(\kappa)-\kappa P'(\kappa)}{\log R} + F_2\frac{-\kappa P'(\kappa)}{(\log R)^2} + O\left(R^{-1/2}\right)\Bigg) \\
&= Q^{-1/2}\Bigg(E_1\log Q\int_{-\infty}^\kappa Q^{u/2}uP(u)du + E_2\int_{-\infty}^\kappa Q^{u/2}P(u)du \\
&\qquad + F_1\frac{\frac{f(1)}{2}-\frac{f'(1)}{\log Q}}{\log Q} - F_2\frac{f'(1)}{2(\log Q)^2} + O\left(R^{-1/2}\right)\Bigg).
\end{aligned} \tag{4.18}$$

We obtain in an analogous way with $R := 2Q^\kappa - \frac{2}{Q}$ that

$$\begin{aligned}
&\int_1^{1+\kappa} P(u-1)Q^{\frac{u}{2}-1}\sum_{r\le 2\frac{Q^u-1}{Q}}\frac{1}{\phi(r)}\left(1-\frac{r}{2Q^{u-1}}\right)du \\
&= Q^{-1/2}\sum_{r\le R}\frac{1}{\phi(r)}\int_{\frac{\log(r/2)}{\log(R/2)}}^1 \kappa P(\kappa v)\left((R/2)^{\frac{v}{2}}-\frac{r}{2(R/2)^{\frac{v}{2}}}\right)dv + O_\epsilon(Q^{-\frac{3}{2}+\epsilon}),
\end{aligned} \tag{4.19}$$

which by Lemma 2.3 is

$$\begin{aligned}
&= \frac{\kappa}{Q^{1/2}}\Bigg(E_1\log(R/2)\int_{-\infty}^{1}(R/2)^{\frac{v}{2}}vP(\kappa v)dv \\
&\quad + (E_2+E_1\log 2)\int_{-\infty}^{1}(R/2)^{\frac{v}{2}}P(\kappa v)dv + \sum_{j=1}^{n}\frac{F_j^{(2)}}{(\log(R/2))^j} + O(R^{-1/2})\Bigg) \\
&= Q^{-1/2}\Bigg(E_1\log Q\int_{-\infty}^{\kappa}Q^{u/2}uP(u)du + (E_2+E_1\log 2)\int_{-\infty}^{\kappa}Q^{u/2}P(u)dv \\
&\qquad + \frac{F_1}{\sqrt{2}}\frac{\frac{f(1)}{2}-\frac{f'(1)}{\log Q}}{\log Q} - \frac{F_2+F_1\log 2}{\sqrt{2}}\frac{f'(1)}{2(\log Q)^2} + O(R^{-1/2})\Bigg). \qquad (4.20)
\end{aligned}$$

We now substitute (4.18) and (4.20) in (4.15), to get that (4.15) is (notice the remarkable cancellations),

$$\begin{aligned}
&= -4C_1\int_{1}^{1+\kappa}P(u-1)Q^{\frac{u}{2}-1}du + 4E_1\log 2Q^{-\frac{1}{2}}\int_{-\infty}^{\kappa}Q^{\frac{u}{2}}P(u)du \\
&\quad + 4Q^{-\frac{1}{2}}\left(-F_1\frac{\frac{f(1)}{2}-\frac{f'(1)}{\log Q}}{\log Q} + F_2\frac{f'(1)}{2(\log Q)^2} + \frac{F_1}{\sqrt{2}}\frac{\frac{f(1)}{2}-\frac{f'(1)}{\log Q}}{\log Q} - \frac{F_2+F_1\log 2}{\sqrt{2}}\frac{f'(1)}{2(\log Q)^2}\right) \\
&\qquad + O\left(Q^{-\frac{1}{2}}\frac{(\log\log Q)^2}{(\log Q)^3} + \frac{Q^{-\frac{1}{2}}}{(\log Q)^{A/2}}\right) \\
&= 4\log 2\frac{\zeta(2)\zeta(3)}{\zeta(6)}\int_{-\infty}^{1}P(u-1)Q^{\frac{u}{2}-1}du \\
&\quad + (2-\sqrt{2})Q^{-\frac{1}{2}}\left(-F_1\frac{f(1)}{\log Q} + \left(F_2 - \frac{\sqrt{2}+4}{3}F_1\right)\frac{f'(1)}{(\log Q)^2}\right) \\
&\qquad + O\left(Q^{-\frac{1}{2}}\frac{(\log\log Q)^2}{(\log Q)^3} + \frac{Q^{-\frac{1}{2}}}{(\log Q)^{A/2}}\right). \qquad (4.21)
\end{aligned}$$

But, yet another cancellation is coming: we have that

$$\int_{-\infty}^{1}P(u-1)Q^{\frac{u}{2}-1}du \;=\; \int_{-\infty}^{1}g(u)Q^{\frac{u}{2}-1}du + O\left(Q^{-\frac{1}{2}}\frac{(\log\log Q)^2}{(\log Q)^3}\right), \qquad (4.22)$$

and so by (4.5) this term cancels (up to the error term $O(Q^{-1})$) with the part of the integral of $T_4(Q)$ with $u \le 1$ (which is coming from a totally different part of the problem, where there are no primes in arithmetic progressions involved)!

Combining all the terms,

$$\frac{1}{Q/2}\sum_{Q/2<q\le Q} T_4(Q) \;=\; (2-\sqrt{2})Q^{-1/2}\left(-F_1\frac{f(1)}{\log Q}+\left(F_2-\frac{\sqrt{2}+4}{3}F_1\right)\frac{f'(1)}{(\log Q)^2}\right) + O\left(\frac{Q^{-1/2}}{(\log Q)^{A/2}}+Q^{-1/2}\frac{(\log\log Q)^2}{(\log Q)^3}\right). \tag{4.23}$$

The proof is completed by taking $A=6$.

□

5. Results under De-averaging Hypothesis (Theorem 1.13)

In this section we assume the de-averaging hypothesis (Hypothesis 1.3), which relates the variance in the distribution of primes congruent to 1 to the average variance over all residue classes. Explicitly, we assume (1.16) holds for some $\eta \in (0,1]$, and show how this allows us to compute the main term in the averaged 1-level density, $D_{1;Q/2,Q}(\widehat{f})$, for test functions f supported in $[-4+2\eta, 4-2\eta]$. (Remember that this hypothesis is trivially true for $\eta=1$, and expected to hold for any $\eta>0$.)

Proof of Theorem 1.13. Starting from (2.23), we have that

$$T_4(q) \;=\; 2\int_0^\infty\left(\frac{f(u)}{2}-\frac{f'(u)}{\log Q}\right)\frac{\psi(Q^u;q,1)-\frac{\psi(Q^u)}{\phi(q)}}{Q^{u/2}}\,du. \tag{5.1}$$

Feeding this into Proposition 2.2, we are left with determining

$$\frac{1}{Q/2}\sum_{Q/2<q\le Q} T_4(q) \;=\; \frac{1}{Q/2}\int_0^\sigma\left(\frac{f(u)}{2}-\frac{f'(u)}{\log Q}\right)Q^{-u/2}\sum_{Q/2<q\le Q}\left(\psi(Q^u;q,1)-\frac{\psi(Q^u)}{\phi(q)}\right)du. \tag{5.2}$$

We have already seen in the proof of Theorem 1.2 that the part of the integral in (5.2) with $0\le u\le 1$ is $O(Q^{-1/2})$. For the part where $u\ge 1$, the Cauchy-Schwartz inequality shows that its contribution to the integral in (5.2) is

$$\ll\; \frac{1}{Q/2}\int_1^\sigma Q^{-u/2}\left|\sum_{Q/2<q\le Q}\left(\psi(Q^u;q,1)-\frac{\psi(Q^u)}{\phi(q)}\right)^2\right|^{1/2}\cdot\left|\sum_{Q/2<q\le Q}1^2\right|^{1/2}du. \tag{5.3}$$

Now, by Hypothesis 1.11_η, this is

$$\ll\; \frac{1}{Q/2}\int_1^\sigma Q^{-u/2}\cdot Q^{\frac{\eta-1}{2}}\left(\sum_{Q/2<q\le Q}\sum_{\substack{1\le a\le q\\(a,q)=1}}\left(\psi(Q^u;q,a)-\frac{\psi(Q^u)}{\phi(q)}\right)^2\right)^{1/2}\cdot Q^{1/2}\,du. \tag{5.4}$$

We now use a result of Goldston and Vaughan [GV], which states that under GRH we have for $1 \le Q \le x$ that

$$\sum_{q\le Q}\sum_{\substack{1\le a\le q\\(a,q)=1}}\left(\psi(x;q,a)-\frac{\psi(x)}{\phi(q)}\right)^2$$
$$= Qx\log Q - cxQ + O_\epsilon\left(Q^2(x/Q)^{\frac14+\epsilon} + x^{3/2}(\log 2x)^{5/2}(\log\log 3x)^2\right), \quad (5.5)$$

where $c := \gamma + \log 2\pi + 1 + \sum_p \frac{\log p}{p(p-1)}$.

We now split the range of integration into the two subintervals $1 \le u \le 2$ and $2 \le u \le \sigma$. In the first range, we have that for $\epsilon > 0$ small enough, $u+1 \ge \max(7/4+u/4+\epsilon(u-1), 3u/2)$, so (5.5) implies that

$$\sum_{q\le Q}\sum_{\substack{1\le a\le q\\(a,q)=1}}\left(\psi(x;q,a)-\frac{\psi(x)}{\phi(q)}\right)^2 \ll Qx(\log x)^3 \quad (5.6)$$

(which, up to x^ϵ, follows from Hooley's original result [Ho]), so we get that the part of (5.4) with $1 \le u \le 2$ is

$$\ll Q^{\frac{\eta}{2}-1}\int_1^2 Q^{-u/2}Q^{\frac{u+1}{2}}(\log Q)^{3/2}\, du \ll Q^{\frac{\eta-1}{2}}(\log Q)^{3/2}, \quad (5.7)$$

which is $o(1)$ if $\eta < 1$.

We now examine the second interval, that is $2 \le u \le \sigma$. In this range, (5.5) becomes

$$\sum_{q\le Q}\sum_{\substack{1\le a\le q\\(a,q)=1}}\left(\psi(x;q,a)-\frac{\psi(x)}{\phi(q)}\right)^2 \ll x^{3/2}(\log x)^{5/2}(\log\log x)^2 \quad (5.8)$$

(which, up to a factor of x^ϵ, follows from Hooley's original result [Ho]). We thus get that the part of (5.4) with $2 \le u \le \sigma$ is

$$\ll \frac{Q^{\frac{\eta}{2}}}{Q/2}\int_2^\sigma Q^{-u/2}Q^{3u/4}(u\log Q)^{5/4}\log\log(Q^u)du \ll Q^{\frac{\sigma+2\eta}{4}-1}(\log Q)^{1/4}\log\log Q. \quad (5.9)$$

If $\sigma < 4 - 2\eta$ then the above is $o(1)$, completing the proof. □

6. Results under Montgomery's Hypothesis (Theorem 1.16)

We continue our investigations beyond the GRH, and assume a smoothed version of Montgomery's hypothesis, Hypothesis (1.3). Interestingly, this assumption allows us to compute the main term of the 1-level density, $D_{1;q}(\widehat{f})$, for test functions of arbitrarily large (but finite) support. While similar results have been previously observed [MilSar], we include a proof both for completeness and because these observations are not in the literature.

Proof of Theorem 1.16. As we are fixing the modulus, we take $Q := q$. By the explicit formula from Proposition 2.1, we have

$$\begin{aligned} D_{1;q}(\widehat{f}) &= \frac{f(0)}{\log q}\left(\log q - \log(8\pi e^{\gamma}) - \sum_{p|q}\frac{\log p}{p-1}\right) + \int_0^\infty \frac{f(0)-f(t)}{q^{t/2}-q^{-t/2}}dt \\ &\quad - \frac{2}{\log q}\left(\sum_{n\equiv 1 \bmod q} - \frac{1}{\phi(q)}\sum_n\right)\frac{\Lambda(n)}{n^{1/2}} f\left(\frac{\log n}{\log q}\right) + O\left(\frac{1}{\phi(q)}\right). \end{aligned} \quad (6.1)$$

Let $\sigma := \sup(\text{supp} f) < \infty$. We proved in §3 that the only terms that are not $O(1/\log q)$ are the leading term $f(0)$ and possibly the prime sum, which we now study. We have

$$T_4(q) \;=\; 2\int_0^\infty \left(\frac{f(u)}{2} - \frac{f'(u)}{\log q}\right)\frac{\psi(q^u;q,1) - \frac{\psi(q^u)}{\phi(q)}}{q^{u/2}}\,du. \quad (6.2)$$

In the proof of Theorem 1.2 we determined that the part of the integral with $0 \le u \le 1$ is $O(q^{-1/2})$. From the proof of Theorem 1.4, the part with $1 \le u \le 2$ is $O(\frac{\log\log q}{\log q})$.

(1) *Proof of Theorem 1.16(1).* For the rest of the integral, we use Hypothesis 1.15*. Note that $u \ge 2$, so $x = q^u \ge q^2$ with $u \le \sigma$, hence we can replace $o_{x\to\infty}$ by $o_{q\to\infty}$. An integration by parts gives that the rest of the integral is

$$\begin{aligned} &= 0 - \left(\frac{f(2)}{2} - \frac{f'(2)}{\log q}\right)\frac{\psi_2(q^2;q,1) - \frac{\psi_2(q^2)}{\phi(q)}}{q} \\ &\quad - 2\int_0^\infty \left(\frac{3f(u)}{4} - \frac{2f'(u)}{\log q} + \frac{f''(u)}{(\log q)^2}\right)\frac{\psi_2(q^u;q,1) - \frac{\psi_2(q^u)}{\phi(q)}}{q^{u/2}}\,du \\ &= \frac{o(q)}{q} + \int_2^\sigma (|f(u)| + |f'(u)| + |f''(u)|)\,\frac{o(q^{u/2})}{q^{u/2}}du \;=\; o(1), \end{aligned} \quad (6.3)$$

proving the claim. Note that we are using the smoothed version of the prime sum.

(2) *Proof of Theorem 1.16(2).* We already know that the part of the integral with $0 \le u \le 1$ is $\ll q^{-1/2}$. Taking $\epsilon := \epsilon'/\sigma$ in Hypothesis 1.15_θ, the rest of the integral is $O\left(\int_1^\sigma q^{\epsilon u - \theta}du\right)$, which is $O\left(q^{\epsilon' - \theta}\right)$ and thus negligible if we may take $\theta > 0$.

□

Remark 6.1. *Depending on our assumptions about the size of the error term in the distribution of primes in residue classes, we may allow σ to grow with Q at various explicit rates.*

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