

# FROM FIBONACCI NUMBERS TO CENTRAL LIMIT TYPE THEOREMS

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## Abstract

A beautiful theorem of Zeckendorf states that every integer can be written uniquely as a sum of non-consecutive Fibonacci numbers  $\{F_n\}_{n=1}^\infty$ . Lekkerkerker proved that the average number of summands for integers in  $[F_n, F_{n+1})$  is  $n/(\varphi^2 + 1)$ , with  $\varphi$  the golden mean. This has been generalized to the following: given nonnegative integers  $c_1, c_2, \dots, c_L$  with  $c_1, c_L > 0$  and recursive sequence  $\{H_n\}_{n=1}^\infty$  with  $H_1 = 1$ ,  $H_{n+1} = c_1 H_n + c_2 H_{n-1} + \dots + c_n H_1 + 1$  ( $1 \leq n < L$ ) and  $H_{n+1} = c_1 H_n + c_2 H_{n-1} + \dots + c_L H_{n+1-L}$  ( $n \geq L$ ), every positive integer can be written uniquely as  $\sum a_i H_i$  under natural constraints on the  $a_i$ 's, the mean and the variance of the numbers of summands for integers in  $[H_n, H_{n+1})$  are of size  $n$ , and the distribution of the numbers of summands converges to a Gaussian as  $n$  goes to the infinity. Previous approaches used number theory or ergodic theory. We convert the problem to a combinatorial one. In addition to re-deriving these results, our method generalizes to a multitude of other problems (in the sequel paper [BM] we show how this perspective allows us to determine the distribution of gaps between summands in decompositions). For example, it is known that every integer can be written uniquely as a sum of the  $\pm F_n$ 's, such that every two terms of the same (opposite) sign differ in index by at least 4 (3). The presence of negative summands introduces complications and features not seen in previous problems. We prove that the distribution of the numbers of positive and negative summands converges to a bivariate normal with computable, negative correlation, namely  $-(21 - 2\varphi)/(29 + 2\varphi) \approx -0.551058$ .

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## 1. INTRODUCTION

**1.1. History.** The Fibonacci numbers have intrigued mathematicians for hundreds of years. One of their most interesting properties is the Zeckendorf decomposition. Zeckendorf [Ze] proved that every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers (called the *Zeckendorf decomposition*), where the Fibonacci numbers<sup>1</sup> are  $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, \dots$ . Lekkerkerker extended this result and proved that the average number of summands needed to represent an integer in  $[F_n, F_{n+1})$  is  $\frac{n}{\varphi^2+1} + O(1) \approx 0.276n$ , where  $\varphi = \frac{\sqrt{5}+1}{2}$  is the golden mean. There is a related question: *how are the number of summands distributed about the mean for integers in  $[F_n, F_{n+1})$ ?* This is a very natural question to ask. Both the question and the answer are reminiscent of the Erdős-Kac Theorem [EK], which states that as  $n \rightarrow \infty$  the number of distinct prime divisors of integers on the order of size  $n$  tends to a Gaussian with mean  $\log \log n$  and standard deviation  $\sqrt{\log \log n}$ .

We first set some notation before describing the previous results.

**Definition 1.1.** We say a sequence  $\{H_n\}_{n=1}^\infty$  of positive integers is a **Positive Linear Recurrence Sequence (PLRS)** if the following properties hold:

- (1) Recurrence relation: There are non-negative integers  $L, c_1, \dots, c_L$  such that

$$H_{n+1} = c_1 H_n + \dots + c_L H_{n+1-L}, \quad (1.1)$$

with  $L, c_1$  and  $c_L$  positive.

- (2) Initial conditions:  $H_1 = 1$ , and for  $1 \leq n < L$  we have

$$H_{n+1} = c_1 H_n + c_2 H_{n-1} + \dots + c_n H_1 + 1. \quad (1.2)$$

We call a decomposition  $\sum_{i=1}^m a_i H_{m+1-i}$  of a positive integer  $N$  (and the sequence  $\{a_i\}_{i=1}^m$ ) **legal** if  $a_1 > 0$ , the other  $a_i \geq 0$ , and one of the following two conditions holds:

**Condition 1.** We have  $m < L$  and  $a_i = c_i$  for  $1 \leq i \leq m$ .

**Condition 2.** There exists  $s \in \{1, \dots, L\}$  such that

$$a_1 = c_1, a_2 = c_2, \dots, a_{s-1} = c_{s-1} \text{ and } a_s < c_s, \quad (1.3)$$

$a_{s+1}, \dots, a_{s+\ell} = 0$  for some  $\ell \geq 0$ , and  $\{b_i\}_{i=1}^{m-s-\ell}$  (with  $b_i = a_{s+\ell+i}$ ) is legal.

If  $\sum_{i=1}^m a_i H_{m+1-i}$  is a legal decomposition of  $N$ , we define the **number of summands** (of this decomposition of  $N$ ) to be  $a_1 + \dots + a_m$ .

Informally, a legal decomposition is one where we cannot use the recurrence relation to replace a linear combination of summands with another summand, and the coefficient of each summand is appropriately bounded; other authors [DG, Ste1] use the phrase  $G$ -ary decomposition for a legal decomposition, and sum-of-digits function for the number of summands. For example, if  $H_{n+1} = 2H_n + 3H_{n-1} + H_{n-2}$ , then  $H_5 + 2H_4 + 3H_3 + H_1$  is legal, while

<sup>1</sup>If we used the standard counting, then 1 would appear twice and numerous decompositions would not be unique.

$H_5 + 2H_4 + 3H_3 + H_2$  is not (we can replace  $2H_4 + 3H_3 + H_2$  with  $H_5$ ), nor is  $7H_5 + 2H_2$  (as the coefficient of  $H_5$  is too large). Note the Fibonacci numbers are just the special case of  $L = 2$  and  $c_1 = c_2 = 1$ .

The following probabilistic language will be convenient for stating some of the results.

**Definition 1.2** (Associated Probability Space to a Positive Linear Recurrence Sequence). *Let  $\{H_n\}$  be a Positive Linear Recurrence Sequence. For each  $n$ , consider the discrete outcome space*

$$\Omega_n = \{H_n, H_n + 1, H_n + 2, \dots, H_{n+1} - 1\} \tag{1.4}$$

with probability measure

$$\mathbb{P}_n(A) = \sum_{\substack{\omega \in A \\ \omega \in \Omega_n}} \frac{1}{H_{n+1} - H_n}, \quad A \subset \Omega_n; \tag{1.5}$$

in other words, each of the  $H_{n+1} - H_n$  numbers is weighted equally. We define the random variable  $K_n$  by setting  $K_n(\omega)$  equal to the number of summands of  $\omega \in \Omega_n$  in its legal decomposition. Implicit in this definition is that each integer has a unique legal decomposition; we prove this in Theorem 1.1, and thus  $K_n$  is well-defined.

We denote the cardinality of  $\Omega_n$  by

$$\Delta_n = H_{n+1} - H_n, \tag{1.6}$$

and we set  $p_{n,k}$  equal to the number of elements in  $[H_n, H_{n+1})$  whose generalized Zeckendorf decomposition has exactly  $k$  summands; thus

$$p_{n,k} = \Delta_n \cdot \text{Prob}(K_n = k). \tag{1.7}$$

We first review previous results and methods, and then describe our new perspective and extensions. See [Bu, Ha, Ho, Ke, Len] for more on generalized Zeckendorf decompositions, [GT] for a proof of Theorems 1.1 and 1.2, and [DG, FGNPT, GTNP, LT, Ste1] for a proof and some generalizations of Theorem 1.3.

**Theorem 1.1** (Generalized Zeckendorf's Theorem for PLRS). *Let  $\{H_n\}_{n=1}^\infty$  be a Positive Linear Recurrence Sequence. Then*

(a) *There is a unique legal decomposition for each positive integer  $N \geq 0$ .*

(b) *There is a bijection between the set  $\mathcal{S}_n$  of integers in  $[H_n, H_{n+1})$  and the set  $\mathcal{D}_n$  of legal decompositions  $\sum_{i=1}^n a_i H_{n+1-i}$ .*

**Theorem 1.2** (Generalized Lekkerkerker's Theorem for PLRS). *Let  $\{H_n\}_{n=1}^\infty$  be a Positive Linear Recurrence Sequence, let  $K_n$  be the random variable of Definition 1.2 and denote its mean by  $\mu_n$ . Then there exist constants  $C > 0$ ,  $d$  and  $\gamma_1 \in (0, 1)$  depending only on  $L$  and the  $c_i$ 's in the recurrence relation of the  $H_n$ 's such that*

$$\mu_n = Cn + d + o(\gamma_1^n). \tag{1.8}$$

**Theorem 1.3** (Gaussian Behavior for PLRS). *Let  $\{H_n\}_{n=1}^\infty$  be a Positive Linear Recurrence Sequence and let  $K_n$  be the random variable of Definition 1.2. The mean  $\mu_n$  and variance  $\sigma_n^2$  of  $K_n$  grow linearly in  $n$ , and  $(K_n - \mu_n)/\sigma_n$  converges weakly to the standard normal  $N(0, 1)$  as  $n \rightarrow \infty$ .*

While the proof of Theorem 1.3 becomes very technical in general, the special case  $L = 1$  is straightforward, and suggests why the result should hold. When  $L = 1$ ,  $H_n = c_1^{n-1}$ . Thus our PLRS is just the geometric series  $1, c_1, c_1^2, \dots$ , and a legal decomposition of  $N$  is just its base  $c_1$  expansion. Hence every positive integer has a unique legal decomposition. Further, the distribution of the number of summands converges to a Gaussian by the Central Limit Theorem, as we essentially have the sum of  $n - 1$  independent, identically distributed discrete uniform random variables.<sup>2</sup>

Previous approaches to this problem used number theory or ergodic theory, often requiring the analysis of certain exponential sums. We recast this as a combinatorial problem. We are able to re-derive the above results from a different perspective. Our method generalizes to a multitude of other problems (in a sequel paper we use the combinatorial vantage to determine the distribution of gaps between summands). For the main part of this paper, we concentrate on one particularly interesting situation where features not present in previous works arise.

**Definition 1.4.** *We call a sum of the  $\pm F_n$ 's a **far-difference representation** if every two terms of the same sign differ in index by at least 4, and every two terms of opposite sign differ in index by at least 3.*

Recently Alpert [Al] proved the analogue of Zeckendorf's Theorem for the far-difference representation. It is convenient to set

$$S_n = \begin{cases} \sum_{0 < n-4i \leq n} F_{n-4i} = F_n + F_{n-4} + F_{n-8} + \dots & \text{if } n > 0 \\ 0 & \text{otherwise.} \end{cases} \quad (1.9)$$

**Theorem 1.5** (Generalized Zeckendorf's Theorem for Far-Difference Representations). *Every integer has a unique far-difference representation. For each  $N \in (S_{n-1} = F_n - S_{n-3} - 1, S_n]$ , the first term in its far-difference representation is  $F_n$ , and the unique far-difference representation of 0 is the empty representation.*

Most results in the literature concern only one quantity, the number of summands. An exception is [Ste2], where the standard Zeckendorf expansion (called the greedy expansion) and the lazy expansion (which uses as many summands as possible) are simultaneously considered. Steiner proves that their joint distribution converges to a bivariate Gaussian with a correlation of  $9 - 5\varphi \approx .90983$ . Unlike the Zeckendorf expansions, the far-difference representations have both positive and negative summands, which opens up the fascinating question of how the number of each are related. In the result below we find a non-zero correlation between the two types of summands.

**Theorem 1.6** (Generalized Lekkerkerker's Theorem and Gaussian Behavior for Far-Difference Representations). *Let  $\mathcal{K}_n$  and  $\mathcal{L}_n$  be random variables denoting the number of positive and negative summands in the far-difference representation for integers in  $(S_{n-1}, S_n]$ . As  $n$  tends to infinity,  $\mathbb{E}[\mathcal{K}_n] = \frac{1}{10}n + \frac{371-113\sqrt{5}}{40} + o(1)$ , and is  $\frac{\sqrt{5}+1}{4} = \frac{\varphi}{2}$  greater than  $\mathbb{E}[\mathcal{L}_n]$ ; the variance of both is of size  $\frac{15+21\sqrt{5}}{1000}n$ ; the standardized joint density of  $\mathcal{K}_n$  and  $\mathcal{L}_n$  converges weakly to a*

<sup>2</sup>Writing  $N = a_1 c_1^n + \dots + a_{n+1} 1$ , we are interested in the large  $n$  behavior of  $a_1 + \dots + a_{n+1}$  as we vary  $N$  in  $[c_1^n, c_1^{n+1})$ . The contribution of  $a_1$  is immaterial, and the remaining  $a_i$ 's can be understood by considering the sum of  $n$  independent, identically distributed discrete uniform random variables on  $\{0, \dots, B-1\}$  (which have mean  $\frac{B-1}{2}$  and standard deviation  $\sqrt{(c_1^2 - 1)/12}$ ). Denoting these by  $A_i$ , by the Central Limit Theorem  $A_2 + \dots + A_{n+1}$  converges to being normally distributed with mean  $\frac{c_1-1}{2}n$  and standard deviation  $n\sqrt{(c_1^2 - 1)/12}$ .

bivariate Gaussian with negative correlation  $\frac{10\sqrt{5}-121}{179} = -\frac{21-2\varphi}{29+2\varphi} \approx -0.551$ ; and  $\mathcal{K}_n + \mathcal{L}_n$  and  $\mathcal{K}_n - \mathcal{L}_n$  are independent.

**1.2. Sketch of Proofs.** By recasting the problem as a combinatorial one and using generating functions, we are able to re-derive and extend the previous results in the literature. The key techniques in our proof are generating functions, partial fractional expansions, differentiating identities and the method of moments. Unfortunately, in order to be able to handle a general Positive Linear Recurrence Sequence, the arguments become quite technical due to the fact that we cannot exploit any special properties of the coefficients of the recurrence relations, but rather must prove certain technical lemmas for *any* choice of the  $c_i$ 's. We therefore quickly look at the special case of the Fibonacci numbers, as this highlights the main ideas of the method without many of the technicalities.<sup>3</sup> In the rest of the paper, we provide details only for the results about far-difference representations, as the other results have been proved by other techniques. The reader interested in the details of applying our method to the known cases should see [MW] for the details.

We first derive a recurrence relation for the  $p_{n,k}$ 's, which in this case is the number of integers in  $[F_n, F_{n+1})$  with precisely  $k$  summands in their legal decomposition (see Definition 1.2). We find  $p_{n+1,k+1} = p_{n,k+1} + p_{n,k}$ . Multiplying both sides of this equation by  $x^k y^n$ , summing over  $n, k > 0$ , and calculating the initial values of the  $p_{n,k}$ 's, namely  $p_{1,1}$ ,  $p_{2,1}$  and  $p_{2,2}$ , we obtain a formula for the generating function  $\sum_{n,k>0} p_{n,k} x^k y^n$ :

$$\mathcal{G}(x, y) := \sum_{n,k>0} p_{n,k} x^k y^n = \frac{xy}{1 - y - xy^2}. \tag{1.10}$$

By partial fraction expansion, we write the right-hand side as

$$-\frac{y}{y_1(x) - y_2(x)} \left( \frac{1}{y - y_1(x)} - \frac{1}{y - y_2(x)} \right), \tag{1.11}$$

where  $y_1(x)$  and  $y_2(x)$  are the roots of  $1 - y - xy^2 = 0$ . Rewriting  $\frac{1}{y - y_i(x)}$  as  $-(1 - \frac{y}{y_i(x)})^{-1}$  and using a power series expansion, we are able to compare the coefficients of  $y^n$  of both sides of (1.10). This gives an explicit formula for  $g(x) = \sum_{k>0} p_{n,k} x^k$ .

Note that

$$g(1) = \sum_{k>0} p_{n,k}, \tag{1.12}$$

which is  $F_{n+1} - F_n$  by definition. Further, we have

$$g'(1) = \sum_{k>0} k p_{n,k} = \mathbb{E}[K_n](F_{n+1} - F_n) = \mathbb{E}[K_n]g(1). \tag{1.13}$$

Therefore, once we determine  $g(1)$  and  $g'(1)$ , we know  $\mathbb{E}[K_n]$ .

Letting  $\mu_n = \mathbb{E}[K_n]$ , we define the random variable  $K'_n = K_n - \mu_n$ . We immediately obtain an explicit, closed form expression for  $h_n(x) = g(x) - \mu_n$ . Arguing as above we find  $h_n(1) = F_{n+1} - F_n$  and  $h'_n(1) = \mathbb{E}[K'_n]h_n(1)$ . Furthermore, we get

$$(xh'_n(x))' = \mathbb{E}[K_n'^2]h_n(1), \quad (x(xh'_n(x)))' = \mathbb{E}[K_n'^3]h_n(1), \quad \dots, \tag{1.14}$$

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<sup>3</sup> The proof can be simplified further for the Fibonacci numbers, as the key quantity  $p_{n,k}$  equals  $\binom{n-k}{k-1}/F_{n-1}$ , which by Stirling's formula tends to the density of a normal random variable; see [KKMW] for details. Unfortunately this approach does not generalize, as the formulas for  $p_{n,k}$  become far more involved.

which allows us to compute the moments of  $K'_n$ .

Let  $\sigma_n$  denote the variance of  $K_n$  (which is of course also the variance of  $K'_n$ ), and recall that the  $2m^{\text{th}}$  moment of the standard normal is  $(2m-1)!! = (2m-1)(2m-3)\cdots 1$ . To show that  $K_n$  converges to being normally distributed with mean  $\mu_n$  and variance  $\sigma_n$ , it suffices to show that the  $2m^{\text{th}}$  moment of  $K'_n/\sigma_n$  converges to  $(2m-1)!!$  and the odd moments converge to 0. We are able to prove this through (1.14), which are repeated applications of differentiating identities to our partial fraction expansion of the generating function.

We prove the Gaussian behavior for the far-difference representation in Section 2. We conclude with some natural problems to consider.

## 2. FAR-DIFFERENCE REPRESENTATION

In this section, we apply the generating function approach to study the distributions of the numbers of positive and negative summands in the far-difference representation of integers in  $(S_n, S_{n+1}]$  (see Definition 1.4). We prove that as  $n \rightarrow \infty$  these two random variables converge to being a bivariate Gaussian with a computable, negative correlation. We do not need to prove that a generalization of Zeckendorf's theorem holds for far-difference representations, as this was done by Alpert [Al] (see Theorem 1.5).

**2.1. Generating Function of the Probability Density.** Let  $p_{n,k,l}$  ( $n > 0$ ) be the number of far-difference representations of integers in  $(S_{n-1}, S_n]$  with  $k$  positive summands and  $l$  negative summands. We have the following formula for the generating function  $\hat{\mathcal{G}}(x, y, z) = \sum_{n>0, k>0, l \geq 0} p_{n,k,l} x^k y^l z^n$ .

**Theorem 2.1.** *We have*

$$\hat{\mathcal{G}}(x, y, z) = \frac{xz + xyz^4}{1 - z - (x+y)z^4 - xyz^6 - xyz^7}. \quad (2.1)$$

*Proof.* We first derive the recurrence relation

$$p_{n,k,l} = p_{n-1,k,l} + p_{n-4,k-1,l} + p_{n-3,l,k-1}, \quad n \geq 5, \quad (2.2)$$

by a combinatorial approach. Next we want to get the generating function. To achieve that, we need to have a recurrence relation with all terms of form  $p_{n-n_0, k-k_0, l-l_0}$  with  $n_0, k_0$  and  $l_0$  constant. We solve this by using the preceding recurrence relation with repeated substitutions.

Let us prove (2.2) first. Clearly,  $p_{n,k,l} = 0$  if  $k \leq 0$  or  $l < 0$ . For every far-difference representation  $N = \sum_{j=1}^m a_j F_{i_j} \in [S_{n-1} + 1, S_n]$ ,  $N' := \sum_{j=2}^m a_j F_{i_j}$  is also a far-difference representation. Theorem 1.5 states that  $i_1 = n$  and  $a_1 = 1$ , therefore  $N' \in [S_{n-1} + 1 - F_n, S_n - F_n]$ . Since

$$\begin{aligned} F_n - S_{n-1} - S_{n-3} &= F_n - F_{n-1} - F_{n-3} - F_{n-5} \cdots = F_{n-2} - F_{n-3} - F_{n-5} - \cdots \\ &= F_{n-4} - F_{n-5} - \cdots = \cdots (= F_3 - F_2) = F_2 - F_1 = 1, \end{aligned} \quad (2.3)$$

we get  $S_{n-1} + 1 - F_n = -S_{n-3}$ . Thus  $p_{n,k,l}$  is the number of far-difference representations of integers in  $[-S_{n-3}, S_{n-4}]$  with  $k-1$  positive summands and  $l$  negative summands.

Let  $n \geq 5$ . We have two cases:  $(k-1, l) \neq (0, 0)$  and  $(k-1, l) = (0, 0)$ .

**Case 1.**  $(k-1, l) = (0, 0)$ .

Since  $F_n - S_{n-1} - S_{n-3} = 1$  by (2.3), we have  $F_{n-1} < S_{n-1} < F_n$  for all  $n > 1$ . Hence there is exactly one Fibonacci number in  $[S_{n-1} + 1, S_n]$  for all  $n > 1$ . Thus  $p_{n,1,0} = p_{n-1,1,0} = 1$ . Further, for  $n \geq 5$ , we have  $p_{n-4,0,0} = p_{n-3,0,0} = 0$ , then (2.2) follows.

**Case 2.**  $(k-1, l) \neq (0, 0)$ .

Then  $N' = N - a_1 F_{i_1} \neq 0$ . Let  $N(J, k, l)$  be the number of far-difference representations of integers in the interval  $J$  with  $k$  positive summands and  $l$  negative summands. Thus

$$\begin{aligned} p_{n,k,l} &= N((0, S_{n-4}], k-1, l) + N([-S_{n-3}, 0), k-1, l) \\ &= N((0, S_{n-4}], k-1, l) + N((0, S_{n-3}], l, k-1) \\ &= \sum_{i=1}^{n-4} p_{i,k-1,l} + \sum_{i=1}^{n-3} p_{i,l,k-1}. \end{aligned} \quad (2.4)$$

For  $n \geq 5$ , replacing  $n$  with  $n-1$  yields

$$p_{n-1,k,l} = \sum_{i=1}^{n-5} p_{i,k-1,l} + \sum_{i=1}^{n-4} p_{i,l,k-1}. \quad (2.5)$$

Subtracting (2.5) from (2.4), we get (2.2).

Let  $n \geq 9$ . Replacing  $(n, k, l)$  in (2.2) with  $(n-3, l, k-1)$  gives

$$p_{n-3,l,k-1} = p_{n-4,l,k-1} + p_{n-7,l-1,k-1} + p_{n-6,k-1,l-1}, \quad n \geq 8. \quad (2.6)$$

Rearranging the terms of (2.2), we obtain

$$p_{n-3,l,k-1} = p_{n,k,l} - p_{n-1,k,l} - p_{n-4,k-1,l}, \quad n \geq 5. \quad (2.7)$$

Replacing  $(n, k, l)$  in (2.2) with  $(n-1, k, l)$  and  $(n-4, k, l-1)$  (since  $n \geq 9$ ,  $n-1 > n-4 \geq 5$ , thus (2.7) applies to  $n-1$  and  $n-4$ ), we get

$$p_{n-4,l,k-1} = p_{n-1,k,l} - p_{n-2,k,l} - p_{n-5,k-1,l} \quad (2.8)$$

and

$$p_{n-7,l-1,k-1} = p_{n-4,k,l-1} - p_{n-5,k,l-1} - p_{n-8,k-1,l-1}. \quad (2.9)$$

Plugging (2.6), (2.8) and (2.9) into (2.2) yields

$$\begin{aligned} p_{n,k,l} &= 2p_{n-1,k,l} - p_{n-2,k,l} + p_{n-4,k-1,l} + p_{n-4,k,l-1} - p_{n-5,k-1,l} \\ &\quad - p_{n-5,k,l-1} + p_{n-6,k-1,l-1} - p_{n-8,k-1,l-1}, \quad n \geq 9. \end{aligned} \quad (2.10)$$

Multiplying both sides of (2.10) by  $x^k y^l z^n$ , we get

$$\begin{aligned} p_{n,k,l} x^k y^l z^n &= 2z p_{n-1,k,l} x^k y^l z^{n-1} - z^2 p_{n-2,k,l} x^k y^l z^{n-2} + xz^4 p_{n-4,k-1,l} x^{k-1} y^l z^{n-4} \\ &\quad + yz^4 p_{n-4,k,l-1} x^{k-1} y^l z^{n-4} - xz^5 p_{n-5,k-1,l} x^{k-1} y^l z^{n-5} \\ &\quad - yz^5 p_{n-5,k,l-1} x^k y^{l-1} z^{n-5} + xyz^6 p_{n-6,k-1,l-1} x^{k-1} y^{l-1} z^{n-6} \\ &\quad - xyz^8 p_{n-8,k-1,l-1} x^{k-1} y^{l-1} z^{n-8}. \end{aligned} \quad (2.11)$$

Summing both sides over  $n \geq 9$  and recalling that  $p_{n,k,l} = 0$  if  $k \geq 0$  or  $l < 0$ , we obtain

$$\hat{\mathcal{G}}(x, y, z) = 2z\hat{\mathcal{G}}(x, y, z) - 2 \sum_{1 < n \leq 8} p_{n-1,k,l} x^k y^l z^n - z^2 \hat{\mathcal{G}}(x, y, z)$$

$$\begin{aligned}
& + \sum_{2 < n \leq 8} p_{n-2,k,l} x^k y^l z^n + xz^4 \hat{\mathcal{G}}(x, y, z) - \sum_{4 < n \leq 8} p_{n-4,k-1,l} x^k y^l z^n \\
& + yz^4 \hat{\mathcal{G}}(x, y, z) - \sum_{4 < n \leq 8} p_{n-4,k,l-1} x^k y^l z^n - xz^5 \hat{\mathcal{G}}(x, y, z) \\
& + \sum_{5 < n \leq 8} p_{n-5,k-1,l} x^k y^l z^n - yz^5 \hat{\mathcal{G}}(x, y, z) + \sum_{5 < n \leq 8} p_{n-5,k,l-1} x^k y^l z^n \\
& + xyz^6 \hat{\mathcal{G}}(x, y, z) - \sum_{6 < n \leq 8} p_{n-6,k-1,l-1} x^k y^l z^n - xyz^8 \hat{\mathcal{G}}(x, y, z) \\
= & (2z - z^2 + xz^4 + yz^4 - xz^5 - yz^5 + xyz^6 - xyz^8) \hat{\mathcal{G}}(x, y, z) \\
& - 2 \sum_{1 < n \leq 8} p_{n-1,k,l} x^k y^l z^n + \sum_{2 < n \leq 8} p_{n-2,k,l} x^k y^l z^n - \sum_{4 < n \leq 8} p_{n-4,k-1,l} x^k y^l z^n \\
& - \sum_{4 < n \leq 8} p_{n-4,k,l-1} x^k y^l z^n + \sum_{5 < n \leq 8} p_{n-5,k-1,l} x^k y^l z^n + \sum_{5 < n \leq 8} p_{n-5,k,l-1} x^k y^l z^n \\
& - \sum_{6 < n \leq 8} p_{n-6,k-1,l-1} x^k y^l z^n. \tag{2.12}
\end{aligned}$$

We calculated all  $p_{n,k,l}$ 's for  $n \leq 8$  and found that the only terms in the right-hand side of (2.12) that are not canceled are  $xz$ ,  $-xz^2$ ,  $xyz^4$  and  $-xyz^5$ , therefore

$$\begin{aligned}
\hat{\mathcal{G}}(x, y, z) & = \frac{x(z - z^2) + xy(z^4 - z^5)}{1 - (2z - z^2 + xz^4 + yz^4 - xz^5 - yz^5 + xyz^6 - xyz^8)} \\
& = \frac{xz + xyz^4}{1 - z - (x + y)z^4 - xyz^6 - xyz^7}. \tag{2.13}
\end{aligned}$$

□

To show that  $\mathcal{K}_n$  and  $\mathcal{L}_n$  are bivariate Gaussian, it suffices to prove the Gaussian behavior of  $a\mathcal{K}_n + b\mathcal{L}_n$  for any  $a, b$  with  $(a, b) \neq (0, 0)$ . Note that the coefficient of  $z^n$  in  $\hat{\mathcal{G}}(x, y, z)$  is  $\sum_{k>0, l \geq 0} p_{n,k,l} x^k y^l$ ; we denote this by  $\langle z^n \rangle \hat{\mathcal{G}}(x, y, z)$ . Setting  $(x, y) = (w^a, w^b)$  and using differentiating identities will give the moments of  $a\mathcal{K}_n + b\mathcal{L}_n$ .

We first prove the a generalized Lekkerkerker's Theorem and Gaussian behavior for  $a\mathcal{K}_n + b\mathcal{L}_n$ , which is a slight generalization of Theorem 1.6. This suffices to deduce Theorem 1.6 as  $\text{cov}(\mathcal{K}_n, \mathcal{L}_n) = \frac{1}{4}\text{var}(\mathcal{K}_n + \mathcal{L}_n) - \frac{1}{4}\text{var}(\mathcal{K}_n - \mathcal{L}_n)$ .

**Theorem 2.2.** *For any real numbers  $(a, b) \neq (0, 0)$ , we have*

(a) *The mean of  $a\mathcal{K}_n + b\mathcal{L}_n$  is*

$$\frac{a+b}{10}n + \frac{371 - 113\sqrt{5}}{40}a + \frac{361 - 123\sqrt{5}}{40}b + o(\hat{\gamma}_{a,b}^n) \text{ for some } \hat{\gamma}_{a,b} \in (0, 1), \tag{2.14}$$

(b) *The variance of  $a\mathcal{K}_n + b\mathcal{L}_n$  is*

$$\frac{\sqrt{5} - 1}{200} \left[ 10(a^2 + b^2) - \frac{20 - \sqrt{5}}{5}(a + b)^2 \right] n + q_{a,b} + o(\hat{\tau}_{a,b}^n) \text{ for some } \hat{\tau}_{a,b} \in (0, 1), \tag{2.15}$$

with  $q_{a,b}$  constant depending on only  $a$  and  $b$ ; further, the standardized distribution of  $a\mathcal{K}_n + b\mathcal{L}_n$  converges weakly to a Gaussian as  $n \rightarrow \infty$ ; in other words,  $\mathcal{K}_n$  and  $\mathcal{L}_n$  are bivariate Gaussian as  $n \rightarrow \infty$ .

**2.2. Generalized Lekkerkerker's Theorem.** As the mean is a crucial input in the proof of Gaussian behavior, we isolate this calculation first.

*Proof of Theorem 2.2(a).* Denote  $\hat{g}(w)$  the coefficient of  $z^n$  in  $\hat{\mathcal{G}}(w^a, w^b, z)$ , i.e.,

$$\hat{g}(w) = \sum_{k>0, l \geq 0} p_{n,k,l} w^{ak+bl}. \quad (2.16)$$

Taking  $w = 1$  in (2.16) gives us the sum of the  $p_{n,k,l}$ 's, which by definition equals  $S_n - S_{n-1} := D_n$ , i.e.,

$$\hat{g}(1) = \sum_{k>0, l \geq 0} p_{n,k,l} = D_n. \quad (2.17)$$

Moreover, taking the derivative of both sides of (2.16) gives

$$\hat{g}'(1) = \sum_{k>0, l \geq 0} (ak + bl) p_{n,k,l}, \quad (2.18)$$

therefore

$$\mu_n := \mathbb{E}[a\mathcal{K}_n + b\mathcal{L}_n] = \frac{\hat{g}'(1)}{\hat{g}(1)}. \quad (2.19)$$

Thus the proof of Part (a) reduces to finding  $\hat{g}(1)$  and  $\hat{g}'(1)$ .

Let  $\hat{A}_w(z)$  be the denominator of  $\hat{\mathcal{G}}(w^a, w^b, z)$ , namely

$$\hat{A}_w(z) = 1 - z - (w^a + w^b)z^4 - w^{a+b}z^6 - w^{a+b}z^7, \quad (2.20)$$

and  $e_1(w), e_2(w), \dots, e_7(w)$  the roots of  $\hat{A}_w(z)$  (i.e., regarding  $\hat{A}_w(z)$  as function of  $z$ ). We want to write  $\frac{1}{\hat{A}_w(z)}$  as a linear combination of the  $\frac{1}{z - e_i(w)}$ 's, i.e., the partial fraction expansion, as we can use power series expansion to find  $\hat{g}(w)$ , the coefficient of  $z^n$  in  $\hat{\mathcal{G}}(w^a, w^b, z)$ . In fact, we have the following proposition.

**Proposition 2.3.** *There exists  $\varepsilon \in (0, 1)$  such that for any  $w \in I_\varepsilon = (1 - \varepsilon, 1 + \varepsilon)$ ,*

- (a) *The 7 roots of  $\hat{A}_w(z)$  are nonzero and distinct.*
- (b) *There exists a root  $e_1(w)$  such that  $|e_1(w)| < 1$  and  $|e_1(w)| < |e_i(w)|$ ,  $1 < i \leq 7$ .*
- (c) *Each root  $e_i(w)$  ( $1 \leq i \leq 7$ ) is continuous and  $\ell$ -times differentiable for any  $\ell \geq 1$ , and*

$$e_i'(w) = -\frac{(aw^{a-1} + bw^{b-1})e_i^4(w) + (a+b)w^{a+b-1}[e_i^6(w) + e_i^7(w)]}{1 + 4(w^a + w^b)e_i^3(w) + 6w^{a+b}e_i^5(w) + 7w^{a+b}e_i^6(w)}. \quad (2.21)$$

$$(d) \quad \frac{1}{\hat{A}_w(z)} = -\frac{1}{w^{a+b}} \sum_{i=1}^7 \frac{1}{(z - e_i(w)) \prod_{j \neq i} (e_j(w) - e_i(w))}. \quad (2.22)$$

*Proof.* Clearly, 0 is not a root of  $\hat{A}_w(z)$ . When  $w = 1$ , we have

$$\hat{A}_1(z) = 1 - z - 2z^4 - z^6 - z^7 = -(z^2 + z - 1)(z^2 + 1)(z^3 + 1). \quad (2.23)$$

Thus  $\hat{A}_1(z)$  has no multiple roots; moreover, except  $\frac{\sqrt{5}-1}{2}$ , any other root  $z$  of  $\hat{A}(z)$  satisfies  $|z| \geq 1 > |\frac{\sqrt{5}-1}{2}|$ . Hence (a), (b) hold for  $w = 1$ .

Note that when  $w \neq 0$ , the leading coefficient of  $\hat{A}_w(z)$  is nonzero, and the coefficients of  $\hat{A}(z)$  are polynomials in one variable and hence continuous, thus the roots of  $\hat{A}_w(z)$  are continuous with respect to  $w$  (see [US] or Appendix A of [MW]). Since (a), (b) hold for  $w = 1$ , they also hold for a sufficiently small neighborhood  $I_\varepsilon$  of 1.

See Appendix A.1 for the proof of (c) and (d).  $\square$

Assume  $w \in I_\varepsilon$ . Combining (2.1) and Proposition 2.3(d), we get

$$\hat{\mathcal{G}}(w^a, w^b, z) = -(z + w^b z^4) \sum_{i=1}^7 \frac{1}{w^b(z - e_i(w)) \prod_{j \neq i} (e_j(w) - e_i(w))}, \quad (2.24)$$

thus

$$\begin{aligned} \hat{g}(w) &= \langle z^{n-4} \rangle \sum_{i=1}^7 \frac{1}{(1 - \frac{z}{e_i(w)}) e_i(w) \prod_{j \neq i} (e_j(w) - e_i(w))} \\ &\quad + \langle z^{n-1} \rangle \sum_{i=1}^7 \frac{1}{w^b (1 - \frac{z}{e_i(w)}) e_i(w) \prod_{j \neq i} (e_j(w) - e_i(w))} \\ &= \sum_{i=1}^7 \frac{1}{e_i^{n-3}(w) \prod_{j \neq i} (e_j(w) - e_i(w))} + \sum_{i=1}^7 \frac{1}{w^b e_i^n(w) \prod_{j \neq i} (e_j(w) - e_i(w))} \\ &= \sum_{i=1}^7 \frac{w^{-b} + e_i^3(w)}{e_i^n(w) \prod_{j \neq i} (e_j(w) - e_i(w))}. \end{aligned} \quad (2.25)$$

Let

$$\hat{q}_i(w) = \frac{w^{-b} + e_i^3(w)}{w \prod_{j \neq i} (e_j(w) - e_i(w))}, \quad \hat{\alpha}_i(w) = \frac{1}{e_i(w)} \text{ and } \hat{g}_i(w) = w \hat{q}_i(w) \hat{\alpha}_i^n(w). \quad (2.26)$$

Then  $\hat{g}(w) = \sum_{i=1}^7 w \hat{q}_i(w) \hat{\alpha}_i^n(w) = \sum_{i=1}^7 \hat{g}_i(w)$ . Since  $e_i(w)$  is nonzero and  $\ell$ -times differentiable for all  $\ell$  and  $i$ , so are  $\hat{q}_i(w)$  and  $\hat{\alpha}_i(w)$ . Further, it follows from Proposition 2.3(b) that  $|\hat{\alpha}_1(w)| > 1$  and  $|\hat{\alpha}_1(w)| > |\hat{\alpha}_i(w)|$ ,  $1 < i \leq 7$ . Hence for fixed  $\ell$ , we have

$$\hat{g}^{(\ell)}(w) = \hat{g}_1^{(\ell)}(w) + \sum_{i=2}^7 [w \hat{q}_i(w) \hat{\alpha}_i^n(w)]^{(\ell)} = \hat{g}_1^{(\ell)}(w) + o(\hat{\gamma}_\ell^n(w)) \hat{\alpha}_1^n(w) \quad (2.27)$$

for some  $\hat{\gamma}_\ell(w) \in (0, 1)$ . Taking  $w = 1$  yields

$$\hat{g}^{(\ell)}(1) = \hat{g}_1^{(\ell)}(1) + o(\hat{\gamma}_\ell^n) \hat{\alpha}_1^n(1), \quad (2.28)$$

where  $\hat{\gamma}_\ell = \hat{\gamma}_\ell(1) \in (0, 1)$ .

Applying (2.28) with  $\ell = 1$ , we obtain

$$\begin{aligned} \frac{\hat{g}'(1)}{\hat{g}(1)} &= \frac{n \hat{q}_1(1) \hat{\alpha}_1'(1) \hat{\alpha}_1^{n-1}(1) + (\hat{q}_1(1) + \hat{q}_1'(1)) \hat{\alpha}_1^n(1) + o(\hat{\gamma}_1^n) \hat{\alpha}_1^n(1)}{\hat{q}_1(1) \hat{\alpha}_1^n(1) + o(\hat{\gamma}_0^n) \hat{\alpha}_1^n(1)} \\ &= \frac{n \hat{q}_1(1) \hat{\alpha}_1'(1) (\hat{\alpha}_1(1))^{-1} + (\hat{q}_1(1) + \hat{q}_1'(1)) + o(\hat{\gamma}_1^n)}{\hat{q}_1(1) + o(\hat{\gamma}_0^n)} \\ &= \frac{\hat{\alpha}_1'(1)}{\hat{\alpha}_1(1)} n + \frac{\hat{q}_1(1) + \hat{q}_1'(1)}{\hat{q}_1(1)} + o(\hat{\gamma}_1^n). \end{aligned} \quad (2.29)$$

Therefore, by (2.19)  $\hat{\mu}_n$  is of the form:

$$\hat{\mu}_n = \hat{C}_{a,b}n + \hat{d}_{a,b} + o(\hat{\gamma}_{a,b}^n), \quad (2.30)$$

with

$$\hat{C}_{a,b} = \frac{\hat{\alpha}'_1(1)}{\hat{\alpha}_1(1)} = -\frac{e'_1(1)}{e_1(1)} \quad \text{and} \quad \hat{d}_{a,b} = 1 + \frac{\hat{q}'_1(1)}{\hat{q}_1(1)}. \quad (2.31)$$

Here we used the definition that  $\hat{\alpha}_1(w) = 1/e_1(w)$  (see 2.26).

Setting  $w = 1$  in (2.21) and using  $e_1(1) = \Phi$  (with  $\Phi = (\sqrt{5} - 1)/2$ ), we get  $\hat{C}_{a,b} = -e'_1(1)/e_1(1) = (a + b)/10$ . It is harder to calculate  $\hat{d}_{a,b}$ , but still tractable. We prove

$$\hat{d}_{a,b} = \frac{371 - 113\sqrt{5}}{40} a + \frac{361 - 123\sqrt{5}}{40} b. \quad (2.32)$$

Recall from (2.26) that

$$\hat{q}_1(w) = \frac{w^{-b} + e_1^3(w)}{w \prod_{j \neq 1} (e_j(w) - e_1(w))}. \quad (2.33)$$

Let

$$\hat{E}(w) = \prod_{j \neq 1} (e_j(w) - e_1(w)), \quad (2.34)$$

then

$$\hat{q}_1(w) = \frac{w^{-b} + e_1^3(w)}{w \hat{E}(w)}. \quad (2.35)$$

Thus

$$\begin{aligned} \hat{d}_{a,b} &= 1 + \frac{\hat{q}'_1(1)}{\hat{q}_1(1)} = 1 + \frac{[(w^{-b} + e_1^3(w))' w \hat{E}(w) - (w \hat{E}(w))' (w^{-b} + e_1^3(w))]}{(w^{-b} + e_1^3(w)) / (w \hat{E}(w))^2} \\ &= 1 + \frac{(w^{-b} + e_1^3(w))'}{w^{-b} + e_1^3(w)} - \frac{(w \hat{E}(w))'}{w \hat{E}(w)} = 1 + \frac{-bw^{-b-1} + 3e_1^2(w)e'_1(w)}{w^{-b} + e_1^3(w)} - \frac{\hat{E}(w) + w \hat{E}'(w)}{w \hat{E}(w)}. \end{aligned} \quad (2.36)$$

Setting  $x = 1$  and using  $e_1(1) = \Phi$  and  $e'_1(1) = -(a + b)\Phi/10$ , we get

$$\hat{d}_{a,b} = \frac{-b - \frac{3}{10}(a + b)\Phi^3}{1 + \Phi^3} - \frac{\hat{E}'(1)}{\hat{E}(1)} = -\frac{\sqrt{5} + 1}{4}b - \frac{9 - 3\sqrt{5}}{40}(a + b) - \frac{\hat{E}'(1)}{\hat{E}(1)}. \quad (2.37)$$

Thus it remains to evaluate  $\hat{E}(1)$  and  $\hat{E}'(1)$ . Consider  $\hat{A}_w(e' + e_1(w))$ :

$$\hat{A}_w(e' + e_1(w)) = 1 - e' - e_1(w) - (w^a + w^b)(e' + e_1(w))^4 - w^{a+b}(e' + e_1(w))^6 - w^{a+b}(e' + e_1(w))^7. \quad (2.38)$$

On the other hand, we have

$$\hat{A}_w(e' + e_1(w)) = -w^{a+b} \prod_{j \neq 1} (e' + e_1(w) - e_j(w)). \quad (2.39)$$

Comparing the coefficients of  $e'$  in (2.38) and (2.39) gives

$$w^{a+b} \prod_{j \neq 1} (e_1(w) - e_j(w)) = 1 + 4(w^a + w^b)e_1^3(w) + 6w^{a+b}e_1^5(w) + 7w^{a+b}e_1^6(w). \quad (2.40)$$

Thus

$$\hat{E}(w) = \prod_{j \neq 1} (e_1(w) - e_j(w)) = w^{-(a+b)} + 4(w^{-b} + w^{-a})e_1^3(w) + 6e_1^5(w) + 7e_1^6(w). \quad (2.41)$$

Setting  $w = 1$ , we get

$$\hat{E}(1) = 1 + 8\Phi^3 + 6\Phi^5 + 7\Phi^6 = 10\Phi^2. \quad (2.42)$$

Differentiating both sides of (2.41) yields

$$\hat{E}'(x) = (a+b)w^{-(a+b+1)} - 4(a w^{-a-1} + b w^{-b-1})e_1^3(w) + 30e_1^4(w)e_1'(w) + 42e_1^5(w)e_1'(w). \quad (2.43)$$

Setting  $x = 1$  and plugging in  $e_1(1) = \Phi$  and  $e_1'(1) = -(a+b)\Phi/10$  yields

$$\hat{E}'(1) = -(a+b) - 4(a+b)\Phi^3 + 30\Phi^4 \frac{(a+b)}{10} \Phi + 42\Phi^5 \frac{(a+b)}{10} \Phi.$$

Thus

$$\frac{\hat{E}'(1)}{\hat{E}(1)} = \frac{29\sqrt{5} - 95}{10}(a+b). \quad (2.44)$$

Plugging (2.44) into (2.37) yields

$$\hat{d}_{a,b} = \frac{371 - 113\sqrt{5}}{40} a + \frac{361 - 123\sqrt{5}}{40} b. \quad (2.45)$$

This completes the proof of Theorem 2.2(a).  $\square$

**2.3. Gaussian Behavior.** We prove  $\mathcal{M}_n = a\mathcal{K}_n + b\mathcal{L}_n$  converges weakly to a Gaussian by calculating its centralized moments and using Markov's Method of Moments. Its variance is a special case, and is determined below. Note that the proof of Theorem 2.2(a) yielded

$$\mathbb{E}[a\mathcal{K}_n + b\mathcal{L}_n] = \hat{C}_{a,b}n + \hat{d}_{a,b} + o(\hat{\gamma}_{a,b}^n) \text{ and } \text{var}(a\mathcal{K}_n + b\mathcal{L}_n) = \hat{h}'(1)n + \hat{q}_1''(1) + o(\hat{\tau}_{a,b}^n) \quad (2.46)$$

with

$$\hat{C}_{a,b} = -\frac{e_1'(1)}{e_1(1)}, \quad \hat{d}_{a,b} = 1 + \frac{\hat{q}_1'(1)}{\hat{q}_1(1)}, \quad \hat{h}(w) = -\frac{we_1'(w)}{e_1(w)} - \hat{C}_{a,b} \quad (2.47)$$

and constants  $\hat{\gamma}_{a,b}, \hat{\tau}_{a,b} \in (0, 1)$  and  $\hat{q}_1''(1)$  depending on only  $a$  and  $b$ .

Let  $\hat{\sigma}_n$  be the standard deviation of  $\mathcal{M}_n = a\mathcal{K}_n + b\mathcal{L}_n$ . First we centralize and normalize  $\mathcal{M}$  to  $\mathcal{M}_n^{(c)} = (\mathcal{M}_n - \hat{\mu}_n)/\hat{\sigma}_n$ . Thus it suffices to show that  $\mathcal{M}_n^{(c)}$  converges to the standard normal. According to Markov's Method of Moments, we only need to show that each moment of  $\mathcal{M}_n^{(c)}$  tends to that of the standard normal distribution, which is equivalent to the following.

**Theorem 2.4.** *Let  $\hat{\mu}_n(m)$  be the  $m^{\text{th}}$  moment of  $\mathcal{M}_n - \hat{\mu}_n$ , then for any integer  $u \geq 1$ , we have*

$$\frac{\hat{\mu}_n(2u-1)}{\hat{\sigma}_n^{2u-1}} \rightarrow 0 \text{ and } \frac{\hat{\mu}_n(2u)}{\hat{\sigma}_n^{2u}} \rightarrow (2u-1)!!, \text{ as } n \rightarrow \infty. \quad (2.48)$$

In the proof, we first point out that it suffices to prove the same result for  $\mathcal{M}_n - \tilde{\mu}_n$  with  $\tilde{\mu}_n = \hat{C}_{a,b}n + \hat{d}_{a,b}$  and  $\hat{C}_{a,b}, \hat{d}_{a,b}$  defined in (2.31). Then we show that the  $m^{\text{th}}$  moment  $\tilde{\mu}_n(m)$  of  $\mathcal{M}_n - \hat{\mu}_n$  equals  $\tilde{g}_m(1)/D_n$  for polynomials  $\tilde{g}_m(x)$  with

$$\tilde{g}_0(x) = \sum_{k,l} p_{n,k,l} w^{ak+bl-\tilde{\mu}_n-1} = \frac{\hat{g}(x)}{x^{\tilde{\mu}_n+1}}, \quad \tilde{g}_{j+1}(x) = (x\tilde{g}_j(x))', \quad j \geq 1. \quad (2.49)$$

By Definitions (2.16) and (2.49), we prove by induction that the main term of  $\tilde{g}_m(1)$  is of the form  $\hat{\alpha}_1^n(x)x^{-\tilde{\mu}_n}\sum_{i=0}^m f_{i,m}(x)n^i$  for some functions  $f_{i,m}(x)$ 's and thus conclude that  $\tilde{\mu}_n(m) = \frac{1}{q_1(1)}\sum_{i=0}^m f_{i,m}(1)n^i + o(\tau_m^n)$  for some  $\tau_m \in (0, 1)$ . Finally, we evaluate the  $f_{i,m}(1)$ 's to obtain (2.48).

We now give the proof. In the course of our analysis we will interrupt the proof to state and prove some simple, needed propositions. Noting that  $\hat{\mu}_n = \tilde{\mu}_n + o(\gamma_1^n)$  by (2.30), by some simple approximations (see Appendix A.2), we see that

$$\hat{\mu}_n(m) = \tilde{\mu}_n(m) + o(\beta_m^n) \quad (2.50)$$

for some  $\beta_m \in (0, 1)$ . In the special case of  $m = 2$ , we have  $\hat{\sigma}_n^2 = \hat{\mu}_n(2) = \tilde{\mu}_n(2) + o(\tau_m^n)$ , therefore (2.48) is equivalent to

$$\frac{\tilde{\mu}_n(2u-1)}{\tilde{\mu}_n^{u-\frac{1}{2}}(2)} \rightarrow 0 \text{ and } \frac{\tilde{\mu}_n(2u)}{\tilde{\mu}_n^u(2)} \rightarrow (2u-1)!!, \text{ as } u \rightarrow \infty. \quad (2.51)$$

We calculate the moments  $\tilde{\mu}_n(m)$ 's by applying the method of differentiating identities to  $\tilde{g}$ . Setting  $x = 1$  in (2.49), we get

$$\tilde{g}_1(x) = (x\tilde{g}_0(x))' = \left( \sum_{k,l} p_{n,k,l} x^{ak+bl-\tilde{\mu}_n} \right)' = \sum_{k,l} p_{n,k,l} (ak+bl-\tilde{\mu}_n) x^{ak+bl-\tilde{\mu}_n-1}. \quad (2.52)$$

When  $m = 2$ , by (2.49) and (2.52), we get

$$\tilde{g}_2(x) = (x\tilde{g}_1(x))' = \sum_{k,l} p_{n,k,l} (ak+bl-\tilde{\mu}_n)^2 x^{ak+bl-\tilde{\mu}_n-1}. \quad (2.53)$$

Setting  $x = 1$ , we get

$$\tilde{g}_2(1) = \sum_{k,l} p_{n,k,l} (ak+bl-\tilde{\mu}_n)^2 = \tilde{\mu}_n(2)D_n. \quad (2.54)$$

By induction on  $m$ , we can prove the following.

**Proposition 2.5.** *For any  $m \geq 0$ , we have*

$$\tilde{g}_m(x) = \sum_{k,l} p_{n,k,l} (ak+bl-\tilde{\mu}_n)^m x^{ak+bl-\tilde{\mu}_n-1} \text{ and } \tilde{g}_m(1) = \tilde{\mu}_n(m)D_n. \quad (2.55)$$

*Proof.* We have proved the statement for  $m = 0, 1, 2$ . If (2.55) holds for  $m$ , then the recurrence relation (2.49) gives

$$\begin{aligned} \tilde{g}_{m+1}(x) &= (x\tilde{g}_m(x))' = \left( \sum_{k,l} p_{n,k,l} (ak+bl-\tilde{\mu}_n)^m x^{ak+bl-\tilde{\mu}_n} \right)' \\ &= \sum_{k,l} p_{n,k,l} (ak+bl-\tilde{\mu}_n)^{m+1} x^{ak+bl-\tilde{\mu}_n-1}. \end{aligned} \quad (2.56)$$

Setting  $x = 1$  gives  $\tilde{g}_{m+1}(1) = \tilde{\mu}_n(m+1)D_n$ . Thus the statement holds for  $m+1$  and hence for any  $m \geq 0$ .  $\square$

Returning to the proof of Theorem 2.4, denote

$$\tilde{g}_{0,i}(x) = \frac{\hat{q}_i(x)\hat{\alpha}_i^n(x)}{x^{\tilde{\mu}_n}}, \text{ and } \tilde{g}_{j+1,i}(x) = (x\tilde{g}_{j,i}(x))' \quad (2.57)$$

for  $x \in I_\varepsilon$  if  $1 < i \leq 7$  and for  $x \in I_\varepsilon \cup \{1\}$  if  $i = 1$ . By Definition (2.57) and using the same approach as in proving (2.27), we can show that

$$\forall x \in I_\varepsilon : \sum_{i=2}^L \tilde{g}_{j,i}(x) = o(\tau_j^n)\alpha_1^n(x) \quad (2.58)$$

for some  $\tau_j \in (0, 1)$ . Thus referring to (2.49), we have

$$\forall x \in I_\varepsilon : \tilde{g}_j(x) = \sum_{i=1}^L \tilde{g}_{j,i}(x) = \tilde{g}_{j,1}(x) + o(\tau_j^n)\hat{\alpha}_1^n(x). \quad (2.59)$$

Taking the limit as  $x$  approaches 1 yields

$$\forall x \in I_\varepsilon, \tilde{g}_j(1) = \tilde{g}_{j,1}(1) + o(\tau_j^n)\hat{\alpha}_1^n(1). \quad (2.60)$$

Denoting  $\tilde{g}_{j,1}(x)$  by  $F_j(x)$ , then

$$F_0(x) = \hat{q}_1(x)\hat{\alpha}_1^n(x)x^{-\tilde{\mu}_n} \text{ and } F_{j+1}(x) = (xF_j(x))'. \quad (2.61)$$

Note that  $\hat{q}_1(x)$  and  $\hat{\alpha}_1(x)$  are  $\ell$ -times differentiable for all  $\ell$ . Thus when  $j = 0$ , we get

$$\begin{aligned} F_1(x) &= (xF_0(x))' = (\hat{q}_1(x)\hat{\alpha}_1^n(x)x^{-\tilde{\mu}_n})' \\ &= nx\hat{q}_1(x)\hat{\alpha}_1'(x)\hat{\alpha}_1^{n-1}(x)x^{-\tilde{\mu}_n} - (\tilde{\mu}_n - 1)\hat{q}_1(x)\hat{\alpha}_1^n(x)x^{-\tilde{\mu}_n} + x\hat{q}_1'(x)\hat{\alpha}_1^n(x)x^{-\tilde{\mu}_n} \\ &= nx\hat{q}_1(x)\hat{\alpha}_1'(x)\hat{\alpha}_1^{n-1}(x)x^{-\tilde{\mu}_n} - (\hat{C}_{a,b}n + d_{a,b} - 1)\hat{q}_1(x)\hat{\alpha}_1^n(x)x^{-\tilde{\mu}_n} + x\hat{q}_1'(x)\hat{\alpha}_1^n(x)x^{-\tilde{\mu}_n} \\ &= \hat{\alpha}_1^n(x)x^{-\tilde{\mu}_n} \left[ \left( \frac{x\hat{\alpha}_1'(x)}{\hat{\alpha}_1(x)} - \hat{C}_{a,b} \right) \hat{q}_1(x)n + (1 - \hat{d}_{a,b})\hat{q}_1(x) + x\hat{q}_1'(x) \right] \\ &= \hat{\alpha}_1^n(x)x^{-\tilde{\mu}_n} [h(x)\hat{q}_1(x)n + d'\hat{q}_1(x) + x\hat{q}_1'(x)], \end{aligned} \quad (2.62)$$

where  $h(x)$  and  $d'$  are defined as

$$h(x) = \frac{x\hat{\alpha}_1'(x)}{\hat{\alpha}_1(x)} - \hat{C}_{a,b} \text{ and } d' = 1 - \hat{d}_{a,b} = -\frac{\hat{q}_1'(1)}{\hat{q}_1(1)} \quad (2.63)$$

(see (2.31) for the definition of  $\hat{d}_{a,b}$ ). By (2.31), we have

$$h(1) = 0. \quad (2.64)$$

Moreover, since  $\hat{\alpha}_1(x)$  is  $\ell$ -times differentiable at 1 and  $\hat{\alpha}_1(1) \neq 0$ , we have

$$h(x) \text{ is } \ell\text{-times differentiable at 1 for any } \ell \geq 1. \quad (2.65)$$

From (2.61) and (2.62), we observe that  $F_m(x)$  can be written as a product of  $\hat{\alpha}_1^n(x)x^{-\tilde{\mu}_n}$  and a sum of other functions of  $n$  and  $x$ . In fact, we have the following.

**Proposition 2.6.** *For any  $m \geq 0$ ,*

(a) *We have  $F_m(x)$  is of the form*

$$F_m(x) = \hat{\alpha}_1^n(x)x^{-\tilde{\mu}_n} \sum_{i=0}^m f_{i,m}(x)n^i, \quad (2.66)$$

where the  $f_{i,m}$ 's are functions of  $x$  and  $\hat{\alpha}_1(x)$  but independent of  $n$ .

(b) The  $f_{i,m}$ 's are  $\ell$ -times differentiable at  $x \in I_\varepsilon$  for any  $\ell \geq 1$ .

(c) Define

$$f_{i,m}(x) = 0 \text{ if } i > m \text{ or } i < 0 \text{ or } m < 0, \quad (2.67)$$

then for  $m > 0$ , we have the following recurrence relation:

$$f_{i,m}(x) = h(x)f_{i-1,m-1}(x) + d'f_{i,m-1}(x) + xf'_{i,m-1}(x). \quad (2.68)$$

*Proof.* We proceed by induction on  $m$ . For  $m = 0$  and 1, (a) holds because of (2.61) and (2.62). Further, (2.61) and (2.62) give the expressions of  $f_{0,0}$ ,  $f_{0,1}$  and  $f_{1,1}$ :

$$f_{0,0}(x) = \hat{q}_1(x), f_{0,1}(x) = d'\hat{q}_1(x) + x\hat{q}'_1(x), f_{1,1}(x) = h(x)\hat{q}_1(x). \quad (2.69)$$

Thus they are differentiable  $\ell$ -times at  $x \in I_\varepsilon$  for any  $\ell \geq 1$ . Hence (b) holds for  $m = 0$  and 1. Finally, with (2.69), it is easy to verify that (c) holds for  $m = 0$  and 1.

If the statement holds for  $m$ , by (2.49) we have

$$F_{m+1}(x) = \left[ \hat{\alpha}_1^n(x)x^{-\tilde{\mu}_n} \sum_{i=0}^m xf_{i,m}(x)n^i \right]' = \sum_{i=0}^m \left[ \hat{\alpha}_1^n(x)x^{-\tilde{\mu}_n} xf_{i,m}(x)n^i \right]'$$

For convenience, we denote  $h_i(x) = \hat{\alpha}_1^n(x)x^{-\tilde{\mu}_n}xf_{i,m}(x)n^i$  for  $0 \leq i \leq m$ . Thus

$$F_{m+1}(x) = \sum_{i=0}^m h'_i(x). \quad (2.70)$$

For each  $0 \leq i \leq m$ , we have

$$\begin{aligned} h'_i(x) &= n^i \left[ \hat{\alpha}'_1(x)\hat{\alpha}_1^{n-1}(x)x^{-\tilde{\mu}_n}xf_{i,m} - (\tilde{\mu}_n - 1)\hat{\alpha}_1^n(x)x^{-\tilde{\mu}_n}f_{i,m}(x) + \hat{\alpha}_1^n(x)x^{-\tilde{\mu}_n}xf'_{i,m}(x) \right] \\ &= n^i\hat{\alpha}_1^n(x)x^{-\tilde{\mu}_n} \left[ nf_{i,m}(x) \left( \hat{\alpha}'_1(x)\hat{\alpha}_1^{-1}(x)x - \hat{C}_{a,b} \right) + (1 - \hat{d}_{a,b})f_{i,m}(x) + xf'_{i,m}(x) \right] \\ &= n^i\hat{\alpha}_1^n(x)x^{-\tilde{\mu}_n} \left[ nh(x)f_{i,m}(x) + d'f_{i,m}(x) + xf'_{i,m}(x) \right] \\ &= \hat{\alpha}_1^n(x)x^{-\tilde{\mu}_n} \left[ n^{i+1}h(x)f_{i,m}(x) + n^i(d'f_{i,m}(x) + xf'_{i,m}(x)) \right] \end{aligned} \quad (2.71)$$

(see (2.63) for the definitions of  $h(x)$  and  $d'$ ). Plugging (2.71) into (2.70) yields

$$\begin{aligned} F_{m+1}(x) &= \hat{\alpha}_1^n(x)x^{-\tilde{\mu}_n} \left[ n^{m+1}h(x)f_{m,m}(x) + \sum_{i=1}^m n^i (h(x)f_{i-1,m}(x) + d'f_{i,m}(x) \right. \\ &\quad \left. + xf'_{i,m}(x)) + d'f_{0,m}(x) + xf'_{0,m}(x) \right]. \end{aligned} \quad (2.72)$$

Hence (2.66) holds for  $m+1$  as desired.

For (b) and (c), from (2.72) we get

$$f_{m+1,m+1}(x) = h(x)f_{m,m}(x), \quad (2.73)$$

$$f_{i,m+1}(x) = h(x)f_{i-1,m}(x) + d'f_{i,m}(x) + xf'_{i,m}(x), \quad 1 \leq i \leq m \quad (2.74)$$

and

$$f_{0,m+1}(x) = d'f_{0,m}(x) + xf'_{0,m}(x). \quad (2.75)$$

By Definition (2.67), we can combine (2.73), (2.74) and (2.75) into one recurrence relation (2.68) (with  $m$  replaced by  $m+1$ ). With this recurrence relation, (2.65) and the induction hypothesis of (b) for  $m$ , we see that (b) also holds for  $m+1$ . This completes the proof.  $\square$

**Proposition 2.7.** *We have*

$$\tilde{\mu}_n(m) = \frac{1}{\hat{q}_1(1)} \sum_{i=0}^m f_{i,m}(1)n^i + o(\tau_m^n) \text{ for some } \tau_m \in (0, 1). \quad (2.76)$$

*Proof.* From (2.55), (2.59), (2.17), (2.28) with  $\ell = 0$ , the definition  $F_m(x) = \tilde{g}_{m,1}(x)$  and Proposition 2.6, we obtain

$$\begin{aligned} \tilde{\mu}_n(m) &= \frac{\tilde{g}_m(1)}{D_n} = \frac{\tilde{g}_{m,1}(1) + o(\tau_m^n)\hat{\alpha}_1^n(1)}{D_n} = \frac{\tilde{F}_m(1) + o(\tau_m^n)\hat{\alpha}_1^n(1)}{D_n} \\ &= \frac{[\sum_{i=0}^m f_{i,m}(1)n^i + o(\tau_m^n)]\hat{\alpha}_1^n(1)}{[\hat{q}_1(1) + o(\tau_m^n)]\hat{\alpha}_1^n(1)} = \frac{1}{\hat{q}_1(1)} \sum_{i=0}^m f_{i,m}(1)n^i + o(\tau_m^n). \end{aligned} \quad (2.77)$$

□

From Proposition 2.7, we see that the main term of  $\tilde{\mu}_n(m)$  only depends on  $\hat{q}_1(1)$  and the  $f_{i,m}(1)$ 's. Note that to prove (2.51), it suffices to find the main term of  $\tilde{\mu}_n(m)$ . Thus the problem reduces to finding the  $f_{i,m}(1)$ 's. We first calculate the variance, namely  $\tilde{\mu}_n(2)$ .

**Proposition 2.8.** *The variance of  $\mathcal{M}_n - \tilde{\mu}_n$*

$$\tilde{\mu}_n(2) = h'(1)n + \hat{q}_1''(1) + o(\tau_2^n) \quad (2.78)$$

with  $h'(1) \neq 0$ ,  $\hat{q}_1''(1)$  and  $\tau_2 \in (0, 1)$  constant depending on only  $L$  and the  $c_i$ 's.

With the estimation (2.50), it follows immediately that the variance of  $\mathcal{M}_n$  is of order  $n$ .

**Theorem 2.9.** *The variance of  $\mathcal{M}_n = a\mathcal{K}_n + b\mathcal{L}_n$*

$$\hat{\mu}_n(2) = h'(1)n + \hat{q}_1''(1) + o(\tau_2^n) \quad (2.79)$$

with  $h'(1) \neq 0$ ,  $\hat{q}_1''(1)$  and  $\tau_2 \in (0, 1)$  constant depending on only  $a$  and  $b$ ; further,  $h'(1)$  is given by

$$\hat{h}'(1) = \frac{\sqrt{5} - 1}{200} \left[ 10(a^2 + b^2) - \frac{20 - \sqrt{5}}{5}(a + b)^2 \right]. \quad (2.80)$$

*This proves (2.15) in Theorem 2.2.*

*Proof of Proposition 2.8.* If  $m = 2$ , by (2.73) and (2.64) we get  $f_{2,2}(1) = h(1)f_{1,1}(1) = 0$ . Applying (2.68) to  $(i, m) = (1, 2)$  and plugging in (2.69) yields

$$\begin{aligned} f_{1,2}(x) &= h(x)f_{0,1}(x) + d'f_{1,1}(x) + xf'_{1,1}(x) \\ &= h(x)f_{0,1}(x) + d'h(x)\hat{q}_1(x) + xh(x)\hat{q}_1'(x) + xh'(x)\hat{q}_1(x). \end{aligned} \quad (2.81)$$

Setting  $x = 1$  and using  $h(1) = 0$  (see (2.64)) yields

$$f_{1,2}(1) = h(1)f_{0,1}(1) + d'h(1)\hat{q}_1(1) + h(1)\hat{q}_1'(1) + h'(1)\hat{q}_1(1) = h'(1)\hat{q}_1(1). \quad (2.82)$$

Using (2.75) and (2.68), we can find  $f_{0,2}(x)$  as follows:

$$f_{0,2}(x) = d'f_{0,1}(x) + xf'_{0,1}(x) = d'^2\hat{q}_1(x) + d'x\hat{q}_1'(x) + d'x\hat{q}_1(x) + x\hat{q}_1'(x) + x^2\hat{q}_1''(x). \quad (2.83)$$

Setting  $x = 1$  and substituting  $d'$  by  $-\frac{\hat{q}_1'(1)}{\hat{q}_1(1)}$  (see (2.63)) yields

$$f_{0,2}(1) = \hat{q}_1''(1). \quad (2.84)$$

Combining the above results with Proposition 2.7 gives (2.78). We can derive a formula for  $\hat{h}'(w)$  in terms of  $e_1(w)$  by using (2.21). Then (2.80) follows by  $e_1(1) = \Phi$ . We can verify that  $\hat{h}'(1) \neq 0$  by simple quadratic inequalities (details can be found in Appendix A.3).  $\square$

From Propositions 2.7 and 2.8, we see that (2.51) (which is what we need to show to finish the proof of Theorem 2.4) is equivalent to

$$f_{i,2u-1}(1) = 0, \quad i \geq u, \quad (2.85)$$

$$f_{i,2u}(1) = 0, \quad i > u, \quad (2.86)$$

and

$$f_{u,2u}(1) = (2u-1)!!\hat{q}_1(1)(h'(1))^u. \quad (2.87)$$

For convenience, we denote

$$t_{i,m}^{(\ell)} = f_{i,m}^{(\ell)}(1), \quad \ell \geq 0. \quad (2.88)$$

Note that if  $\ell = 0$ , then the definition is just  $t_{i,m} = f_{i,m}(1)$ .

**Proposition 2.10.** *For any  $0 \leq m < 2i$  and  $\ell \geq 0$ , we have*

$$t_{i,m-\ell}^{(\ell)} = f_{i,m-\ell}^{(\ell)}(1) = 0. \quad (2.89)$$

*Proof.* If  $\ell > m$  or  $i > m - \ell$ , according to Definition (2.67), we have  $f_{i,m-\ell}(x) = 0$ . Thus  $f_{i,m-\ell}^{(\ell)}(x) = 0$  and (2.89) follows. Therefore, it suffices to prove for  $0 \leq \ell \leq m < 2i$  and  $i \leq m - \ell$ , i.e.,

$$0 \leq \ell \leq m - i < i. \quad (2.90)$$

We proceed by induction on  $m$ . If  $m = 0$ , then there is no  $i$  that satisfies (2.90). Thus the statement holds. If  $m = 1$ , the only choice for  $i$  and  $\ell$  that satisfies (2.90) is  $i = 1$  and  $\ell = 0$ . By (2.69) and (2.64), we get  $t_{i,m-\ell}^{(\ell)} = t_{1,1} = f_{1,1}(1) = h(1)\hat{q}_1(1) = 0$ . Thus the statement holds for  $m = 1$ . Assume that the statement holds for any  $m' < m$  ( $m \geq 2$ ). For any  $(i, m, \ell)$  that satisfies (2.90) and  $1 \leq j \leq \ell$ , we have

$$2(i-1) = 2i-2 > m-2 \geq m-1-j, \quad (2.91)$$

thus we can apply the induction hypothesis (2.89) to  $(i-1, m-1-j, \ell-j)$ ,  $(i, m-1, \ell)$  and  $(i, m-1-\ell+j, j)$  with  $1 \leq j \leq \ell$  and obtain

$$f_{i-1,m-1-\ell}^{(\ell-j)}(1) = f_{i,m-1-\ell}^{(\ell)}(1) = f_{i,m-1-\ell}^{(j)}(1) = 0. \quad (2.92)$$

Taking the  $\ell^{\text{th}}$  derivative of both sides of (2.68), we get

$$\begin{aligned} f_{i,m-\ell}^{(\ell)}(x) &= h(x)f_{i-1,m-1-\ell}^{(\ell)}(x) + \sum_{j=1}^{\ell} \binom{\ell}{j} h^{(j)}(x)f_{i-1,m-1-\ell}^{(\ell-j)}(x) \\ &\quad + d'f_{i,m-1-\ell}^{(\ell)}(x) + xf_{i,m-1-\ell}^{(\ell+1)}(x) + \sum_{j=1}^{\ell} f_{i,m-1-\ell}^{(j)}(x). \end{aligned} \quad (2.93)$$

Setting  $x = 1$  and using (2.92) and (2.64) yields

$$f_{i,m-\ell}^{(\ell)}(1) = f_{i,m-1-\ell}^{(\ell+1)}(1), \quad \text{i.e., } t_{i,m-\ell}^{(\ell)} = t_{i,m-1-\ell}^{(\ell+1)}. \quad (2.94)$$

Applying (2.94) to  $\ell = 0, 1, \dots, m$ , we get

$$t_{i,m}^{(0)} = t_{i,m-1}^{(1)} = t_{i,m-2}^{(2)} = \dots = t_{i,0}^{(m)} = t_{i,-1}^{(m+1)} = 0, \quad (2.95)$$

where the last step follows from (2.67).

Thus the statement holds for  $m$  as well. This completes the proof.  $\square$

**Corollary 2.11.** *For any  $u \geq 1$ , we have (2.85) and (2.86), i.e.,*

$$t_{i,2u-1} = 0, \quad i \geq u \text{ and } t_{i,2u} = 0, \quad i > u. \quad (2.96)$$

*Proof.* Applying Proposition 2.10 with  $(i, m, \ell) = (i, 2u - 1, 0)$  ( $i \geq u$ ) and  $(i, m, \ell) = (i, 2u - 1, 0)$  ( $i > u$ ).  $\square$

Thus it remains to show (2.87).

**Proposition 2.12.** *For any  $u \geq 1$  we have*

(a)  $f_{u,u+v}(x)$  with  $0 \leq v \leq u$  is of the form

$$f_{u,u+v}(x) = r_{u,v} \hat{q}_1(x) x^v h^{u-v}(x) (h'(x))^v + s_{u,v}(x) h^{u+1-v}(x), \quad (2.97)$$

where  $r_{u,v}$  is a constant determined by  $u$  and  $v$ ,  $s_{u,v}(x)$  is a polynomial of the  $h^{(\ell)}(x)$ 's and the  $\hat{q}_1^{(\ell)}(x)$ 's ( $\ell \geq 0$ ) with coefficients polynomials of  $x$ .

(b)  $r_{u,0} = 1$  and

$$r_{u,v} = r_{u-1,v} + (u - v + 1)r_{u,v-1}, \quad r_{u,u} = r_{u,u-1}, \quad 1 \leq v < u. \quad (2.98)$$

$$(c) \quad r_{u,u} = (2u - 1)!!. \quad (2.99)$$

*Proof.* We proceed by induction on  $u + v$ .

By (2.69) and (2.73), we get

$$f_{u,u}(x) = \hat{q}_1(x) h^u(x), \quad u \geq 1. \quad (2.100)$$

Hence (a) holds for  $v = 0$  and  $r_{u,0} = 1$ .

Since the only  $(u, v)$  with  $u + v = 1$  and  $0 \leq v \leq u$  is  $(0, 1)$ , (a) holds for  $u + v = 1$ . Assume that (a) holds for  $u + v \leq t$  ( $t \geq 1$ ). We will simultaneously prove (a) and (b). If  $u + v = t + 1$ , we have shown that the statement holds for  $v = 0$ . For  $1 \leq v \leq u$ , we have three cases:  $v = 1$ ,  $1 < v < u$  and  $1 < v = u$ .

When  $1 \leq v < u$ , applying (2.68) to  $(i, m, \ell) = (u, u + v, 0)$  and using the induction hypothesis for  $(u - 1, v)$ ,  $(u, v - 1)$ , we get

$$\begin{aligned} f_{u,u+v}(x) &= h(x) f_{u-1,u+v-1} + d' f_{u,u+v-1} + x f'_{u,u+v-1} \\ &= h(x) [r_{u-1,v} \hat{q}_1(x) x^v h^{u-1-v}(x) (h'(x))^v + s_{u-1,v}(x) h^{u-v}(x)] \\ &\quad + d' [r_{u,v-1} \hat{q}_1(x) x^{v-1} h^{u-v+1}(x) (h'(x))^{v-1} + s_{u,v-1}(x) h^{u+2-v}(x)] \\ &\quad + x [r_{u,v-1} \hat{q}_1(x) x^{v-1} h^{u-v+1}(x) (h'(x))^{v-1} + s_{u,v-1}(x) h^{u+2-v}(x)]' \\ &= r_{u-1,v} \hat{q}_1(x) x^v h^{u-v}(x) (h'(x))^v + [s_{u-1,v}(x) \\ &\quad + d' r_{u,v-1} \hat{q}_1(x) x^{v-1} (h'(x))^{v-1} + d' s_{u,v-1}(x) h(x)] h^{u+1-v}(x) \\ &\quad + x [r_{u,v-1} \hat{q}_1(x) x^{v-1} h^{u-v+1}(x) (h'(x))^{v-1} + s_{u,v-1}(x) h^{u+2-v}(x)]'. \end{aligned} \quad (2.101)$$

Denote the last line of (2.101) by  $W$ .

**Case 1.**  $v = 1$ . We have

$$\begin{aligned}
W &= x [r_{u,v-1}\hat{q}'_1(x)h^{u-v+1}(x) + (u-v+1)r_{u,v-1}\hat{q}_1(x)h'(x)h^{u-v}(x) \\
&\quad + (u+2-v)s_{u,v-1}(x)h'(x)h^{u+1-v}(x)] \\
&= x [r_{u,v-1}\hat{q}'_1(x) + (u+2-v)s_{u,v-1}(x)h'(x)] h^{u-v+1}(x) \\
&\quad + x(u-v+1)r_{u,v-1}\hat{q}_1(x)h'(x)h^{u-v}(x).
\end{aligned} \tag{2.102}$$

Noting that  $v = 1$ , thus the above equation can be written as

$$\begin{aligned}
W &= x [r_{u,v-1}\hat{q}'_1(x) + (u+2-v)s_{u,v-1}(x)h'(x)] h^{u-v+1}(x) \\
&\quad + (u-v+1)r_{u,v-1}\hat{q}_1(x)x^v h^{u-v}(x) (h'(x))^v.
\end{aligned} \tag{2.103}$$

Plugging this into (2.101) yields

$$\begin{aligned}
f_{u,u+v}(x) &= r_{u-1,v}\hat{q}_1(x)x^v h^{u-v}(x) (h'(x))^v + [s_{u-1,v}(x) \\
&\quad + d'r_{u,v-1}\hat{q}_1(x)x^{v-1} (h'(x))^{v-1} + d's_{u,v-1}(x)h(x)] h^{u+1-v}(x) \\
&\quad + x [r_{u,v-1}\hat{q}'_1(x) + (u+2-v)s_{u,v-1}(x)h'(x)] h^{u-v+1}(x) \\
&\quad + (u-v+1)r_{u,v-1}\hat{q}_1(x)x^v h^{u-v}(x) (h'(x))^v \\
&= [r_{u-1,v} + (u-v+1)r_{u,v-1}]\hat{q}_1(x)x^v h^{u-v}(x) (h'(x))^v + [s_{u-1,v}(x) \\
&\quad + d'r_{u,v-1}\hat{q}_1(x)x^{v-1} (h'(x))^{v-1} + d's_{u,v-1}(x)h(x) + xr_{u,v-1}\hat{q}'_1(x) \\
&\quad + x(u+2-v)s_{u,v-1}(x)h'(x)] h^{u-v+1}(x).
\end{aligned} \tag{2.104}$$

Hence  $f_{u,u+v}(x)$  is of the form (2.97) and (2.98) holds.

**Case 2.**  $1 < v < u$ . We have

$$\begin{aligned}
W &= (u-v+1)r_{u,v-1}\hat{q}_1(x)x^v h^{u-v}(x) (h'(x))^v \\
&\quad + [r_{u,v-1}\hat{q}'_1(x)x^v + (v-1)r_{u,v-1}\hat{q}_1(x)x^{v-1} (h'(x))^{v-1} \\
&\quad + (v-1)r_{u,v-1}\hat{q}_1(x)x^v (h'(x))^{v-2} h''(x) \\
&\quad + (u+2-v)xs_{u,v-1}(x)h'(x)] h^{u+1-v}(x).
\end{aligned} \tag{2.105}$$

Plugging this into (2.101) yields

$$\begin{aligned}
f_{u,u+v}(x) &= [r_{u-1,v} + (u-v+1)r_{u,v-1}]\hat{q}_1(x)x^v h^{u-v}(x) (h'(x))^v + [s_{u-1,v}(x) \\
&\quad + d'r_{u,v-1}\hat{q}_1(x)x^{v-1} (h'(x))^{v-1} + d's_{u,v-1}(x)h(x) + r_{u,v-1}\hat{q}'_1(x)x^v \\
&\quad + (v-1)r_{u,v-1}\hat{q}_1(x)x^{v-1} (h'(x))^{v-2} (h'(x) + xh''(x)) \\
&\quad + (u+2-v)xs_{u,v-1}(x)h'(x)] h^{u+1-v}(x).
\end{aligned} \tag{2.106}$$

Hence  $f_{u,u+v}(x)$  is of the form (2.97) and (2.98) holds in this case too.

**Case 3.**  $1 < v = u$ . Thus  $u \geq 2$ . From the recurrence relation (2.68) and the initial condition (2.69), we see that each  $f_{i,m}$  is a polynomial of the  $h^{(\ell)}(x)$ 's and the  $\hat{q}_1^{(\ell)}(x)$ 's ( $\ell \geq 0$ ) with coefficients polynomials of  $x$ . By (2.101) and the induction hypothesis (2.97) for  $(u, v) = (u, u-1)$ , we get

$$\begin{aligned}
f_{u,u+v}(x) &= f_{u,2u-1}(x) = h(x)f_{u-1,2u-1} + d'f_{u,2u-1} + xf'_{u,2u-1} \\
&= h(x)f_{u-1,2u-1} + r_{u,u-1}\hat{q}_1(x)x^{u-1}h(x) (h'(x))^{u-1} + s_{u,u-1}(x)h^2(x) \\
&\quad + x[r_{u,u-1}\hat{q}_1(x)x^{u-1}h(x) (h'(x))^{u-1} + s_{u,u-1}(x)h^2(x)]'
\end{aligned}$$

$$\begin{aligned}
&= r_{u,u-1}\hat{q}_1(x)x^u (h'(x))^u + [f_{u-1,2u-1} + r_{u,u-1}\hat{q}_1(x)x^{u-1} (h'(x))^{u-1} + s_{u,u-1}(x)h(x) \\
&\quad + r_{u,u-1}\hat{q}'_1(x)x^u (h'(x))^{u-1} + (u-1)r_{u,u-1}\hat{q}_1(x)x^{u-1} (h'(x))^{u-2} (h'(x) + xh''(x)) \\
&\quad + xs'_{u,u-1}(x)h(x) + 2xs_{u,u-1}(x)h'(x)]h(x). \tag{2.107}
\end{aligned}$$

Hence  $f_{u,u+v}(x)$  is of the form (2.97) and (2.98) holds in this case, completing the proof of (a) and (b).

We use generating functions to prove (c). The proof of (c) is an immediate consequence of Lemma 2.13 (see Remark 2.1 for the details).  $\square$

**Lemma 2.13.** *Define*

$$T_v(x) = \sum_{u=v}^{\infty} r_{u,v}x^{u-v}, \quad v \geq 0. \tag{2.108}$$

Then we have

(a)

$$T_v(x) = \frac{T'_{v-1}(x)}{1-x}, \quad v \geq 1. \tag{2.109}$$

(b)

$$T_0(x) = \frac{1}{1-x} \text{ and } T_v(x) = \frac{(2v-1)!!}{(1-x)^{2v+1}}, \quad v \geq 1. \tag{2.110}$$

*Proof.* (a) According to Definition (2.108),

$$\begin{aligned}
(1-x)T_v(x) &= \sum_{u=v}^{\infty} r_{u,v}x^{u-v} - \sum_{u=v}^{\infty} r_{u,v}x^{u-v+1} = \sum_{u=v}^{\infty} r_{u,v}x^{u-v} - \sum_{u=v+1}^{\infty} r_{u-1,v}x^{u-v} \\
&= r_{v,v} + \sum_{u=v+1}^{\infty} (r_{u,v} - r_{u-1,v})x^{u-v}. \tag{2.111}
\end{aligned}$$

By the recurrence relation (2.98), we get

$$r_{u,v} - r_{u-1,v} = (u-v+1)r_{u,v-1} \text{ for } u \geq v+1, \text{ and } r_{v-1,v} = r_{v,v}. \tag{2.112}$$

Thus

$$\begin{aligned}
(1-x)T_v(x) &= r_{v,v} + \sum_{u=v+1}^{\infty} (u-v+1)r_{u,v-1}x^{u-v} = r_{v-1,v} + \sum_{u=v+1}^{\infty} (u-v+1)r_{u,v-1}x^{u-v} \\
&= \sum_{u=v}^{\infty} (u-v+1)r_{u,v-1}x^{u-v}. \tag{2.113}
\end{aligned}$$

On the other hand, taking the derivative of both sides of Definition (2.108), we see that  $T'_{v-1}(x)$  also equals (2.113). Therefore (2.109) holds.

(b) Since  $r_{u,0} = 1$  (see Proposition 2.12(b)), we have

$$T_0(x) = \sum_{u=0}^{\infty} r_{u,0}x^u = \sum_{u=0}^{\infty} x^u = \frac{1}{1-x}. \tag{2.114}$$

Applying (a) to  $v = 1$ , we get

$$T_1(x) = \frac{T'_0(x)}{1-x} = \frac{1}{1-x} \left( \frac{1}{1-x} \right)' = \frac{1}{(1-x)^3}. \tag{2.115}$$

Thus (2.110) holds for  $v = 1$ .

Assume that (2.110) holds for  $v-1$  ( $v \geq 2$ ). It follows from (a) and the induction hypothesis that

$$T_v(x) = \frac{T'_{v-1}(x)}{1-x} = \frac{1}{1-x} \left( \frac{(2v-3)!!}{(1-x)^{2v-1}} \right)' = \frac{(2v-1)!!}{(1-x)^{2v+1}}.$$

Hence (2.110) holds for  $v$  and therefore for any  $v \geq 1$ .  $\square$

**Remark 2.1.** *The proof of part (c) of Proposition 2.12 is immediate, as any  $u \geq 1$ ,*

$$r_{u,u} = T_u(0) = (2u-1)!! \quad (2.116)$$

by Definition (2.108) and Lemma 2.13.

Setting  $v = u$  and  $x = 1$  in Proposition 2.12(a) and using (2.64) and (2.99), we get

$$f_{u,2u}(1) = r_{u,u} \hat{q}_1(1) (h'(1))^u = (2u-1)!! \hat{q}_1(1) (h'(1))^u, \quad (2.117)$$

as desired, completing the proof of Theorem 2.4 and Theorem 2.2(b).  $\square$

Applying Theorem 2.2 to the special cases  $(a, b) = (1, 0)$  and  $(0, 1)$ , we obtain the following results.

**Theorem 2.14.** *The expected values and variances of  $\mathcal{K}_n$  and  $\mathcal{L}_n$  are*

$$\mathbb{E}[\mathcal{K}_n] = \frac{1}{10}n + \frac{371 - 113\sqrt{5}}{40} + o(\hat{\gamma}_{1,0}^n), \quad \text{var}(\mathcal{K}_n) = \frac{29\sqrt{5} - 25}{1000}n + O(1), \quad (2.118)$$

$$\mathbb{E}[\mathcal{L}_n] = \frac{1}{10}n + \frac{361 - 123\sqrt{5}}{40} + o(\hat{\gamma}_{0,1}^n), \quad \text{var}(\mathcal{L}_n) = \frac{15 + 21\sqrt{5}}{1000}n + O(1). \quad (2.119)$$

Additionally, we have

$$\mathbb{E}[\mathcal{K}_n] - \mathbb{E}[\mathcal{L}_n] = \frac{1 + \sqrt{5}}{4} + o(\hat{\gamma}'^n) = \frac{\varphi}{2} + o(\hat{\gamma}'^n) \approx 0.809016994 \text{ for some } \hat{\gamma}' \in (0, 1) \quad (2.120)$$

In words, on average there are approximately 0.809 more positive terms than negative terms in the far-difference representation.

Applying Theorem 2.2 to  $a = b = 1$ , we get

$$\text{var}(\mathcal{K}_n + \mathcal{L}_n) = \frac{2\sqrt{5}}{125}n + O(1), \text{ and } \text{var}(\mathcal{K}_n - \mathcal{L}_n) = \frac{\sqrt{5} - 1}{10}n + O(1). \quad (2.121)$$

Hence

$$\begin{aligned} \text{cov}(\mathcal{K}_n, \mathcal{L}_n) &= \frac{\text{var}(\mathcal{K}_n + \mathcal{L}_n) - \text{var}(\mathcal{K}_n - \mathcal{L}_n)}{4} \\ &= \frac{25 - 21\sqrt{5}}{1000}n + O(1) \approx -0.0219574275n + O(1). \end{aligned} \quad (2.122)$$

With Theorem 2.14 and (2.122), we compute the correlation between  $\mathcal{K}_n$  and  $\mathcal{L}_n$ :

$$\begin{aligned} \text{corr}(\mathcal{K}_n, \mathcal{L}_n) &= \frac{\text{cov}(\mathcal{K}_n, \mathcal{L}_n)}{\sqrt{\text{var}(\mathcal{K}_n)\text{var}(\mathcal{L}_n)}} = \frac{\frac{25-21\sqrt{5}}{1000}n + O(1)}{\sqrt{\left(\frac{29\sqrt{5}-25}{1000}n + O(1)\right)\left(\frac{29\sqrt{5}-25}{1000}n + O(1)\right)}} \\ &= \frac{\frac{25-21\sqrt{5}}{1000}n + O(1)}{\frac{29\sqrt{5}-25}{1000}n + O(1)} = \frac{25 - 21\sqrt{5}}{29\sqrt{5} - 25} + o(1) \end{aligned}$$

$$= \frac{10\sqrt{5} - 121}{179} + o(1) \approx -0.551057655 + o(1). \quad (2.123)$$

Since  $\text{var}(\mathcal{K}_n)$  and  $\text{var}(\mathcal{L}_n)$  are of size  $n$  and have the same coefficients of  $n$ , we have

$$\begin{aligned} \text{cov}(\mathcal{K}_n + \mathcal{L}_n, \mathcal{K}_n - \mathcal{L}_n) &= E[(\mathcal{K}_n - E[\mathcal{K}_n] + (\mathcal{L}_n - E[\mathcal{L}_n]))(\mathcal{K}_n - E[\mathcal{K}_n] - (\mathcal{L}_n - E[\mathcal{L}_n]))] \\ &= E[(\mathcal{K}_n - E[\mathcal{K}_n])^2 - (\mathcal{L}_n - E[\mathcal{L}_n])^2] \\ &= \text{var}(\mathcal{K}_n) - \text{var}(\mathcal{L}_n) = O(1). \end{aligned} \quad (2.124)$$

Further, we have the values of  $\text{var}(\mathcal{K}_n + \mathcal{L}_n)$  and  $\text{var}(\mathcal{K}_n - \mathcal{L}_n)$  from (2.121) and (2.121), thus

$$\begin{aligned} \text{corr}(\mathcal{K}_n + \mathcal{L}_n, \mathcal{K}_n - \mathcal{L}_n) &= \frac{\text{cov}(\mathcal{K}_n + \mathcal{L}_n, \mathcal{K}_n - \mathcal{L}_n)}{\sqrt{\text{var}(\mathcal{K}_n + \mathcal{L}_n)\text{var}(\mathcal{K}_n - \mathcal{L}_n)}} \\ &= \frac{O(1)}{\sqrt{\left(\frac{2\sqrt{5}}{125}n + O(1)\right)\left(\frac{\sqrt{5}-1}{10}n + O(1)\right)}} = o(1). \end{aligned}$$

Since  $\mathcal{K}_n$  and  $\mathcal{L}_n$  are bivariate Gaussian,  $\mathcal{K}_n + \mathcal{L}_n$  and  $\mathcal{K}_n - \mathcal{L}_n$  are independent as  $n \rightarrow \infty$ .

### 3. CONCLUSION AND FUTURE RESEARCH

Our combinatorial perspective has extended previous work, allowing us to prove Gaussian behavior for the number of summands for a large class of expansions in terms of solutions to linear recurrence relations. This is just the first of many questions one can ask. Others, which we hope to return to at a later date, include:

- (1) Are there similar results for linearly recursive sequences with arbitrary integer coefficients (i.e., negative coefficients are allowed in the defining relation, which is different than allowing negative summands)?
- (2) What happens if we consider sequences where either uniqueness of representation fails, or some numbers are not representable? What is true for a ‘generic’ number?
- (3) Lekkerkerker’s theorem, and the Gaussian extension, are for the behavior in intervals  $[F_n, F_{n+1})$ . Do the limits exist if we consider other intervals, say  $[F_n + g_1(F_n), F_n + g_2(F_n))$  for some functions  $g_1$  and  $g_2$ ? If yes, what must be true about the growth rates of  $g_1$  and  $g_2$ ?
- (4) For the generalized recurrence relations, what happens if instead of looking at  $\sum_{i=1}^n a_i$  we study  $\sum_{i=1}^n \min(1, a_i)$ ? In other words, we only care about how many distinct  $H_i$ ’s occur in the decomposition.
- (5) What can we say about the distribution of the largest gap between summands in generalized Zeckendorf decomposition? Appropriately normalized, how does the distribution of gaps between the summands behave?

The last question has been solved in some cases in [BM]. They prove

**Theorem 3.1** (Base  $B$  Gap Distribution). *For base  $B$  decompositions, as  $n \rightarrow \infty$  the probability of a gap of length 0 between summands for numbers in  $[B^n, B^{n+1})$  tends to  $\frac{(B-1)(B-2)}{B^2}$ , and for gaps of length  $k \geq 1$  to  $\frac{(B-1)(3B-2)}{B^2} B^{-k}$ .*

**Theorem 3.2** (Zeckendorf Gap Distribution). *For Zeckendorf decompositions, for integers in  $[F_n, F_{n+1})$  the probability of a gap of length  $k \geq 2$  tends to  $\frac{\varphi(\varphi-1)}{\varphi^k}$  for  $k \geq 2$ , with  $\varphi = \frac{1+\sqrt{5}}{2}$  the golden mean.*

## APPENDIX A. NEEDED TECHNICAL RESULTS

## A.1. Proof of Proposition 2.3.

*Proof of Proposition 2.3.* For (c), since  $e_i(w)$  is a root of  $\hat{A}_w(z)$ , we have

$$0 = 1 - e_i(w) - (w^a + w^b)e_i^4(w) - w^{a+b}e_i^6(w) - w^{a+b}e_i^7(w). \quad (\text{A.1})$$

For a small increment  $\Delta w$ , we have

$$\begin{aligned} 0 = & 1 - e_i(w + \Delta w) - [(w + \Delta w)^a + (w + \Delta w)^b]e_i^4(w + \Delta w) \\ & - (w + \Delta w)^{a+b}e_i^6(w + \Delta w) - (w + \Delta w)^{a+b}e_i^7(w + \Delta w). \end{aligned} \quad (\text{A.2})$$

Subtracting (A.2) from (A.1) yields

$$\begin{aligned} 0 = & e_i(w + \Delta w) - e_i(w) + (w^a + w^b)[e_i^4(w + \Delta w) - e_i^4(w)] \\ & + [(w + \Delta w)^a + (w + \Delta w)^b - w^a - w^b]e_i^4(w + \Delta w) \\ & + w^{a+b}[e_i^6(w + \Delta w) - e_i^6(w)] + [(w + \Delta w)^{a+b} + w^{a+b}]e_i^6(w + \Delta w) \\ & + w^{a+b}[e_i^7(w + \Delta w) - e_i^7(w)] + [(w + \Delta w)^{a+b} - w^{a+b}]e_i^7(w + \Delta w) \\ = & [e_i(w + \Delta w) - e_i(w)] \left[ 1 + (w^a + w^b) \sum_{j=0}^3 e_i^j(w + \Delta w) e_i^{3-j}(w) \right. \\ & \left. + w^{a+b} \sum_{j=0}^5 e_i^j(w + \Delta w) e_i^{5-j}(w) + w^{a+b} \sum_{j=0}^6 e_i^j(w + \Delta w) e_i^{6-j}(w) \right] \\ & + \Delta w \left[ \left( \frac{(w + \Delta w)^a - w^a}{\Delta w} + \frac{(w + \Delta w)^b - w^b}{\Delta w} \right) e_i^4(w + \Delta w) \right. \\ & \left. + \frac{(w + \Delta w)^{a+b} - w^{a+b}}{\Delta w} (e_i^6(w + \Delta w) + e_i^7(w + \Delta w)) \right]. \end{aligned} \quad (\text{A.3})$$

Since  $e_i(w)$  is continuous, the coefficient of  $[e_i(w + \Delta w) - e_i(w)]$  converges as  $\Delta w \rightarrow 0$  and its limit is

$$1 + 4(w^a + w^b)e_i^3(w) + 6w^{a+b}e_i^5(w) + 7w^{a+b}e_i^6(w), \quad (\text{A.4})$$

which is exactly  $-\hat{A}'_w(z)$  (with respect to  $z$ ) at  $e_i(w)$  and therefore nonzero since  $\hat{A}_w(z)$  has no multiple roots. Since  $w^a$ ,  $w^b$  and  $w^{a+b}$  are differentiable at  $w = 1$ , the coefficient of  $\Delta w$  in (A.3) also converges as  $\Delta w \rightarrow 0$  and its limit is

$$(aw^{a-1} + bw^{b-1})e_i^4(w) + (a+b)w^{a+b-1}[e_i^6(w) + e_i^7(w)]. \quad (\text{A.5})$$

Thus  $e'_i(w)$  exists and

$$\begin{aligned} e'_i(w) &= \lim_{\Delta w \rightarrow 0} \frac{e_i(w + \Delta w) - e_i(w)}{\Delta w} \\ &= -\frac{(aw^{a-1} + bw^{b-1})e_i^4(w) + (a+b)w^{a+b-1}[e_i^6(w) + e_i^7(w)]}{1 + 4(w^a + w^b)e_i^3(w) + 6w^{a+b}e_i^5(w) + 7w^{a+b}e_i^6(w)}. \end{aligned} \quad (\text{A.6})$$

Since the denominator of  $e'_i(w)$  is not zero, by repeated use of the quotient rule and the differentiability of  $e_i(w)$ , we see that  $e_i^{(\ell)}(w)$  exists for any  $\ell \geq 1$ .

Finally for (d), since the leading coefficient of  $\hat{A}_w(z)$  is  $-w^{a+b}$  and the roots of  $\hat{A}_w(z)$  are  $e_1(w), e_2(w), \dots, e_7(w)$ , we have

$$\hat{A}_w(z) = -w^{a+b} \prod_{i=1}^7 (z - e_i(w)). \quad (\text{A.7})$$

For any  $w \in I_\varepsilon$ , the  $e_i(w)$ 's are distinct, thus we can interpolate the Lagrange polynomial of  $\mathcal{L}(z) = 1$  at  $e_1(w), e_2(w), \dots, e_7(w)$ :

$$\sum_{i=1}^7 \frac{\prod_{j \neq i} (z - e_j(w))}{\prod_{j \neq i} (e_j(w) - e_i(w))} = 1. \quad (\text{A.8})$$

Dividing both sides by  $\prod_{i=1}^7 (z - e_i(w))$  and combining with (A.7) yields (2.22).  $\square$

**A.2. Proof that  $\hat{\mu}_n(m) = \tilde{\mu}_n(m) + o(\beta_m^n)$ .** In (2.30), we proved that  $\hat{\mu}_n = \hat{C}_{a,b}n + \hat{d}_{a,b} + o(\hat{\gamma}_{a,b}^n)$  and we set  $\tilde{\mu}_n = \hat{C}_{a,b}n + \hat{d}_{a,b}$  ( $\hat{C}_{a,b}$  and  $\hat{d}_{a,b}$  are defined in (2.31)). Thus  $\hat{\mu}_n = \tilde{\mu}_n + o(\hat{\gamma}_1^n)$ . We defined  $\tilde{\mu}_n(m) = \sum_{k,l} p_{n,k,l} (ak + bl - \tilde{\mu}_n)^m / D_n$ . In this section we prove the following.

**Lemma A.1.** *For any  $m$ , we have  $\hat{\mu}_n(m) = \tilde{\mu}_n(m) + o(\beta_m^n)$  for some  $\beta_m \in (0, 1)$ .*

*Proof.* In the argument below, we will need an upper bound for the number of positive (negative) summands an  $N \in (S_{n-1}, S_n)$  can have. As there are  $n$  Fibonacci numbers and each one can be taken at most once, the maximum number of positive (negative) summands such an  $N$  can have is  $n$ .

Since

$$\begin{aligned} \mu_n(m) &= \sum_{k,l} \frac{p_{n,k,l} (ak + bl - \hat{\mu}_n)^m}{D_n} = \sum_{k,l} \text{Prob}(n, k, l) (ak + bl - \hat{\mu}_n)^m \\ \tilde{\mu}_n(m) &= \sum_{k,l} \frac{p_{n,k,l} (ak + bl - \tilde{\mu}_n)^m}{D_n} = \sum_k \text{Prob}(n, k, l) (ak + bl - \tilde{\mu}_n)^m \\ \mu_n &= \tilde{\mu}_n + o(\hat{\gamma}_1^n) \text{ by Theorem 1.2,} \end{aligned} \quad (\text{A.9})$$

when  $n$  is large, we have

$$\begin{aligned} |\mu_n(m) - \tilde{\mu}_n(m)| &= \left| \sum_{k,l} \text{Prob}(n, k, l) [(ak + bl - \tilde{\mu}_n + o(\hat{\gamma}_1^n))^m - (ak + bl - \tilde{\mu}_n)^m] \right| \\ &= \left| o(\hat{\gamma}_1^n) \sum_{k,l} \text{Prob}(n, k, l) \sum_{i=1}^m \binom{m}{i} (ak + bl - \tilde{\mu}_n)^{m-i} (o(\hat{\gamma}_1^n))^{i-1} \right| \\ &\leq \left| o(\hat{\gamma}_1^n) \sum_k \text{Prob}(n, k, l) (ak + bl + \tilde{\mu}_n + 1)^m \right| \\ &\leq \left| o(\hat{\gamma}_1^n) (|a|n + |b|n + C_{a,b}n + d_{a,b} + 2011)^m \sum_k \text{Prob}(n, k) \right| \\ &\leq |o(\hat{\gamma}_1^n) (|a| + |b| + |C_{a,b}| + |d_{a,b}| + 2011)^m n^m \cdot 1| = o(\beta_m^n) \end{aligned} \quad (\text{A.10})$$

for some  $\beta_m \in (0, 1)$ .  $\square$

**A.3. Proof that  $h'(1) \neq 0$ .** This is important in the proof of the Gaussian behavior in Section 2.3, as this tells us that the variances grows like  $n$ .

*Proof.* By (2.21), we have

$$\frac{we'_1(w)}{e_1(w)} = -\frac{(aw^a + bw^b) e_1^3(w) + (a+b)w^{a+b}[e_1^5(w) + e_1^6(w)]}{1 + 4(w^a + w^b)e_1^3(w) + 6w^{a+b}e_1^5(w) + 7w^{a+b}e_1^6(w)}. \quad (\text{A.11})$$

Thus

$$\begin{aligned} \hat{h}'(w) &= \left[ \frac{(aw^a + bw^b) e_1^3(w) + (a+b)w^{a+b}(e_1^5(w) + e_1^6(w))}{1 + 4(w^a + w^b)e_1^3(w) + 6w^{a+b}e_1^5(w) + 7w^{a+b}e_1^6(w)} \right]' \\ &= \left[ [(aw^a + bw^b) e_1^3(w) + (a+b)w^{a+b}(e_1^5(w) + e_1^6(w))] \right]' \\ &\quad \cdot [1 + 4(w^a + w^b)e_1^3(w) + 6w^{a+b}e_1^5(w) + 7w^{a+b}e_1^6(w)] \\ &\quad - [(aw^a + bw^b) e_1^3(w) + (a+b)w^{a+b}(e_1^5(w) + e_1^6(w))] \\ &\quad \cdot [1 + 4(w^a + w^b)e_1^3(w) + w^{a+b}(6e_1^5(w) + 7e_1^6(w))] \Big]' \\ &\quad \cdot [1 + 4(w^a + w^b)e_1^3(w) + w^{a+b}(6e_1^5(w) + 7e_1^6(w))]^{-2}. \end{aligned} \quad (\text{A.12})$$

Setting  $w = 1$  in (A.11) and using  $e_1(1) = \Phi$ , we get

$$\frac{e'_1(1)}{e_1(1)} = -\frac{(a+b)(\Phi^3 + \Phi^5 + \Phi^6)}{1 + 8\Phi^3 + 6\Phi^5 + 7\Phi^6} = -\frac{a+b}{10}. \quad (\text{A.13})$$

Thus

$$e'_1(1) = -\frac{a+b}{10} \Phi. \quad (\text{A.14})$$

Plugging  $e_1(1) = \Phi$  and (A.14) into (A.12) with  $w = 1$  yields

$$\begin{aligned} \hat{h}'(1) &= [\Phi^5 [10(a^2 + b^2) + (a+b)^2(-3 + 10\Phi - 5\Phi^2 - 6\Phi^3)] \\ &\quad - \Phi^5(a+b)^2(1.6 + 3\Phi^2 + 2.8\Phi^3)] / (100\Phi^4) \\ &= \frac{\sqrt{5}-1}{200} \left[ 10(a^2 + b^2) - \frac{20-\sqrt{5}}{5}(a+b)^2 \right] \end{aligned} \quad (\text{A.15})$$

Since  $\frac{20-\sqrt{5}}{5} < 4$  and  $a^2 + b^2 > 0$ , we have

$$\frac{20-\sqrt{5}}{5}(a^2 + b^2) < 4(a+b)^2 \leq 8(a^2 + b^2) < 10(a^2 + b^2). \quad (\text{A.16})$$

Hence  $\hat{h}'(1) \neq 0$ . □

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