Closed-Form Bayesian Inferences for the Logit Model via Polynomial Expansions

Steven J. Miller, Eric T. Bradlow, Kevin Dayaratna*
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Abstract

Articles in Marketing and choice literatures have demonstrated the need for incorporating person-level heterogeneity into behavioral models (e.g., logit models for multiple binary outcomes as studied here). However, the logit likelihood extended with a population distribution of heterogeneity doesn't yield closed-form inferences, and therefore numerical integration techniques are relied upon (e.g., MCMC methods).

We present here an alternative, closed-form Bayesian inferences for the logit model, which we obtain by approximating the logit likelihood via a polynomial expansion, and then positing a distribution of heterogeneity from a flexible family that is now conjugate and integrable. For problems where the response coefficients are independent, choosing the Gamma distribution leads to rapidly convergent closed-form expansions; if there are correlations among the coefficients one can still obtain rapidly convergent closed-form expansions by positing a distribution of heterogeneity from a Multivariate Gamma distribution. The solution then comes from the moment generating function of the Multivariate Gamma distribution or in general from the multivariate heterogeneity distribution assumed.

Closed-form Bayesian inferences, derivatives (useful for elasticity calculations), population distribution parameter estimates (useful for summarization) and starting values (useful for complicated algorithms) are hence directly available. Two simulation studies demonstrate the efficacy of our approach.

Keywords: Closed-Form Bayesian Inferences, logit model, Generalized Multivariate Gamma Distribution

JEL Classification: C6, C8, M3

*Steven J. Miller (Email: sjmiller@math.brown.edu) is a Tamarkin Assistant Professor of Mathematics at Brown University. Eric T. Bradlow (Email: ebradlow@wharton.upenn.edu) is the K. P. Chao Professor, Professor of Marketing and Statistics, and Academic Director of the Wharton Small Business Development Center at the Wharton School of the University of Pennsylvania. Kevin Dayaratna (Email: kdayarat@rhsmith.umd.edu) is a doctoral student in the Marketing Department at the University of Maryland. The authors would like to thank Peter S. Fader and Dan Stone for comments on this manuscript.

INTRODUCTION

Whether it's the 20,000+ hits based on a www.google.com search or the 1000+ hits on www.jstor.org or the hundreds of published papers in a variety of disciplines from Marketing to Economics (Hausman and McFadden 1984) to Statistics (Albert and Chib, 1993) to Transportation (Bierlaire et. al, 1997), the logit model plays a very prominent role in many literatures as a basis for probabilistic inferences for binary outcome data. In part, this is due to the ubiquitous nature of binary outcome data, whether it is choices to buy in a given product category or not, the choice to go to a given medical provider or not, and the like; and, in part, it may be due to the link between random utility theory and the logit model in which binary choices following the logit model are the outcome of a rational economic maximization of latent utility with extreme value distributed errors (McFadden, 1974).

One of the recent advances regarding this class of models, which has made its use even more widespread, is its ability to incorporate heterogeneity into the response coefficients, reflecting the fact that individuals are likely to vary on the attribute coefficients that influence their choices (Rossi and Allenby, 1993; McCulloch and Rossi, 1994). Whether this heterogeneity is modelled in an hierarchical Bayesian fashion allowing for complete variation (Gelfand et. al, 1990), in a latent-class way allowing for discrete segments (Kamakura and Russell, 1989), or by using a finite mixture approach (Train and McFadden, 2000), incorporating person-level heterogeneity is now the "expected" rather than the "exception".

Unfortunately, the added flexibility that heterogeneity allows comes with a price – numerical computation and complexity. That is, once one combines the logit choice kernel, a Bernoulli random variable with logit link function, with a heterogeneity distribution, closed-form inference is unavailable due to the non-conjugacy of the product Bernoulli likelihood and the heterogeneity distribution (prior). Therefore, numerical methods such as quadrature, simulated maximum likelihood (Revelt and Train, 1998), or Markov chain Monte Carlo methods (Gelman et. al., 1995) are commonly employed to integrate over the heterogeneity distribution and obtain inferences for the parameters that govern the heterogeneity distribution (the so-called, population level parameters). For instance, in the case of a Gaussian heterogeneity distribution this requires the marginal integration of the product Bernoulli logit likelihood with the Gaussian distribution, to obtain means, variances, and possibly covariances of the prior. While faster computing and specialized software has made this feasible, this research considers an alternative to these approaches, a "closed-form" solution.

That is, in this research we consider a closed-form solution to the heterogeneous binary logit choice problem that involves approximating the product Bernoulli logit likelihood via a polynomial expansion (to any specified accuracy), and then specifying a rich and flexible class of heterogeneity distributions for the response coefficients (slopes). If the response coefficients within individuals are independent, we model them as arising from the Gamma distribution (albeit we demonstrate how are results can be obtained for any multivariate distribution) or, more generally, a mixture of Gamma distributions (McDonald and Butler, 1990); if the response coefficients are not independent we model them as arising from a Multivariate Gamma distribution, which allows correlations among the coefficients. We then integrate, now possible in closed-form, the approximated logit model with respect to these families. Once the model is integrated with respect to the heterogeneity distribution, we then can either: (a) maximize the marginal likelihood and obtain Maximum Marginal Likelihood (MML) estimates of the population parameters and utilize them for conditional inferences (the empirical Bayes approach: Morrison and Schmittlein, 1981; Morris, 1983; Schmittlein, Morrison, and Columbo, 1987) or (b) in the case where the parameters of the heterogeneity distribution are set informatively based on prior information, historical data, subjective beliefs, or the like, fully Bayesian inferences are obtainable.

In this manner, as in Bradlow, Hardie, and Fader (2002) for the negative-Binomial distribution, and in Everson and Bradlow (2002) for the beta-binomial distribution, one can effectively incorporate prior information and allow shrinkage that Bayesian models attend to, but can also obtain closed-form inferences *without* Monte Carlo simulation efforts or quadrature that can be sensitive to the starting values and/or contain significant simulation error. We demonstrate the efficacy of our approach using two simulated studies, therefore supporting its use as an alternative method. In addition, we also demonstrate that as a by-product of the method, closed-form derivatives of the marginal distribution are obtained which are often of interest in that they inform how the distribution (possibly in particular the mean and variance) of population effects would change as a function of a change in the decision inputs (i.e. covariates). These derivatives are also commonly (and directly) used in the computation of probability elasticities, thus providing the opportunity for optimization decisions.

The remainder of this paper is laid out as follows. In Section 1 we systematically lay out the problem formulation by deriving the likelihood, which provides the basis of our polynomial expansion (an application of a geometric series expansion), as well as discuss the types of data for which our method is applicable. In particular, the results presented here (albeit they are generalizable) are most applicable (as we discuss in Section 1) to product categories for which the binary response rate is either rare (e.g. durable goods purchases (Bayus, 1992) and mail catalog responses (Anderson and Simester (2001)), or those for which the frequency of purchase is high (e.g. orange juice).

Section 2 is an in-depth analysis of the case when the response coefficients are drawn from independent Gamma distributions. Sections 2.1 and 2.2 contain our key integration results demonstrating the conjugacy of the approximation to the binary data likelihood and the Gamma family of distributions (Theorem 2.2). Details of the integration lemma and plots of the robustness of the Gamma family and its generalizations that we consider are in Appendix A. We discuss computational issues related to our series expansion in Section 2.3. In Section 2.3.1, details of the method to maximize the marginal likelihood are given, and in addition we provide computational efficiency gains and guidelines as to the number of calculations that will occur using our method. In particular, we initially obtain closed-form solutions involving infinite sums. Using combinatorial results on systems of equations with integer coefficients, we show in Theorem 2.3 how these sums may be re-grouped to a lengthy (to be discussed and evaluated via simulation) initial calculation independent of the parameter values, and then a fast parameter-specific computation which makes the entire approach tractable. Thus subsequent computations of the marginal likelihood at different parameter values (necessary for optimization) is rapid. Additional details of these combinatorial savings are provided in Appendix C. In Section 2.3.2 we provide some simulations to demonstrate the efficacy of our approach. In Section 2.3.3 we compare our closed form series expansion with previous numerical techniques used to analyze these types of Bayesian inference problems. In the case of one observation per household, our series expansions have a comparable run-time to Monte Carlo Markov Chain methods (in fact, the series expansions are faster); however, for multiple observations per household these numerical methods are typically faster, though our series expansions can still be implemented in a reasonable amount of time. In this manner, our approach is an alternative, albeit for many practical problems not one that is faster, but rather one that can be used to verify other (e.g. MCMC) methods. Some areas for future research and limitations of our approach, in particular the extension of our findings to a more general class of priors (Theorem 2.4), and a more general class of covariates, are described in Section 2.4. We show that, at the cost of introducing new special functions, we can handle any one-sided probability distribution for the priors.

In Section 3 we generalize the results of Section 2 to allow for covariances. In Section 3.1.1 we derive a closed-form series expansion for an arbitrary multivariate distribution; however, if the distribution has a good closed-form expression for its moment generating function, more can be done. We concentrate on the case where the response coefficients are drawn from a Multivariate Gamma distribution, which allows us to have correlations among the response coefficients. The key observation is Theorem 3.1, where we interpret the resulting integrals as evaluations of the Moment Generating Function which exists in closed-form for the Multivariate Gamma distribution. Thus we may mirror the arguments from Section 2 and again obtain a rapidly convergent series expansion (Theorem 3.3), and the combinatorial results of Section 2.3.1 and Appendix C are still applicable.

Section 4 contains some concluding remarks.

1 PROBLEM FORMULATION

As the logit model and its associated likelihood are well understood, we briefly describe them in Section 1.1 and focus mainly here (in Section 1.2) on the geometric series expansion of the model. If we assume the parameters are independent (all zero covariances), then a tractable model is obtained by assuming that each is drawn from a Gamma distribution. This is described in detail in Section 2; in Section 3 we generalize the model by assuming the parameters are drawn from a multivariate Gamma distribution, which allows us to handle covariances among the parameters. In both cases we obtain closed-form series expansions. Further, a careful analysis of the resulting combinatorics leads to computational gains that make the approach feasible and attractive.

1.1 Notation

To describe the model, and to be specific about the data structures addressed (and not addressed) in this research, we utilize the following notation. The jargon is drawn from the Marketing domain and is done for purely explicative purposes. As we demonstrate, our approach is applicable for a wide class of general data structures.

Consider a data set obtained from $i \in \{1, \ldots, I\}$ households (units) containing $j \in \{1, \ldots, J\}$ product categories (objects; e.g. coffee) measured on $t \in \{1, \ldots, N_i\}$ purchase occasions (repeated measures). At each purchase occasion, for each category j each household i decides whether or not to purchase in that category.

As is standard, we define

$$y_{ijt} \;=\; \begin{cases} 1 & \text{if household } i \text{ buys in category } j \text{ at time } t \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

where $p_{ijt} = \text{Prob}(y_{ijt} = 1)$ is the probability of purchase of the j^{th} category by the i^{th} household on its t^{th} purchase occasion. Further, let P denote a set of attributes describing the categories, with corresponding values $x_{ijt,p}$ such that $X_{ijt}^T = (x_{ijt,1}, \ldots, x_{ijt,P})$. To account for differences in base-level preferences for categories, we define $x_{ijt,1} = 1$ defining category-level intercepts. Thus, multiplying over all households, categories, and occasions, we obtain that the standard logit likelihood of the data, $Y = (y_{ijt})$, is given by

$$P(Y|\beta) \;=\; \prod_{i=1}^{I}\prod_{j=1}^{J}\prod_{t=1}^{N_i} \frac{e^{-X_{ijt}^T\beta_i y_{ijt}}}{1+e^{-X_{ijt}^T\beta_i}}, \tag{2}$$

where $\beta_i = (\beta_{i,1}, \ldots, \beta_{i,P})$ is the coefficient vector for the i^{th} household with variable p specific coefficient, $\beta_{i,p}$. It is the heterogeneity across households i in their $\beta_{i,p}$ that we model in Section 2 as coming from the Gamma family of distributions, and in Section 3 as coming from a multivariate family of distributions.

The marginalization of the likelihood, which is the problem we address here, is that we want to "hit" $P(Y|\beta)$ (integrate with respect to) a set of distributions $g(\beta_{i,p}|\Omega)$ depending on parameters Ω such that

$$P(Y|\Omega) \;=\; \int P(Y|\beta)g(\beta|\Omega)d\beta \tag{3}$$

is available in closed-form. To accomplish this, we require a properly chosen series expansion of $P(Y|\beta)$, and we describe its basic building block next, an application of the geometric series expansion. The goal is to obtain a good closed-form expansion of the marginalization of the likelihood for each choice of parameters Ω. To do so requires finding an appropriate conjugate distribution leading to tractable integration; we shall see that the Gamma (Theorem 2.2) and Multivariate Gamma (Theorem 3.1) distributions lead to integrals which can be evaluated in closed-form. Using such expansions, we then determine the value of Ω which maximizes this likelihood; this will allow us to make inferences about the population heterogeneity distributions.

1.2 Geometric Series Expansion

To obtain closed-form Bayesian inferences for the logit model, we expand the likelihood $P(Y|\beta)$ given in (2) by using the geometric series expansion:

$$\frac{1}{1-z} = \sum_{k=0}^{\infty} z^k. \tag{4}$$

This is directly applicable for our problem as $P(Y|\beta)$ is the product of terms of the form $\frac{e^{uy}}{1+e^u}$. Our interest will be in expanding the denominator when $u < 0$. Note terms with $u > 0$ can be handled by writing $\frac{1}{1+e^u}$ as $\frac{e^{-u}}{1+e^{-u}}$.

While in theory we can unify the two cases (positive and negative values of u) by using the sgn function ($\text{sgn}(u) = 1$ if $u > 0$, 0 if $u = 0$ and -1 otherwise), the sgn function is only practical in the case of one attribute (i.e. x is one dimensional and P=1); otherwise it is undesirable (untenable) in the expansions. In high dimensions (lots of households with lots of categories and attributes), the sgn function leads to numerous, complicated subdivisions of the integration space. This greatly increases the difficulty in performing the integration and obtaining tractable closed-form expansions, and hence is not entertained here. Details of the unification are available from the authors upon request, and applying it in practice is an area for future research. Instead, we describe below a specific set of restrictions that we employ, and the class of problems (data sets) where our expansions can then directly be applied. In Section 6, areas for future research to generalize our work to richer data sets are discussed.

As mentioned above, to eliminate the need for the sgn function and to allow for straightforward expansions, we limit our investigations to the common set of Marketing problems (as described in Section 1.1 and throughout) in which all

1. $X_{ijt,p} \geq 0$,
2. $\beta_{i,p} > 0$,
3. $\sum_{p=1}^{P} \beta_{i,p} X_{ijt,p} > 0$.

From a practical perspective, these restrictions indicate as follows. First, each $X_{ijt,p} \geq 0$ is not particularly restrictive, as commonly utilized descriptor variables – prices, dummy variables for feature and display, etc..., as in standard SCANPRO models (Wittink et al. 1988), are all non-negative and are straightforwardly accounted for in our framework. Those variables which take signs counter to previously signed variables can be coded as $f(X_{ijt,p})$, for instance $-X_{ijt,p}$ or $\exp(-X_{ijt,p})$.

Secondly, restriction of $\beta_{i,p} > 0$ may or may not be restrictive. If the variables which comprise $X_{ijt,p}$ are ones in which we want to enforce monotonicity constraints (Allenby, Aurora, and Ginter, 1995), or naturally one would expect upward sloping demand at the category-level (which may be much more likely than at the brand level), then this constraint is not at all restrictive, and in fact may improve the predictions of the model. To implement this, especially in the case of dummy coded $X_{ijt,p}$, the least preferred level should be coded as 0 so that all other corresponding dummy variables have $X_{ijt,p} = 1$ and it is expected that $\beta_{i,p} > 0$.

Our third constraint, which is not restrictive as long one of the X's (e.g. price, coupon, etc...) is non-zero, is required so that we are not expanding $\frac{1}{2} = \frac{1}{1+1}$ in a polynomial series; if this condition fails, trivial book-keeping suffices.

There are two important things to note. First, if all the $\beta_{i,p}$'s are negative, we may explicitly factor out the negative sign of each $\beta_{i,p}$, yielding terms such as $-X_{ijt,p}|\beta_{i,p}|$. If this is the case, for simplicity we change variables and let $\beta_{i,p} = |\beta_{i,p}|$; thus, $\beta_{i,p}$ and $X_{ijt,p}$ are now both non-negative, and we have minus signs in the exponents above. Thus we do not need to assume all persons have positive or all persons have negative coefficients, but rather (by recoding X to $-X$) that each person's coefficients are all positive or all negative. Secondly, in totality, the restrictions above suggest that our model works only for categories in which the probability of purchasing in that category on any given occasion is strictly greater than $\frac{1}{2}$ (if coded as before) or less than $\frac{1}{2}$ (if coded as in this paragraph), where again this can very person-by-person. Certainly an area for future research would be the ability, if possible, to relax some of these assumptions, and to empirically investigate the set of product categories for which these restrictions are not particularly binding (such as long-lasting durable goods).

Thus (after possibly recoding), due to our restrictions, we only need to use (4) when $X_{ijt}^T\beta_i > 0$, which yields

$$\frac{1}{1+e^{-X_{ijt}^T\beta_i}} \quad = \quad \sum_{k_{ijt}=0}^{\infty} (-1)^{k_{ijt}}\ e^{-k_{ijt}X_{ijt}^T\beta_i}, \qquad X_{ijt}^T\beta_i \ > \ 0. \tag{5}$$

This is combined, as described next, with the Gamma or Multivariate Gamma family of distributions in a conjugate way. It is the constancy of the sign of $X_{ijt}^T\beta_i$ that allows us to use the same series expansion for all β_i.

1.3 Expansion of $P(Y|\beta)$

Using the likelihood for the logit model given in (2), we have as follows:

$$\begin{aligned} P(Y|\beta) &= \prod_{i=1}^{I}\prod_{j=1}^{J}\prod_{t=1}^{N_i} \frac{e^{-X_{ijt}^T\beta_i y_{ijt}}}{1+e^{-X_{ijt}^T\beta_i}} \\ &= \prod_{i=1}^{I}\prod_{j=1}^{J}\prod_{t=1}^{N_i} e^{-y_{ijt}X_{ijt}^T\beta_i} \prod_{i=1}^{I}\prod_{j=1}^{J}\prod_{t=1}^{N_i} \frac{1}{1+e^{-X_{ijt}^T\beta_i}} \\ &= \prod_{i=1}^{I}\prod_{p=1}^{P} e^{-\ \sum_{j=1}^{J}\sum_{t=1}^{N_i} y_{ijt}x_{ijt,p}\ \ \beta_{i,p}} \cdot \prod_{i=1}^{I}\prod_{j=1}^{J}\prod_{t=1}^{N_i} \frac{1}{1+e^{-X_{ijt}^T\beta_i}} \\ &= P_1(Y|\beta)P_2(Y|\beta). \end{aligned} \tag{6}$$

Note that the first term, $P_1(Y|\beta)$, is already an exponential function. This combines nicely with the Gamma and Multivariate Gamma distributions, and for each variable $\beta_{i,p}$, we simply have the exponential of a multiple of $\beta_{i,p}$. In fact, as we show later in Theorem 3.1, it is this exponential form that leads to the result that all closed-form integrals are obtainable using the Moment Generating Functions of the heterogeneity distribution. It is the second term, $P_2(Y|\beta)$, that we expand by using the geometric series. We describe this now.

The real difficulty in coming up with a conjugate family to the logit model is in the expansion of $P_2(Y|\beta)$. Using the geometric series expansion, we obtain

$$\begin{aligned} P_2(Y|\beta) &= \prod_{i=1}^{I}\prod_{j=1}^{J}\prod_{t=1}^{N_i} \frac{1}{1+e^{-X_{ijt}^T\beta_i}} \\ &= \prod_{i=1}^{I}\prod_{j=1}^{J}\prod_{t=1}^{N_i} \sum_{k_{ijt}=0}^{\infty} (-1)^{k_{ijt}} e^{-k_{ijt}X_{ijt}^T\beta_i} \\ &= \prod_{i=1}^{I}\prod_{j=1}^{J}\prod_{t=1}^{N_i} \sum_{k_{ijt}=0}^{\infty} (-1)^{k_{ijt}} e^{-k_{ijt}\sum_{p=1}^{P} x_{ijt,p}\beta_{i,p}}. \end{aligned} \tag{7}$$

For fixed household i, replacing $\prod_{j=1}^{J}\prod_{t=1}^{N_i}\sum_{k_{ijt}=0}^{\infty}$ with $\sum_{k_{i11}=0}^{\infty}\cdots\sum_{k_{iJN_i}=0}^{\infty}$ yields

$$P_2(Y|\beta) = \prod_{i=1}^{I}\left(\sum_{k_{i11}=0}^{\infty}\cdots\sum_{k_{iJN_i}=0}^{\infty} (-1)^{\sum_{j=1}^{J}\sum_{t=1}^{N_i} k_{ijt}} \prod_{p=1}^{P} e^{-\ \sum_{j=1}^{J}\sum_{t=1}^{N_i} k_{ijt}x_{ijt,p}\ \ \cdot\beta_{i,p}}\right). \tag{8}$$

Our problem therefore reduces to finding a good expansion for the integral of $P_1(Y|\beta)P_2(Y|\beta)g(\beta|\Omega)$. We assume $g(\beta|\Omega)$ is given by a product of Gamma or Multivariate Gamma distributions (or their generalizations as described below), rich families of probability densities defined for non-negative inputs $\beta_{i,p}$ which for certain choices of parameters have good, closed-form expressions for integrals against exponentials. For other parameter distribution choices and other probability distributions, at the cost of introducing new special functions we still have closed-form series expansions for the integrals (as we discuss in Section 2.4). The reason the Gamma and Multivariate Gamma distributions lead to closed-form expansions is that both have a good closed-form expression for their moment generating function; in Theorem 3.1 we generalize our results to any multivariate distribution with a good closed-form moment generating function.

2 UNIVARIATE CASE: GENERALIZED GAMMA

In this section we combine all of the pieces in the case when the $\beta_{i,p}$ are independently drawn from Gamma distributions with parameters (b_p, n_p) (independent of i): the logit likelihood given in (2), the geometric series expansion in (4), and an integration lemma given in (11) below which allows us to obtain series expansions for $P(Y|\Omega)$. Then in Section 2.3.1 we discuss how to re-group the resulting series expansions for computational savings. In Section 3 we consider the more general case of choosing the $\beta_{i,p}$'s from a Multivariate Gamma distribution, where now for a given i there may be correlations among the $\beta_{i,p}$'s.

2.1 The Gamma Distribution and its Generalization

As $\beta_{i,p}$ is assumed greater than 0, one flexible distribution to draw the $\beta_{i,p}$'s from is the three parameter Generalized Gamma distribution (McDonald and Butler, 1990). The Generalized Gamma distribution is extremely rich, and by appropriate choices of its parameters, many standard functions are obtainable. It is defined for z non-negative by

$$GG(z; a, b, n) = \frac{|a|}{b\Gamma(n)} \left(\frac{z}{b}\right)^{an-1} e^{-\left(\frac{z}{b}\right)^a}. \tag{9}$$

For example, the following assignments of the parameters a, b and n yield well known distributions: $\lim_{a\to 0} GG(z; a, b, n)$ is lognormal; $GG(z; a, b, 1)$ is Weibull; $GG(z; 1, b, n)$ is Gamma; $GG(z; 1, b, 1)$ is Exponential; $GG(z; 2, b, 1)$ is Rayleigh. For fixed a and n, the effect of b is to re-scale the units of z. That is, as z only appears as $\frac{z}{b}$, b may be interpreted as fixing the scale (i.e. the commonly interpreted scale parameter); a and n change the general shape of the Generalized Gamma. Hence, as opposed to the more familiar Gamma family of distributions commonly used in Marketing problems, which we focus on here, the Generalized Gamma has a second shape parameter, a, allowing for more flexible shapes.

We provide in Appendix A.1 some plots of the Gamma family of distributions for various parameter values, and of a mixture of Gamma distributions, an even more flexible class to demonstrate its flexibility in providing a rich yet parsimoniously parameterized set of priors for the $\beta_{i,p}$. Although the results reported directly in this paper correspond to the heterogeneity distribution following a single Gamma distribution, they straightforwardly extend to a mixture of Gammas, where the mixture is a weighted sum of component Gammas. For each component of the mixture we can integrate its expansion term by term, and hence the entire weighted sum. From a practical point of view, this allows us to handle the situation of latent class modelling, in which the $\beta_{i,p}$ come from a latent segment, each of which has its own Gamma parameters.

As the geometric series expansion of the logit likelihood, as described in Sections 1.1 and 1.2, will lead to terms involving exponential functions, our key integration result arises from integrating an exponential function against a Gamma distribution. For simplicity we consider only the case of a Gamma distribution ($a = 1$), and discuss its generalization below and in Section 2.4.

This assumption allows us to not only obtain closed-form expansions, but these expansions will be rational functions of the arguments of the Gamma distribution (which allow us to obtain tractable closed-form expansions for the *derivatives* as well). For notational convenience let $G(z;b,n) = GG(z;1,b,n)$ denote a Gamma distribution with parameters b and n.

Lemma 2.1 (Exponential against a Gamma distribution)**.** *Consider a Gamma distribution $G(z;b,n)$. For $d \geq 0$,*

$$e^{-zd}G(z;b,n) \quad = \quad (1+bd)^{-n} G\left(z, \frac{b}{1+bd}, n\right). \tag{10}$$

As $G(z;\frac{b}{1+bd},n)$ is a probability distribution, we obtain

$$\int_{z=0}^{\infty} e^{-zd}G(z;b,n)dz \quad = \quad \frac{1}{(1+bd)^n}. \tag{11}$$

See Appendix A.2 for a proof. This closed-form integration result allows us to avoid resorting to Monte Carlo or other numerical techniques to approximate the integral for $P(Y|\Omega)$, and is our integration "engine". Later in Theorem 3.1 of Section 3 we generalize Lemma 2.1 by interpreting it as evaluating the moment generating function of the Gamma distribution at $-d$.

2.2 Series Expansion for $P(Y|\Omega)$ for the Gamma Distribution

We assume the response coefficients are drawn from the Gamma distribution. Summarizing our inference problem, we need to investigate the integral for $P(Y|\Omega)$:

$$\int_0^\infty \cdots \int_0^\infty P(Y|\beta)g(\beta|\Omega)d\beta \; = \; \prod_{i=1}^{I} \int_0^\infty \cdots \int_0^\infty P_{1i}(Y|\beta)P_{2i}(Y|\beta) \prod_{p=1}^{P} G(\beta_{i,p}; b_p, n_p)d\beta_{i,p}, \tag{12}$$

where

$$\begin{aligned} P_{1i}(Y|\beta) &= \prod_{p=1}^{P} e^{-\sum_{j=1}^{J}\sum_{t=1}^{N_i} y_{ijt}x_{ijt,p}\;\;\beta_{i,p}} \\ P_{2i}(Y|\beta) &= \sum_{k_{i11}=0}^{\infty} \cdots \sum_{k_{iJN_i}=0}^{\infty} (-1)^{\sum_{j=1}^{J}\sum_{t=1}^{N_i} k_{ijt}} \prod_{p=1}^{P} e^{-\sum_{j=1}^{J}\sum_{t=1}^{N_i} k_{ijt}x_{ijt,p}\;\;\cdot\beta_{i,p}} \\ g(\beta|\Omega) &= \prod_{p=1}^{P} G(\beta_{i,p}; b_p, n_p). \end{aligned} \tag{13}$$

Because of the conditional independence across i, we can evaluate each integral in (12) separately. We denote each of the i-integrals above by H_i (for the i^{th} household), where

$$\begin{aligned} \mathrm{H}_i &= \int_0^\infty \cdots \int_0^\infty \prod_{p=1}^{P} \sum_{k_{i11}=0}^{\infty} \cdots \sum_{k_{iJN_i}=0}^{\infty} (-1)^{\sum_{j=1}^{J}\sum_{t=1}^{N_i} k_{ijt}} \\ &\quad \cdot e^{-\sum_{j=1}^{J}\sum_{t=1}^{N_i} y_{ijt}x_{ijt,p}\;\;\beta_{i,p}} e^{-\sum_{j=1}^{J}\sum_{t=1}^{N_i} k_{ijt}x_{ijt,p}\;\;\beta_{i,p}} G(\beta_{i,p}; b_p, n_p)d\beta_{i,p} \\ &= \sum_{k_{i11}=0}^{\infty} \cdots \sum_{k_{iJN_i}=0}^{\infty} (-1)^{\vec{k}\cdot\vec{1}} \prod_{p=1}^{P} \int_{\beta_{i,p}=0}^{\infty} e^{-K_{i,p}\beta_{i,p}} G(\beta_{i,p}; b_p, n_p)d\beta_{i,p}, \end{aligned} \tag{14}$$

where

$$K_{i,p} \; = \; \sum_{j=1}^{J}\sum_{t=1}^{N_i}(y_{ijt}+k_{ijt})x_{ijt,p}, \quad \vec{k}\cdot\vec{1} \; = \; \sum_{j=1}^{J}\sum_{t=1}^{N_i} k_{ijt}. \tag{15}$$

Therefore

$$P(Y|\Omega) \;=\; \int_0^\infty \cdots \int_0^\infty P(Y|\beta) g(\beta|\Omega) d\beta \;=\; \prod_{i=1}^{I} \mathrm{H}_i. \tag{16}$$

Applying the integration lemma (Lemma 2.1) to (14) yields

$$\int_{\beta_{i,p}=0}^{\infty} e^{-K_{i,p}\beta_{i,p}} G(\beta_{i,p}; b_p, n_p) d\beta_{i,p} \quad = \quad \frac{1}{(1+b_p K_{i,p})^{n_p}}. \tag{17}$$

By combining the expansion in (14) with (12), we obtain our final result for H_i:

Theorem 2.2. *Assume the* $\beta_{i,p}$ *are independently drawn from Gamma distributions with parameters* (b_p, n_p)*. Then* $P(Y|\Omega) = \prod_{i=1}^{I} H_i$*, where*

$$H_i \;=\; \sum_{k_{i11}=0}^{\infty} \cdots \sum_{k_{iJN_i}=0}^{\infty} (-1)^{\sum_{j=1}^{J} \sum_{t=1}^{N_i} k_{ijt}} \prod_{p=1}^{P} \frac{1}{(1+b_p K_{i,p})^{n_p}}, \quad K_{i,p} \;=\; \sum_{j=1}^{J} \sum_{t=1}^{N_i} (y_{ijt} + k_{ijt}) x_{ijt,p}. \tag{18}$$

Hence the log marginal distribution, $\log L = \log P(Y|\Omega) = \sum_i \log(\mathrm{H}_i)$, can be computed as the sum of the logarithm of (18). This yields the desired closed-form solution.

2.3 Computational and Implementation Issues

2.3.1 Computational Issues and Gains from Diophantine Analysis

While Theorem 2.2 yields a closed-form expansion for the marginal posterior distribution when the response coefficients are independently drawn from Gamma distributions, to be useful we must be able to efficiently determine the optimal values of the parameters Ω. As written, the number of terms needed in the series expansions are computationally expensive/impossible (i.e. the upper bounds are at ∞). If every sum ranged from 0 to R, to have good expansions R would have to be prohibitively large. In this section we describe a more efficient way to group the summands to significantly decrease computational time and maximize the marginal posterior which will make this more computationally tractable for the Marketing scientist. We also note that due to the high degree of non-linearity and the infinite series expansion, there does not exist a closed-form solution for the optimal parameter values, $\hat{\Omega}$, by simply solving the first-order condition equation $\frac{\partial \log L}{\partial \Omega} = 0$. We therefore use numerical methods to obtain the maximum marginal a posteriori values.

One common approach to determining the optimal values is to use a multivariate Newton's method. Unfortunately, in many of the simulations investigated here, the flatness of the surface around the mode and the multi-modality of the marginal posterior led to poor convergence; however, we expect for larger (and different) data sets, Newton's method may become feasible and hence we include the closed-form first, second, and cross derivatives in Appendix D. We also note that one reason for our choice of the Gamma distribution was that the resulting expansions (see Theorem 2.2) are elementary functions of the parameters b_p and n_p, and hence have elementary closed-form expansions for their derivatives. This facilitates calculations of elasticities, shown to be crucial in determining optimal marketing strategies (Russell and Bolton, 1988).

We therefore instead resorted to evaluating (18) in a grid over the parameter space, and then choosing the value that maximized the marginal posterior. That is, the beauty and value of our expansions is that it allows us to calculate H_i rapidly even for many grid points. However, this is assuming that we can truncate each of the summations at a computationally feasible value, an approach we now describe.

As the expansion stands in (18), without careful thought, only moderate sizes for J and N_i are feasible. In any numerical calculation, the infinite sums must be truncated. For simplicity and for explicative purposes, assume each sum ranges from 0 to $R-1$. As there are JN_i summations, we have a total of R^{JN_i} terms to evaluate. Additionally, we have a product over $p \in \{1, \ldots, P\}$ attributes, and then a product or sum over $i \in \{1, \ldots, I\}$ households. If we ssume all $N_i = N$ for purposes of approximating the computational complexity, the number of computations required is therefore of order $P \prod_i R^{JN_i} = PI \cdot R^{JN}$.

As we want to determine the values for the parameters b_p, n_p that maximize the integral given in (18), two common approaches, Newton's Method or evaluating in a grid, can theoretically be done (especially as we have explicit formulas); however, the number of terms makes direct computation from this expansion (i.e. without computational savings as described below) impractical at present computing speeds. For each parameter, we need to calculate on the order of $PI \cdot R^{JN}$ terms for just *one* iteration of Newton's Method or evaluation of $\log L$ for a grid approach. We discuss a way to re-group the terms in the expansion which greatly reduces the computational time and allows us to handle larger triples (R, J, N_i).

We show below in detail that what allows us to succeed is that it is possible to re-group the computations in such a way that we have a lengthy initial computation, whose results we store in a data file. From this, it is possible to evaluate the log-likelihood (or derivatives if using Newton's method) at all points of interest extremely rapidly. The reason such a savings as described below is possible, in some sense, is that the computations factor into two components, and most of the computations are the same for all values of the parameters and hence only need to be done once.

Consider the case of P attributes: $p \in \{1, 2, \ldots, P\}$. We then have x-vectors

$$x_{i,p} \;=\; (x_{i11,p}, \ldots, x_{iJN_i,p}), \quad p \in \{1, \ldots, P\}. \tag{19}$$

Assume all $x_{ijt,p}$ are integers; this is not a terribly restrictive assumption[1], and can be simply accomplished by changing the scale we use to measure the $x_{ijt,p}$'s. The advantage of having integer X's is that we now have Diophantine equations, and powerful techniques are available to count the number of solutions to such equations and hence "judge" the feasible values of (R, J, N_i).

For notational convenience let $\vec{k} = (k_{i11}, \ldots, k_{iJN_i})$, $\vec{1} = (1, \ldots, 1)$ and

$$\begin{aligned} Y_{i,p} &= \sum_j \sum_t y_{ijt} x_{ijt,p} \\ K_{i,p} &= \sum_{j=1}^{J} \sum_{t=1}^{N_i} (y_{ijt} + k_{ijt}) x_{ijt,p} \;=\; Y_{i,p} + \sum_{j=1}^{J} \sum_{t=1}^{N_i} k_{ijt} x_{ijt,p}. \end{aligned} \tag{20}$$

Recall from (18) that when the $\beta_{i,p}$ are independently drawn from Gamma distributions that

$$\mathrm{H}_i \;=\; \sum_{k_{i11}=0}^{\infty} \cdots \sum_{k_{iJN_i}=0}^{\infty} (-1)^{\sum_{j=1}^{J} \sum_{t=1}^{N_i} k_{ijt}} \prod_{p=1}^{P} \frac{1}{(1 + b_p K_{i,p})^{n_p}}. \tag{21}$$

Fix an $i \in \{1, \ldots, I\}$. We see that (21) depends weakly on $\vec{k}$; by (20), all that matters are the dot products $\vec{k} \cdot \vec{x}_{i,p}$ and the parity of $(-1)^{\vec{k} \cdot \vec{1}}$. Let $r = (r_1, \ldots, r_P)$ be a P-tuple of non-negative integers. For each $\vec{k}$ we count the number of solutions to the system of Diophantine equations $\vec{k} \cdot \vec{x}_{i,p} = r_p$ ($p \in \{1, \ldots, P\}$) while recording the sign of $(-1)^{\sum_j \sum_t k_{ijt}} = (-1)^{\vec{k} \cdot \vec{1}}$. Explicitly, we may re-write (18) from Theorem 2.2 as

Theorem 2.3. *Set* $Y_{i,p} = \sum_j \sum_t y_{ijt} x_{ijt,p}$ *and*

$$\begin{aligned} S(M) &= \{v : v = (v_1, \ldots, v_M), v_l \in \{0, 1, 2, 3, \ldots\}\} \\ K_i(x, r, +) &= \#\{k \in S(JN_i) : \forall p \in \{1, \ldots, P\}, \vec{k} \cdot \vec{x}_{i,p} = r_p, (-1)^{\vec{k} \cdot \vec{1}} = +1\} \\ K_i(x, r, -) &= \#\{k \in S(JN_i) : \forall p \in \{1, \ldots, P\}, \vec{k} \cdot \vec{x}_{i,p} = r_p, (-1)^{\vec{k} \cdot \vec{1}} = -1\}. \end{aligned} \tag{22}$$

Assume the $\beta_{i,p}$ *are independently drawn from Gamma distributions with parameters* (b_p, n_p). *Then* $P(Y|\Omega) = \prod_{i=1}^{I} H_i$ *with*

$$H_i \;=\; \sum_{r \in S(P)} \frac{K_i(x, r, +) - K_i(x, r, -)}{\prod_p (1 + b_p Y_{i,p} + b_p r_p)^{n_p}}. \tag{23}$$

[1] This is not restrictive even for an attribute like price. Many studies are done with a discrete set of integer prices and in other cases, even if there were a fairly moderate number, the model can handle it albeit with increased computation.

There is a large computational startup cost in solving (22), but future computations are significantly faster. We calculate $K_i(x, r, \pm)$ *once*, and store the results in a data file. Then, in subsequent calculations, we need only input the new values for b_p and n_p (or even better calculate the values for multiple b_p and n_p simultaneously if we are evaluating over a grid). The advantage of such an expansion is that successive terms involving larger r decay with $\vec{k} \cdot \vec{x}_{i,p}$. Thus, we do not want to truncate the sum $\sum_{k_{ij1}} \cdots \sum_{k_{ijP}}$ by having each sum range from 0 to $R-1$; instead, we want to consider the k-tuples where the dot products are small, as the k-dependence is weak (the expansion depends only on the value of $\vec{k} \cdot \vec{x}_{i,p}$). From a computational point of view, there is enormous savings in such grouping.

To determine how many terms are needed for this truncation to be a good approximation to the infinite expansion requires an analysis of $K_i(x, r, +) - K_i(x, r, -)$. We sketch some straightforward, general bounds in Appendix C. We do not exploit the gain from the factors of $(1 + b_p Y_{i,p} + b_p r_p)^{-n_p}$ so that our bounds will apply to the more general cases that we consider later (explicitly, the multivariate distributions with good closed-form moment generating functions of Section 3). One other point to note and which will greatly improve the convergence of the truncated expansions is to introduce translations in the Gamma distributions, which will give exponentially convergent factors. Assume each $\beta_{i,p} \geq \epsilon$; for ϵ small (e.g. 0.0001), from a practical point of view such an assumption is harmless as a coefficient restricted to this range is not practically different than one restricted to be greater than or equal to 0. Explicitly, we draw $\beta_{i,p}$ from $G(z - \epsilon; b_p, n_p)$ rather than $G(z; b_p, n_p)$. Similar arguments as before yield

$$\mathrm{H}_i \;=\; \sum_{r \in S(P)} \prod_{p=1}^{P} e^{-Y_{i,p}\epsilon} \frac{(K_i(x, r, +) - K_i(x, r, -)) e^{-r_p \epsilon}}{(1 + b_p Y_{i,p} + b_p r_p)^{n_p}}. \tag{24}$$

As $K_i(x, r, +) - K_i(x, r, -)$ grows at most polynomially (see Theorem C.4 in Appendix B), it is clear the above expansion converges (and for reasonable values, it will converge more rapidly than when $\epsilon = 0$).

2.3.2 Numerical Simulations

To demonstrate the efficacy of our approach given in (21), we ran a series of numerical simulations. The results reported here are from two sets of the many simulations conducted, the remainder of which are available upon request. The first simulation design was chosen to be computational feasible; however, without loss of generality it contains all the elements that are required to generalize our results. In some sense, due to its maximal sparseness in information, it is the most strict test of our approach.

Specifically, we report here first on a series of simulations with the following design:

- $P = 1$, one attribute per observation,
- $JN_i = 1$, one brand and one observation per household,
- $I = 1000$, one thousand households,
- $0 \leq k_i \leq R$ with R (the number of polynomial expansion terms) equal to 100,
- $a = 1$ (the Gamma distribution) for various choices of b and n, and
- untranslated Gamma distribution (i.e. $\epsilon = 0$ in (24)).

All simulations were run using Matlab on a 1.9 GHZ athlon processor with 192 Mb of RAM, a very modest computing machine in today's standards.

For each b_1 and n_1 pair, 25 simulates were run by: (i) choosing $I = 1000$ values of $\beta_{ip=1}$ from a $G(z; b_1, n_1)$. The values of x_{ijt} were selected from the values $(1, 2, 3)$ with equal probability, and then arbitrarily scaled by a constant c to make the values of $\beta_{i,p} \cdot x_{ijt}$ reasonable so as to allow for enough 0/1 variation in the y_{ijt}. Then for each of the 25 simulates, we numerically approximated $P(Y|\Omega)$ as given by (21) and then maximized the resulting marginal likelihood, as a function of Ω, using a grid of values. In particular, we utilized a grid size of dimension 5×7 centered at (b_1, n_1) with spacings of .1 (this was reached after considerable empirical testing to ensure enough fineness and that the solutions were not occurring on the boundary of the grid).

We summarize our results in the table below: the true values of b_1 and n_1, the mean and standard deviation over 25 replicates of the estimated values, and the t-statistics for both b and n.

$(b_1,\ n_1)$	$(\overline{b_1},\ \overline{n_1})$	$(\sigma_{b_1},\ \sigma_{n_1})$	t-stat (b_1)	t-stat (n_1)
(5, 14)	(5.21, 14.86)	(1.40, 3.36)	0.76	1.29
(10, 28)	(10.72, 26.38)	(1.46, 3.17)	2.46	-2.55
(9, 9)	(9.06, 9.64)	(2.41, 2.37)	0.12	1.36
(18, 18)	(17.39, 18.62)	(2.28, 2.40)	-1.34	1.30
(11.5, 6.5)	(10.65, 7.38)	(2.46, 2.03)	-1.73	2.16
(23, 13)	(23.93, 12.63)	(2.35, 1.43)	1.97	-1.30

To assess whether the simulate values are in accordance with the true values, we conducted t-tests for each of the parameters and simulated conditions. This resulted in 12 significance tests, all of which correspond to a t-distribution with 24 degrees of freedom (note we did 25 simulates). Using the common, albeit conservative, Bonferroni adjustment method for multiple comparisons, we note the critical value of 3.167 in absolute value of which none of the comparisons is close (the corresponding value for one comparison is 2.064, which 9 of the 12 are less than). This suggests a very adequate fit of our approach and therefore the size of R in our polynomial expansions. Other simulations, not shown, suggested higher values of R provided even greater accuracy.

The six set of simulations were chosen to be indicative of three possible settings, $b_1 > n_1, b_1 = n_1$ and $b_1 < n_1$. We then replicated these three settings by scaling each of the values of b and n by a factor of 2. In this way we are able to show that it is not a particular ordering of b_1 and n_1 that matters nor the relative sizes of them. Note again, as above, that this simulation test of our approach is ultra-conservative in that we have tested our method using $J \cdot N_i = 1$ and $I = 1000$, modest values. That is, with simply one observation per household and 1000 households, our approach is accurately able to reconstruct the heterogeneity distribution from which the $\beta_{i,p}$ were derived. This result was also not dependent on I=1000, as shown below, and hence would have led to even faster processing time. Our belief is that this is a strong signal of the efficacy of our approach.

A second series of simulations with more general conditions was conducted in which the number of attributes was increased to $P = 2$. The purpose was to see how well *multiple* Gamma distributions could be detected. There are now four parameters (b_1, n_1, b_2, n_2). To have these simulations run in a comparable time as the previous, we chose $I = 250$, $R = 40$, a grid of size $4 \times 4 \times 4 \times 4$ centered at (b_1, n_1, b_2, n_2) with a grid spacing of .5 units, and 10 simulates for each condition. We summarize the results below.

(b_1, n_1, b_2, n_2)	$(\overline{b_1}, \overline{n_1}, \overline{b_2}, \overline{n_2})$	$(\sigma_{b_1}, \sigma_{n_1}, \sigma_{b_2}, \sigma_{n_2})$	t-stat (b_1)	t-stat (n_1)	t-stat (b_2)	t-stat (n_2)
(9, 9, 18, 18)	(8.60, 8.75, 17.9, 18.1)	(2.12, 2.20, 1.96, 2.25)	-.60	-.36	-.16	.14
(11.5, 6.5, 23, 13)	(11.3, 6.80, 22.3, 12.8)	(1.77, 1.95, 2.08, 1.96)	-.36	.49	-1.14	-.40
(5, 14, 23, 13)	(4.85, 13.9, 24.15 14.05)	(1.56, 1.76, 1.55, 1.28)	-.30	-.18	2.35	2.60

This resulted in 12 significance tests, all which correspond to a t-distribution with 9 degrees of freedom (note we did 10 simulates). Using the common, albeit conservative, Bonferroni adjustment method for multiple comparisons, we note the critical value of 3.81 in absolute value of which none of the comparisons is close (the corresponding value for one comparison is 2.26, which ten of the twelve values are less than). This suggests a very adequate fit of our approach and therefore the size of R in our polynomial expansions.

Our findings suggest again the general efficacy of our approach as none of the significance tests indicate divergence between the true and estimated parameter values.

2.3.3 Comparison with Monte Carlo Markov Chain Methods

As mentioned previously, and described in detail in Appendix C, one aspect of our theoretical results that requires study is its computational feasibility due to the large number of summands. As the *exact* results in Theorem 2.2 or Theorem 2.3 have upper sum limits at infinity, we conducted an additional small scale simulation to assess the efficacy of our method under the truncation approximation. To act as a further baseline to our approach, we also ran a Bayesian MCMC sampler (a Bayesian multinomial logit model with non-conjugate gamma priors as per Section 2) to assess both the computation accuracy for our approach and its accuracy per unit time compared to established extant methods. All analyses were run on a Pentium IV 3.3MHZ processor with 2GB of RAM. For a more accurate comparison of times we wrote a C program rather than a Matlab program (as in 2.3.2) for evaluating the truncated sums.

In particular, we simulated data for $I = 1000$ households, $N_i = 1$ or 5 observations per household, generated by a multinomial logit model (see (2)) with $P = 1$ covariates. Each household's value of β_i was drawn from a Gamma distribution[2] with $b = 5$ and $n = 14$. To analyze our approach, we evaluated the approximated marginal likelihood (marginalized over β_i) over a grid (of size 21×21) using the Diophantine computation savings by only looking at sums with $k_1 + \cdots + k_{N_i} \leq R$ for various choices of R (as compared to N_i sums where each went from 0 to R, which leads to the inclusion of many summands of negligible size).

When $N_i = 5$ the Bayesian MCMC sampler for 6000 iterations for 3 chains (0.01667 seconds per iteration) took about 50 seconds, where the convergence diagnostic of Gelman and Rubin (1992) indicated convergence after approximately 3000 iterations (hence 9000 observations available for estimation after burn-in). For $N_i = 1$ the Bayesian MCMC sampler for 6000 iterations for 3 chains (0.01667 seconds per iteration) took about 20 seconds.

For the C program based on our truncated series expansions, the approximations depend on the parity of R (if R is even then the final summands all have a factor of $+1$, while if R is odd the final summands all have a factor of -1). Thus if the resulting values at the grid points are stable for two consecutive values of R, we have almost surely included enough terms in our truncation. For $N_i = 1$ there was about a 2% difference in values when $R = 100$ and 101 (about 12 seconds); there was about a .2% difference in values when $R = 200$ and 201 (about 24 seconds). These run-times compare favorably with those of the Bayesian MCMC sampler. For more observations per household, however, the Bayesian MCMC sampler does better. The problem, as shown in Appendix C, is that the number of summands with $k_1 + \cdots + k_{N_i} \leq R$ is a polynomial in R of degree N_i. When $N_i = 5$ and $R = 6$ the program ran for about 40 seconds, and when $N_i = 5$ and $R = 7$ the run-time was about 64 seconds; while these values of R are too small to see convergence in the truncated series, for these data sets the series expansion is still implementable, though at a cost of a significantly greater run-time.

Thus our series expansions, with the present computing power, are comparable to existing numerical methods only in the case of one observation per household, though they can still be implemented in a reasonable amount of time for multiple observations.

2.4 Generalizations of the Univariate Gamma Distribution

We describe several natural generalizations of our model. We assume for each i that $\beta_{i,1}, \ldots, \beta_{i,P}$ are independent below; see Section 3 for removing this assumption as well.

At the expense of using special functions, we may easily remove the assumption that the $\beta_{i,p}$ are drawn from a Gamma distribution; however, as the research currently stands, we still must assume the $\beta_{i,p}$ are drawn from one-sided distributions. In the case of just one attribute, it is straightforward to generalize our methods to handle $\beta_{i,p}$ drawn from *any* distribution (we split the integration into three parts, $\beta_{i,p} < \epsilon, |\beta_{i,p}| \leq \epsilon, \beta_{i,p} > \epsilon$); a natural topic for future research is to handle $\beta_{i,p}$ drawn from two-sided distributions with multiple attributes.

2.4.1 Weakening $\beta_{i,p} > \epsilon$

The assumption that $\beta_{i,p} > \epsilon$ is problematic if we desire to test the hypothesis that $\beta_{i,p} = 0$. To this end, for each $\beta_{i,p}$ we consider instead of an ϵ-translated Gamma distribution a 0-point mass Gamma Distribution

[2] Computation time was essentially invariant over the exact values of b and n chosen. The run-time is a polynomial in R of degree N_i; further empirical testing is needed to ascertain how well our approach works in these settings.

given by

$$w_p\delta(\beta_{i,p}) + (1-w_p)G(\beta_{i,p}-\epsilon; b_p, n_p). \tag{25}$$

In the above, $\delta(x)$ is the Dirac Delta Functional with unit mass concentrated at the origin; $w_p \in [0,1]$ is a weight and can be interpreted as a "weight of evidence" for $\beta_{i,p} = 0$. It is easier, though by no means necessary, to obtain closed-form integrals if we assume instead that we have

$$w_p \prod_{i=1}^{I} \delta(\beta_{i,p})d\beta_{i,p} \;+\; (1-w_p)\prod_{i=1}^{I} G(\beta_{i,p}-\epsilon; b_p, n_p)d\beta_{i,p}. \tag{26}$$

That is, for a given attribute, either $\beta_{i,p} = 0$ for *all* households, or they are *all* drawn from a translated Gamma distribution.

Note, we now have *either* translated Gamma distributions *or* delta masses. If everything were a delta mass, we would be left with $(-1)^{\vec{k}\cdot\vec{1}}$. In this case, we would not use the geometric series expansion, as the integration is trivial.

The expansions are more involved if we have some delta masses and some non-delta masses (varying across attributes). We would have to go through the same arguments as above to estimate convergence, but instead of having P terms in the exponentials, we would have $P-1$, $P-2$, and so on.

A stronger assumption, leading to the easiest integration, is the following:

$$w \prod_{i=1}^{I}\prod_{p=1}^{P} \delta(\beta_{i,p})d\beta_{i,p} \;+\; (1-w)\prod_{i=1}^{I}\prod_{p=1}^{P} G(\beta_{i,p}-\epsilon; b_p, n_p)d\beta_{i,p}. \tag{27}$$

That is, *either* everything is from a delta mass, *or* everything is from some translated Gamma distribution, with a translation of ϵ. In this instance, our approach can be directly applied.

2.4.2 Linear Combinations of Gamma Distributions

We can increase the flexibility of the model by considering linear combinations of Gamma distributions:

$$w_{p,1}G(\beta_{i,p}-\epsilon; b_{p,1}, n_{p,1}) + \cdots + w_{p,C}G(\beta_{i,p}-\epsilon; b_{p,C}, n_{p,C}), \tag{28}$$

where

$$\forall p : w_{p,1} + \cdots + w_{p,C} \;=\; 1, \quad w_{p,c} \in [0,1]. \tag{29}$$

We can regard the weights as either new, additional parameters, or fixed, and (17) becomes

$$\int_{\beta_{i,p}=0}^{\infty} e^{-K_{i,p}\beta_{i,p}} \sum_{c=1}^{C} w_{p,c}G(\beta_{i,p}-\epsilon; b_{p,c}, n_{p,c})d\beta_{i,p} \;=\; \sum_{c=1}^{C} \frac{w_{p,c}\; e^{-K_{i,p}\epsilon}}{(1+b_{p,c}K_{i,p})^{n_{p,c}}}. \tag{30}$$

The essential point is that, in the above integration, $K_{i,p}$ *does not depend on* c. Thus, we will still have the computational savings, and need only count the solutions to the Diophantine system once. The difference is we now have more terms to evaluate, but we still have rapid savings, and (24) becomes

Theorem 2.4. *Notation as in Theorem 2.3, let the $\beta_{i,p}$ be independently drawn from linear combinations of Gamma distributions as in* (28)*. Then $P(Y|\Omega) = \prod_{i=1}^{I} H_i$ with*

$$H_i \;=\; \sum_{c=1}^{C}\sum_{r\in S(P)}\prod_{p=1}^{P} e^{-Y_{i,p}\epsilon}\frac{w_{p,c}(K_i(x,r,+)-K_i(x,r,-))e^{-r_p\epsilon}}{(1+b_{p,c}Y_{i,p}+b_{p,c}r_p)^{n_{p,c}}}. \tag{31}$$

2.4.3 More General One-Sided Distributions

There is no a priori reason or necessity to choose $\beta_{i,p}$ from a Gamma distribution $G(\beta_{i,p}; b_p, n_p)$ (or linear combinations of these). Because we were assuming the $\beta_{i,p} \geq 0$ (later, when we wanted $\beta_{i,p} \geq \epsilon$, this merely caused us to study translated Gamma distributions), it is natural to choose a one-sided, flexible distribution such as the Gamma distribution. If we take any one-sided distribution and translate, we obtain a similar formula as in (23) or (24). The only difference would be the functional form of the non-Diophantine piece. The exponential decay (arising from the requirement that $\beta_{i,p} \geq \epsilon$) is still present; it came solely from the geometric series expansions.

As we have not been using properties of the integration of an exponential against a Gamma distribution to obtain our convergence bounds, our arguments are still applicable; however, in general we don't have simple closed-form expansions with elementary functions. At the cost of introducing new special functions, we could handle significantly more general one-sided distributions. Our integration lemma (Lemma 2.1) is trivially modified, and we still have computational savings. As we shall see in Theorem 3.1, our method is directly applicable to any distribution (univariate or multivariate) with a closed-form moment generating function.

There are two costs. The first is the introduction of new special functions in the expansions of the H_i; however, by tabulating these functions once, subsequent evaluations can be done efficiently. The second difficulty is that, if one attempts to use Newton's Method, closed-form elementary expansions of the derivatives are no longer available in many cases; for cases where the expansions exist, one must calculate the partial derivatives in a manner similar to that in Appendix D (for the Gamma distribution).

3 INCORPORATING COVARIANCES: THE MULTIVARIATE GAMMA MODEL

Our previous investigations have assumed that the households' $\beta_{i,1}, \dots, \beta_{i,P}$ are independent and that $\beta_{i,1}, \dots, \beta_{i,P}$ are independently drawn from Gamma distributions (with different parameters for each $\beta_{i,p}$). In §2.4 we have seen how to generalize to the case when the $\beta_{i,p}$ are still independent but drawn from other distributions. We now discuss another generalization, namely removing the independence assumption of the $\beta_{i,p}$ and thus allowing non-zero covariances.

Let us assume that the households are still independent; however, $\beta_{i,1}, \dots, \beta_{i,P}$ are no longer assumed to be independent. Let us assume that for each household these are drawn from some distribution

$$G(\beta_{i,1}, \dots, \beta_{i,P}; \vec{b}) \;=\; G(\beta_i; \vec{b}), \tag{32}$$

where $\vec{b}$ is some set of parameters. Our previous work in Section 2 is the case

$$\begin{aligned} \vec{b} &= (b_{i,1}, \dots, b_{i,P}, n_{i,1}, \dots, n_{i,P}) \\ G(\beta_i; \vec{b}) &= \prod_{i=1}^{P} \frac{1}{\Gamma(n_{i,p})} \left(\frac{\beta_{i,p}}{b_{i,p}}\right)^{n_{i,p}} e^{-\beta_{i,p}/b_{i,p}}. \end{aligned} \tag{33}$$

By using a multivariate distribution we can capture correlations between the coefficients of different brands (the univariate distribution of (33) has all covariances zero), or in general the coefficients of the covariates.

As we no longer assume that G factors into distributions for each $\beta_{i,p}$, (14) is no longer valid and we now must analyze, for each household i,

$$\begin{aligned}
\mathrm{H}_i &= \int_0^\infty \cdots \int_0^\infty \sum_{k_{i11}=0}^\infty \cdots \sum_{k_{iJN_i}=0}^\infty (-1)^{\sum_{j=1}^J \sum_{t=1}^{N_i} k_{ijt}} \\
&\quad \cdot \prod_{p=1}^P e^{-\ \sum_{j=1}^J \sum_{t=1}^{N_i} (y_{ijt}+k_{ijt}) x_{ijt,p}\ \ \beta_{i,p}} G(\beta_i; \vec{b}) d\beta_{i,p} \\
&= \sum_{k_{i11}=0}^\infty \cdots \sum_{k_{iJN_i}=0}^\infty (-1)^{\vec{k}\cdot\vec{1}} \int_0^\infty \cdots \int_0^\infty e^{-K_{i,1}\beta_{i,1}} \cdots e^{-K_{i,P}\beta_{i,P}} \cdot G(\beta_i; \vec{b}) d\beta_{i,1} \cdots d\beta_{i,P}, \qquad (34)
\end{aligned}$$

where as before

$$K_{i,p} \ = \ \sum_{j=1}^J \sum_{t=1}^{N_i} (y_{ijt} + k_{ijt}) x_{ijt,p}, \quad \vec{k}\cdot\vec{1} \ = \ \sum_{j=1}^J \sum_{t=1}^{N_i} k_{ijt}. \qquad (35)$$

Of course, for general G it will be difficult to evaluate (34) in a tractable form for numerical computation. One of the advantages of our previous method is that the integral of an exponential and a gamma distribution was another gamma distribution, and thus the integrals which arose were simple expressions of the parameters.

There are two natural ways to proceed. For a general multivariate distribution G we will be unable to develop a closed-form expression for the integral in (34) that is analogous to the one we found for the case of the $\beta_{i,p}$'s independently drawn from Gamma distributions (Lemma 2.1). Instead we could series expand the remaining exponentials, recognizing the resulting integrals as the moments of the multivariate distribution.

Alternatively, if G has a known closed-form expression for its moment generating function, then we may recognize (34) as simply evaluating this moment generating function at $(t_1, \dots, t_P) = (-K_{i,1}, \dots, -K_{i,P})$. Unfortunately sometimes the moment generating functions only exist for suitably restricted $(t_1, \dots, t_P)$, in which case we combine this approach with the series expansion for the remaining P-tuples. We present these details below.

3.1 Series Expansion for $P(Y|\Omega)$

3.1.1 General Multivariate G

For each of the P exponential terms $e^{-K_{i,p}\beta_{i,p}}$ we may expand in a geometric series,

$$e^{-K_{i,p}\beta_{i,p}} \ = \ \sum_{\ell_p=0}^\infty \frac{(-K_{i,p}\beta_{i,p})^{\ell_p}}{\ell_p!}. \qquad (36)$$

Thus (34) becomes

$$\begin{aligned}
\mathrm{H}_i &= \sum_{k_{i11}=0}^\infty \cdots \sum_{k_{iJN_i}=0}^\infty (-1)^{\vec{k}\cdot\vec{1}} \int_0^\infty \cdots \int_0^\infty e^{-K_{i,1}\beta_{i,1}} \cdots e^{-K_{i,P}\beta_{i,P}} \cdot G(\beta_i; \vec{b}) d\beta_{i,1} \cdots d\beta_{i,P} \\
&= \sum_{k_{i11}=0}^\infty \cdots \sum_{k_{iJN_i}=0}^\infty (-1)^{\vec{k}\cdot\vec{1}} \\
&\quad \int_0^\infty \cdots \int_0^\infty \sum_{\ell_1,\dots,\ell_P=0}^\infty \frac{(-K_{i,1}\beta_{i,1})^{\ell_1}}{\ell_1!} \cdots \frac{(-K_{i,P}\beta_{i,P})^{\ell_P}}{\ell_P!} \ G(\beta_i; \vec{b}) d\beta_{i,1} \cdots d\beta_{i,P} \\
&= \sum_{k_{i11}=0}^\infty \cdots \sum_{k_{iJN_i}=0}^\infty (-1)^{\vec{k}\cdot\vec{1}} \sum_{\ell_1,\dots,\ell_P=0}^\infty \frac{(-K_{i,1})^{\ell_1} \cdots (-K_{i,P})^{\ell_P}}{\ell_1! \cdots \ell_P!} \mu_{\ell_1,\dots,\ell_P}, \qquad (37)
\end{aligned}$$

where

$$\mu_{\ell_1,\ldots,\ell_P} \;=\; \int_0^\infty \cdots \int_0^\infty \beta_{i,1}^{\ell_1} \cdots \beta_{i,P}^{\ell_P} \cdot G(\beta_{i,1},\ldots,\beta_{i,P};\vec{b}) d\beta_{i,1}\cdots d\beta_{i,P}. \tag{38}$$

We thus obtain a closed-form expression again, except now we have additional summations over $\ell_1,\ldots,\ell_P$. Here $\mu_{\ell_1,\ldots,\ell_P}$ is the $(\ell_1,\ldots,\ell_P)$ non-centered moment of the distribution G. For a general distribution these may be difficult to evaluate explicitly; we need a one-sided distribution (with some parameters $\vec{b}$) that is flexible in terms of shape as well as having good formulas for the moments $\mu_{\ell_1,\ldots,\ell_P}$.

Our combinatorial results from Section 2.3.1 (where we were able to re-arrange calculations to save computational time) depended crucially on the fact that the exponential versus gamma integrals from before led to simple expansions such as $(1 + b_p K_{i,p})^{-n_p}$; these expansions did not depend on the actual values of $k_{i11},\ldots,k_{iJN_i}$ but only some linear combinations (dot products). Thus we still have combinatorial savings in the k_{ijt} sums.

3.1.2 Multivariate G with Closed Form Moment Generating Functions

Let $\beta_i = (\beta_{i,1},\ldots,\beta_{i,P})$ be distributed according to a multivariate density $G(\beta_i,\vec{b})$. The moment generating function of G is given by

$$M_{\beta_i}(t_1,\ldots,t_P) \;=\; \mathbb{E}\left[e^{t_1\beta_{i,1}+\cdots+t_P\beta_{i,P}}\right], \tag{39}$$

where the expectation is with respect to G; i.e.,

$$M_{\beta_i}(t_1,\ldots,t_P) \;=\; \int_{\beta_{i,1}} \cdots \int_{\beta_{i,P}} e^{t_1\beta_{i,1}+\cdots+t_P\beta_{i,P}}\; G(\beta_{i,1},\ldots,\beta_{i,P};\vec{b}) d\beta_{i,1}\cdots d\beta_{i,P}. \tag{40}$$

Depending on the distribution, the moment generating function may exist for all P-tuples $(t_1,\ldots,t_P)$, or instead only for suitably restricted P-tuples. We immediately obtain

Theorem 3.1. *Assume the moment generating function $M_{\beta_i}(t_1,\ldots,t_P)$ for the multivariate distribution $G(\beta_i,\vec{b}) = G(\beta_{i,1},\ldots,\beta_{i,P},\vec{b})$ exists for all $(t_1,\ldots,t_P)$. Then $P(Y|\Omega) = \prod_{i=1}^I H_i$, where*

$$H_i \;=\; \sum_{k_{i11}=0}^{\infty} \cdots \sum_{k_{iJN_i}=0}^{\infty} (-1)^{\vec{k}\cdot\vec{1}} M_{\beta_i}(-K_{i,1},\ldots,-K_{i,P}), \tag{41}$$

with

$$K_{i,p} \;=\; \sum_{j=1}^{J}\sum_{t=1}^{N_i}(y_{ijt}+k_{ijt})x_{ijt,p}, \quad \vec{k}\cdot\vec{1} \;=\; \sum_{j=1}^{J}\sum_{t=1}^{N_i} k_{ijt}. \tag{42}$$

3.2 Multivariate Gamma Distributions

We list several versions of Multivariate Gamma distributions (with non-zero covariances) that have closed-form expressions for their moment generating functions, and thus satisfy the conditions of Theorem 3.1. For additional multivariate distributions see Appendix B. All page and equation references in Sections 3.2.1 and 3.2.2 and are from Kotz, Balakrishnan and Johnson 2000.

3.2.1 (Cheriyan and Ramabhadran's) Bivariate Gamma (pages 432–435)

Recall the Gamma distribution with parameter $\theta > 0$ is given by

$$p_Y(y) \;=\; \begin{cases} \Gamma(\theta)^{-1} y^{\theta-1} e^{-y} & \text{if } y > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{43}$$

It has mean θ, variance θ, and its moment generating function is

$$M_Y(t) \;=\; (1-t)^{-\theta}, \tag{44}$$

which exists for all $t < 1$. Let Y_i for $i \in \{0,1,2\}$ be independent Gamma distributed random variables with parameters θ_i, and for $i \in \{1,2\}$ set $X_i = Y_0 + Y_i$. The density function of (X_1, X_2) is

$$p_{X_1,X_2}(x_1,x_2) \;=\; \frac{e^{-(x_1+x_2)}}{\Gamma(\theta_0)\Gamma(\theta_1)\Gamma(\theta_2)} \int_0^{\min(x_1,x_2)} y_0^{\theta_0-1}(x_1-y_0)^{\theta_1-1}(x_2-y_0)^{\theta_2-1}e^{y_0}dy_0, \tag{45}$$

the bivariate gamma density, equation (48.5). The correlation coefficient of X_1 and X_2 is

$$\mathrm{Corr}(X_1,X_2) \;=\; \frac{\theta_0}{\sqrt{(\theta_0+\theta_1)(\theta_0+\theta_2)}}. \tag{46}$$

As $\theta_0 > 0$ (since Y_0 is Gamma distributed) the correlation coefficient is positive, see (48.7). The moment generating function is

$$M_{X_1,X_2}(t_1,t_2) \;=\; (1-t_1-t_2)^{-\theta_0}(1-t_1)^{-\theta_1}(1-t_2)^{-\theta_2} \tag{47}$$

and exists for all (t_1,t_2) with $t_1+t_2 < 1$ and $t_i < 1$, see (48.10).

3.2.2 Multivariate Gamma Distributions

We may generalize the arguments from Section 3.2.1 and consider the joint distribution of $X_p = \lambda_p(Y_0+Y_p)$ for $i \in \{1,\dots,P\}$ and $\lambda_p > 0$ with $Y_0,\dots,Y_P$ independent Gamma distributed random variables with parameters $\theta_0,\dots,\theta_P$. If $P = 2$ Ghirtis has called this the double-gamma distribution. For general P it is similar to Mathai and Moschopoulos' Multivariate Gamma distribution (pages 465–470), and taking $\theta_p = 1$ we obtain Freund's Multivariate Exponential distribution (pages 388–391). The moment generating function is

$$\begin{aligned} M_{X_1,\dots,X_P}(t_1,\dots,t_P) &= \mathbb{E}\left[e^{t_1X_1+\cdots+t_PX_P}\right] \\ &= \mathbb{E}\left[e^{\lambda_1t_1(Y_0+Y_1)+\cdots+\lambda_Pt_P(Y_0+Y_P)}\right] \\ &= \mathbb{E}\left[e^{(\lambda_1t_1+\cdots+\lambda_Pt_P)Y_0}\right]\cdot\mathbb{E}\left[e^{\lambda_1t_1Y_1}\right]\cdots\mathbb{E}\left[e^{\lambda_Pt_PY_P}\right] \\ &= (1-\lambda_1t_1-\cdots-\lambda_Pt_P)^{-\theta_0}(1-\lambda_1t_1)^{-\theta_1}\cdots(1-\lambda_Pt_P)^{-\theta_P}, \end{aligned} \tag{48}$$

which exists for all $(t_1,\dots,t_P)$ such that $\lambda_1t_1+\cdots+\lambda_Pt_P < 1$ and each $t_p < \lambda_p^{-1}$. For our applications such restrictions are harmless, as in Theorem 3.1 we evaluate the moment generating function at $(-K_{i,1},\dots,-K_{i,P})$ and each $K_{i,p} \geq 0$.

In fact, we may generalize even further.

Lemma 3.2 ((Generalized) Multivariate Gamma Distribution)**.** *Let $Y_{0,1},\dots,Y_{0,M},Y_1,\dots,Y_P$ be independent Gamma distributions with parameters $\theta_{0,1},\dots,\theta_{0,M},\theta_1,\dots,\theta_P$. For $\lambda_{p,m},\lambda_p \geq 0$ let*

$$X_p \;=\; (\lambda_{p,1}Y_{0,1}+\cdots+\lambda_{p,M}Y_{0,M}) \;+\; \lambda_pY_p, \quad p \in \{1,\dots,P\}. \tag{49}$$

Then the moment generating function is

$$M_{X_1,\dots,X_P}(t_1,\dots,t_P) \;=\; \prod_{m=1}^{M}\left(1-\sum_{p=1}^{P}\lambda_{p,m}t_p\right)^{-\theta_{0,m}} \cdot \prod_{p=1}^{P}(1-\lambda_pt_p)^{-\theta_p} \tag{50}$$

and exists for all tuples $(t_1,\dots,t_P)$ where $\sum_{p=1}^{P}\lambda_{p,m}t_p < 1$ for each m and $t_p < \lambda_p^{-1}$ for each p. For $r \neq s$ the covariances are

$$\mathrm{Covar}(X_r,X_s) \;=\; \sum_{m=1}^{M}\lambda_{r,m}\lambda_{s,m}\theta_{0,m}, \tag{51}$$

and the correlation coefficients are

$$\mathrm{Corr}(X_r, X_s) \;=\; \frac{\sum_{m=1}^{M} \lambda_{r,m}\lambda_{s,m}\theta_{0,m}}{\sqrt{\lambda_{r,1}^2\theta_{0,1} + \cdots + \lambda_{r,M}^2\theta_{0,M} + \lambda_r^2 Y_r^2}\;\sqrt{\lambda_{s,1}^2\theta_{0,1} + \cdots + \lambda_{s,M}^2\theta_{0,M} + \lambda_s^2 Y_s^2}}. \tag{52}$$

Proof. The moment generating function is

$$\begin{aligned}
M_{X_1,\dots,X_P}(t_1,\dots,t_P) &= \mathbb{E}\left[e^{t_1X_1+\cdots+t_PX_P}\right] \\
&= \mathbb{E}\left[e^{\sum_{p=1}^{P}(\lambda_{p,1}Y_{0,1}+\cdots+\lambda_{p,M}Y_{0,M}+\lambda_pY_p)t_p}\right] \\
&= \mathbb{E}\left[e^{\sum_{p=1}^{P}\lambda_{p,1}t_pY_{0,1}}\right]\cdots\mathbb{E}\left[e^{\sum_{p=1}^{P}\lambda_{p,M}t_pY_{0,M}}\right]\cdot\mathbb{E}\left[e^{\lambda_1t_1Y_1}\right]\cdots\mathbb{E}\left[e^{\lambda_Pt_PY_P}\right] \\
&= \prod_{m=1}^{M}\left(1-\sum_{p=1}^{P}\lambda_{p,m}t_p\right)^{-\theta_{0,m}} \cdot \prod_{p=1}^{P}(1-\lambda_pt_p)^{-\theta_p}, \qquad (53)
\end{aligned}$$

which exists for tuples $(t_1,\dots,t_P)$ where $\sum_{p=1}^{P}\lambda_{p,m}t_p < 1$ for each m and $t_p < \lambda_p^{-1}$ for each p. For our applications such restrictions are harmless, as in Theorem 3.1 we evaluate the moment generating function at $(-K_{i,1},\dots,-K_{i,P})$ and each $K_{i,p} \ge 0$. The covariances and correlation coefficients are easily determined in this case. As the $Y_{0,m}$ and Y_p are independent, for $r \neq s$

$$\begin{aligned}
\mathrm{Covar}(X_r,X_s) &= \mathbb{E}\left[(\lambda_{r,1}Y_{0,1}+\cdots+\lambda_{r,M}Y_{0,M}+\lambda_rY_r)(\lambda_{s,1}Y_{0,1}+\cdots+\lambda_{s,M}Y_{0,M}+\lambda_sY_s)\right] \\
&= \;-\,\mathbb{E}\left[\lambda_{r,1}Y_{0,1}+\cdots+\lambda_{r,M}Y_{0,M}+\lambda_rY_r\right]\cdot\mathbb{E}\left[\lambda_{s,1}Y_{0,1}+\cdots+\lambda_{s,M}Y_{0,M}+\lambda_sY_s\right] \\
&= \mathbb{E}\left[(\lambda_{r,1}Y_{0,1}+\cdots+\lambda_{r,M}Y_{0,M})(\lambda_{s,1}Y_{0,1}+\cdots+\lambda_{s,M}Y_{0,M})\right] \\
&= \;-\,\mathbb{E}\left[\lambda_{r,1}Y_{0,1}+\cdots+\lambda_{r,M}Y_{0,M}\right]\cdot\mathbb{E}\left[\lambda_{s,1}Y_{0,1}+\cdots+\lambda_{s,M}Y_{0,M}\right] \\
&= \sum_{u=1}^{M}\sum_{v=1}^{M}\lambda_{r,u}\lambda_{s,v}\left(\mathbb{E}[Y_{0,u}Y_{0,v}]-\mathbb{E}[Y_{0,u}]\cdot\mathbb{E}[Y_{0,v}]\right) \\
&= \sum_{m=1}^{M}\lambda_{r,m}\lambda_{s,m}\mathrm{Var}(Y_{0,m}) \\
&= \sum_{m=1}^{M}\lambda_{r,m}\lambda_{s,m}\theta_{0,m} \qquad (54)
\end{aligned}$$

and the correlation coefficient follows immediately. □

Finally, to obtain an even more flexible distribution, we may consider linear combinations of multivariate Gamma functions. The methods of Section 2.4.2 are immediately applicable and yield an extension of Theorem 3.1.

3.3 Computational Savings for Multivariate Distributions

For a general multivariate distribution G as in §3.1.1, the efficiency of our expansion is related to the rate of growth of the moments, which determines the number of terms needed in the series expansions. However, if G has a good closed-form expansion for its moment generating function (as in §3.1.2), then substantial computational savings exist. We study the computational savings for such G below; we may take G to be the bivariate gamma distribution with MGF given by (47), or the multivariate generalizations of (48) or (50).

Our assumptions on the moment generating function of G imply the conditions for Theorem 3.1 are satisfied. Thus we obtain a closed-form series expansion for H_i:

$$H_i \;=\; \sum_{k_{i11}=0}^{\infty}\cdots\sum_{k_{iJN_i}=0}^{\infty}(-1)^{\vec{k}\cdot\vec{1}}M_{\beta_i}(-K_{i,1},\dots,-K_{i,P}), \tag{55}$$

where as always

$$K_{i,p} \ = \ \sum_{j=1}^{J}\sum_{t=1}^{N_i}(y_{ijt}+k_{ijt})x_{ijt,p}, \quad \vec{k}\cdot\vec{1} \ = \ \sum_{j=1}^{J}\sum_{t=1}^{N_i}k_{ijt}. \tag{56}$$

Note again that H_i depends weakly on $\vec{k}$; all that matters are the dot products $\vec{k}\cdot\vec{x}_{i,p}$ and the parity of $(-1)^{\vec{k}\cdot\vec{1}}$. We argue as in Theorem 2.3. Let $r = (r_1, \dots, r_P)$ be a P-tuple of non-negative integers. For each $\vec{k}$ we count the number of solutions to the system of Diophantine equations $\vec{k}\cdot\vec{x}_{i,p} = r_p$ ($p \in \{1, \dots, P\}$) while recording the sign of $(-1)^{\sum_j\sum_t k_{ijt}} = (-1)^{\vec{k}\cdot\vec{1}}$. Then we have

Theorem 3.3. *Set*

$$\begin{aligned} S(M) &= \{v : v = (v_1, \dots, v_M), v_l \in \{0,1,2,\dots\}\} \\ K_i(x,r,+) &= \#\{k \in S(JN_i) : \forall p \in \{1,\dots,P\}, \vec{k}\cdot\vec{x}_{i,p} = r_p, (-1)^{\vec{k}\cdot\vec{1}} = +1\} \\ K_i(x,r,-) &= \#\{k \in S(JN_i) : \forall p \in \{1,\dots,P\}, \vec{k}\cdot\vec{x}_{i,p} = r_p, (-1)^{\vec{k}\cdot\vec{1}} = -1\}. \end{aligned} \tag{57}$$

Assume the $\beta_{i,p}$ are drawn from a one-sided multivariate distribution with parameters $\vec{b}$ and moment generating function $M_{\beta_i}(t_1, \dots, t_P)$ defined when each $t_p \leq 0$. Then $P(Y|\Omega) = \prod_{i=1}^{I} H_i$ with

$$H_i \ = \ \sum_{r\in S(P)} (K_i(x,r,+) - K_i(x,r,-)) \cdot M_{\beta_i}(-K_{i,1}, \dots, -K_{i,P}), \tag{58}$$

and the combinatorial and Diophantine estimates and bounds from Appendix C are still applicable, leading again to enormous computational savings (after an initial one time cost of determining the $K_i(x,r,\pm)$).

To gain additional savings in Theorem 3.3 we may replace the multivariate distribution with a translated one as in Section 2.3.1.

Further (at least if we use the multivariate distributions from §3.2), H_i is a sum of the moment generating function, and the moment generating function is readily differentiable in terms of its parameters. Thus we again obtain closed-form expressions for the derivatives (see §2.3.1 and Appendix D), and thus for certain data sets (where now the parameters may be correlated) there is the possibility of using Newton's Method to determine the optimal values.

Probably the most tractable and useful multivariate density will be the multivariate gamma distribution from Lemma 3.2. While all covariances will be non-negative, the moment generating function, covariances and correlation coefficients are given by very simple formulas, and are easily evaluated and easily differentiated. Moreover the multivariate gamma distribution can take on a variety of shapes, and as discussed in §2.4 we may further increase the admissible shapes by considering linear combinations of multivariate gamma distributions.

4 CONCLUSION

In this research we obtain closed-form expansions for the marginalization of the logit likelihood, allowing us to make direct inferences about the population. In general these expansions involve new special functions; however, in the case where the distribution of heterogeneity follows a Gamma or Multivariate Gamma distribution (or, in full generality, any linear combination of multivariate distributions with a closed-form moment generating function defined for all non-positive inputs), by re-grouping the terms in the expansions we obtain a rapidly converging series expansion of elementary functions. We separate the calculations into two pieces. The first piece is counting solutions to a system of Diophantine equations (we are finding non-negative integer solutions $\vec{k}$ to $\vec{k} \cdot \vec{x}_{i,p} = r_p$; these are linear equations with integer coefficients); the second is evaluating certain integrations, which depend only on Ω and the values of the Diophantine sums.

The advantage of this approach is clear – we need only do the first calculations once. Thus, if we have 10^9 or so operations there, it is a one-time cost. When we need to evaluate the functions at related points (say for the Newton's Method maximization or at the grid points), we need only evaluate the summations on $r = (r_1, \ldots, r_P)$ in (23), (24) or (58). This grouping of terms is an enormous savings; we count the solutions to these systems of equations once, and save the results as expansion coefficients.

While this research has focused on one specific case, the logit model, and two specific set of priors, the Gamma (if the response coefficients are independent) and Multivariate Gamma (if there may be correlations among the response coefficients) distributions, our hope is that this research spurs others to consider deriving closed-form solutions via expansions that can be made arbitrarily close. In fact, closed-form expansions exist for any multivariate distribution that has a closed-form moment generating function. Thus our expansions can incorporate correlations among the coefficients without sacrificing the computational gains.

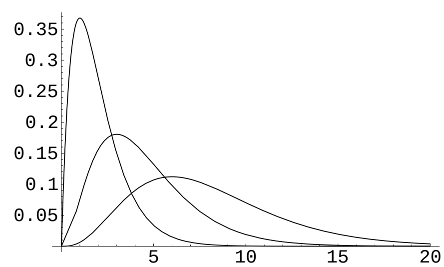
As experience with pure simulation approaches shows, i.e. those that are alternatives to that considered here, it is never a bad thing to have an approach that can be used to explore the parameter space (e.g. mode finding) in advance of running a simulation routine. Whether it is to get good starting values, or simply to understand the potentially multimodal nature of a posterior surface, we hope that research such as this provides value to researchers doing applied problems.

A GAMMA FAMILY OF DISTRIBUTIONS

Given the positivity restriction described in Section 1.2 for the $\beta_{i,p}$, we desired a family of distributions defined on the positive real line that would be extremely flexible, allowing for a variety of shapes of the heterogeneity distribution; and, of course, be conjugate to the geometric series expansion to the logit model. The Generalized Gamma family of distributions satisfies those requirements. As this work concentrated on the Gamma distribution, we only describe this case below.

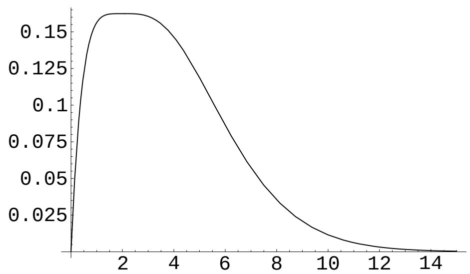
A.1 Plots

We give a few plots of the Gamma distribution to illustrate the richness of the family.



G(z;1,2), G(z;1.5,3), G(z;2,4).

While we develop the theory for $\beta_{i,p}$ drawn from a Gamma distribution, we could use a weighted sum of Gamma distributions as well.



Sum of Weighted Gamma distributions: $\frac{4}{10} \cdot G(z;1,2) + \frac{6}{10} \cdot G(z;1,5)$:

A.2 Integration Lemma

We prove Lemma 2.1:

Proof. We have

$$\begin{aligned} e^{-zd} G(z;b,n) &= e^{-zd} \cdot \frac{1}{b\Gamma(n)} \left(\frac{z}{b}\right)^{n-1} e^{-\frac{z}{b}} \\ &= \frac{1}{b\Gamma(n)} \left(\frac{z}{b}\right)^{n-1} e^{-\frac{z}{b/(1+bd)}} \\ &= (1+bd)^{-n} \cdot \frac{1}{\frac{b}{1+bd}\Gamma(n)} \cdot \left(\frac{z}{b/(1+bd)}\right)^{n-1} \cdot e^{\frac{z}{b/(1+bd)}} \\ &= (1+bd)^{-n} G\left(z, \frac{b}{1+bd}, n\right). \end{aligned} \tag{59}$$

As $b > 0$ and $d \geq 0$, $1 + bd > 0$, and the above is well defined. Note $G(z; \frac{b}{1+bd}, n)$ is another Gamma distribution and therefore integrates to 1. □

B MULTIVARIATE DENSITIES WITH CLOSED FORM MOMENT GENERATING FUNCTIONS

In addition to the Multivariate Gamma distribution discussed in detail in Section 3, we describe two additional multivariate distributions that have closed-form expressions for their moment generating functions. As such, these distributions satisfy the conditions of Theorem 3.1, and thus lead to closed-form series expansions. All page references and equation numbers are from Kotz, Balakrishnan and Johnson 2000. By no means is this list exhaustive, but rather representative of those multivariate distributions which are well suited to our needs. Other distributions are Moran-Downton's Bivariate Exponential (pages 371–377, especially (47.75) and (47.76)), Freund's Multivariate Exponential (pages 388–391, especially (47.85)), Kibble-Moran's Bivariate Gamma (pages 436–437), Farlie-Gumble-Morgenstern Type Bivariate Gamma (pages 441–442, especially (48.19) and (48.20)), and Mathai-Moschopoulos' Multivariate Gamma (pages 465–470, especially (48.61) and (48.62)). Other interesting distributions include truncated multivariate normal distributions; however, as we require one-sided distribution these are not as useful as those related to the Gamma distributions.

B.1 (Arnold and Strauss's) Bivariate Exponential (pages 370–371)

Consider the joint probability density

$$p_{X_1,X_2}(x_1,x_2) \;=\; \begin{cases} A_{12}e^{-\lambda_{12}x_1x_2-\lambda_1x_1-\lambda_2x_2} & \text{if } x_1,x_2>0 \\ 0 & \text{otherwise,} \end{cases} \tag{60}$$

where $\lambda_1, \lambda_2, \lambda_{12} > 0$ and A_{12} is the normalization constant. The moment generating function is given by

$$\begin{aligned} M_{X_1,X_2}(t_1,t_2) &= \mathbb{E}\left[e^{t_1X_1+t_2X_2}\right] \\ &= A_{12}\int_0^\infty\int_0^\infty e^{-\lambda_{12}x_1x_2-\lambda_1x_1-\lambda_2x_2+t_1x_1+t_2x_2}dx_1dx_2 \\ &= A_{12}\int_0^\infty e^{-(\lambda_2-t_2)x_2}\left[\int_0^\infty e^{-(\lambda_{12}x_2+\lambda_1-t_1)x_1}dx_1\right]dx_2 \\ &= A_{12}\int_0^\infty e^{-(\lambda_2-t_2)x_2}\frac{dx_2}{\lambda_{12}x_2+\lambda_1-t_1} \\ &= \frac{A_{12}}{\lambda_{12}}\int_0^\infty e^{-(\lambda_2-t_2)x_2}\frac{dx_2}{x_2+(\lambda_1-t_1)\lambda_{12}^{-1}} \\ &= \frac{A_{12}}{\lambda_{12}}\int_0^\infty e^{-u}\frac{du}{u+(\lambda_1-t_1)(\lambda_2-t_2)\lambda_{12}^{-1}} \\ &= \frac{A_{12}}{\lambda_{12}}\,e^{-(\lambda_1-t_1)(\lambda_2-t_2)/\lambda_{12}}\;\mathrm{Ei}\left(-\frac{(\lambda_1-t_1)(\lambda_2-t_2)}{\lambda_{12}}\right), \end{aligned} \tag{61}$$

where

$$\mathrm{Ei}(z) \;=\; -\int_{-z}^\infty e^{-t}\frac{dt}{t} \tag{62}$$

is the exponential integral function (the principal value is taken). The moment generating function exists for $t_p < \lambda_p$. For our applications such restrictions are harmless, as in Theorem 3.1 we evaluate the moment generating function at $(-K_{i,1},\ldots,-K_{i,P})$ and each $K_{i,p} \geq 0$. The normalization constant can be determined by setting $t_1 = t_2 = 0$:

$$A_{12} \;=\; \lambda_{12}e^{\lambda_1\lambda_2/\lambda_{12}}\;\mathrm{Ei}\left(-\frac{\lambda_1\lambda_2}{\lambda_{12}}\right)^{-1}. \tag{63}$$

B.2 (Freund's) Bivariate Exponential (pages 355–356)

Freund considered the following situation: a two component instrument has components with lifetimes having independent density functions (when both are operating) of

$$p_{X_p} \;=\; \begin{cases} \alpha_p\;e^{-\alpha_px_p} & \text{if } x_p>0 \\ 0 & \text{otherwise,} \end{cases} \tag{64}$$

where $\alpha_p > 0$; however, when one component fails the parameter of the life distribution of the other changes to α'_k. Thus X_1 and X_2 are dependent with joint density function

$$p_{X_1,X_2} \;=\; \begin{cases} \alpha_1\alpha'_2e^{-\alpha'_2x_2-\gamma_2x_1} & \text{if } 0\leq x_1<x_2 \\ \alpha'_1\alpha_2e^{-\alpha'_1x_1-\gamma_1x_2} & \text{if } 0\leq x_2<x_1, \end{cases} \tag{65}$$

where $\gamma_p = \alpha_1 + \alpha_2 - \alpha'_p$; see (47.25). If $\gamma_p \neq 0$ then the marginal density of X_p is

$$p_{X_p}(x_p) \;=\; \frac{(\alpha_p-\alpha'_p)(\alpha_1+\alpha_2)}{\gamma_p}e^{-(\alpha_1+\alpha_2)x_p} \;+\; \frac{\alpha'_p\alpha_{3-p}}{\gamma_p}e^{-\alpha'_px_p}, \quad x_p\geq 0. \tag{66}$$

As these are mixtures of exponentials, this distribution is also called the bivariate mixture exponential. The moment generating function is given by

$$M_{X_1,X_2}(t_1,t_2) \;=\; \frac{1}{\alpha_1+\alpha_2-t_1-t_2}\left(\frac{\alpha_1'\alpha_2}{\alpha_1'-t_1}+\frac{\alpha_1\alpha_2'}{\alpha_2'-t_2}\right), \tag{67}$$

which converges for $t_p < \alpha_p'$ and $t_1+t_2 < \alpha_1+\alpha_2$; see (47.28). For our applications such restrictions are harmless, as in Theorem 3.1 we evaluate the moment generating function at $(-K_{i,1},\dots,-K_{i,P})$ and each $K_{i,p} \ge 0$. The correlation coefficient is given by

$$\mathrm{corr}(X_1,X_2) \;=\; \frac{\alpha_1'\alpha_2'-\alpha_1\alpha_2}{\sqrt{(\alpha_1'^2+2\alpha_1\alpha_2+\alpha_2^2)(\alpha_2'^2+2\alpha_1\alpha_2+\alpha_1^2)}} \;\in\; \left(-\frac{1}{3},\ 1\right), \tag{68}$$

see (47.31). Thus unlike many of the other multivariate distributions, this model allows us to study one-sided distributions with negative correlation.

C COMBINATORIAL AND DIOPHANTINE BOUNDS

We use the notation of Theorem 2.3 and Theorem 2.4:

$$\begin{aligned}
S(M) &= \{v : v=(v_1,\dots,v_M), v_l \in \{0,1,2,3,\dots\}\}\\
K_i(x,r) &= \#\{k \in S(JN_i) : \forall p \in \{1,\dots,P\}, \vec{k}\cdot\vec{x}_{i,p}=r_p\}\\
K_i(x,r,+) &= \#\{k \in S(JN_i) : \forall p \in \{1,\dots,P\}, \vec{k}\cdot\vec{x}_{i,p}=r_p, (-1)^{\vec{k}\cdot\vec{1}}=+1\}\\
K_i(x,r,-) &= \#\{k \in S(JN_i) : \forall p \in \{1,\dots,P\}, \vec{k}\cdot\vec{x}_{i,p}=r_p, (-1)^{\vec{k}\cdot\vec{1}}=-1\},
\end{aligned} \tag{69}$$

and let $K_i(r) = K_i(\vec{1},r)$.

For each i we bound the number of solutions to $\vec{k}\cdot\vec{x}_{i,p} = r_p$ for $p \in \{1,\dots,P\}$. Solutions to Diophantine equations of this nature often crucially depend upon the coefficients $x_{ijt,p}$. In expanding $P(Y|\beta)$ we can trivially handle any terms with an $x_{ijt,p} = 0$. Thus, as we assume $x_{ijt,p}$ is integral, in all arguments below we may assume $x_{ijt,p} \ge 1$; if this assumption fails than trivial book-keeping in our earlier expansions remove the sum over k_{ijt}. The following result is immediate:

Lemma C.1. *Let $x_{i,p} = (x_{i11,p},\dots,x_{iJN_i,p})$ be a $J\cdot N_i$ tuple of positive integers. Then $K_i(x,r,\pm) \le K_i(r)$.*

Thus by Lemma C.1 instead of analyzing $K_i(x,r,\pm 1)$ it suffices to bound the simpler $K_i(r)$.

For ease of exposition, we confine ourselves to the case where the $\beta_{i,p}$ are drawn from a translated Gamma distribution, $G(z-\epsilon; b_p, n_p)$, and we assume $x_{ijt,p} \ge \delta$; for example, we may take $\delta = 1$. Such bounds do not exploit the cancellation in $K_i(x,r,+) - K_i(x,r,-)$ (though it is not unreasonable to expect square-root cancellation). It is straightforward to generalize these arguments to the Multivariate Gamma distribution (or linear combinations thereof) from Lemma 3.2.

Central in the arguments below are combinatorial results about counting the number of representations of an integer as a sum of a fixed number of integers. We briefly recall two useful results.

Lemma C.2. *The number of ways to write a non-negative integer r as a sum of P non-negative integers is $\binom{r+P-1}{P-1}$.*

Sketch of the proof. Consider $r+P-1$ objects in a row. Choosing $P-1$ objects partitions the remaining r objects into P non-negative sets, and there are $\binom{r+P-1}{P-1}$ ways to choose $P-1$ objects from $r+P-1$ objects. □

Lemma C.3. *The number of ways to write a non-negative integer at most R as a sum of P non-negative integers is* $\sum_{r=0}^{R}\binom{r+P-1}{P-1}=\binom{R+P}{P}$.

Sketch of the proof. Partition R into $P+1$ sets as in Lemma C.2. As the last partition runs through all numbers from 0 to R we get partitions of all numbers at most R into P non-negative sets. □

To exploit the exponential decay in (24) from the $\beta_{i,p}$ being drawn from translated Gamma distributions, we must show that $K_i(r)$ does not grow too rapidly; we shall show it grows at most polynomially in r. Note such arguments ignore the decay of the $(1+b_pY_{i,p}+b_pr_p)^{-n_p}$ factors. Assume we truncate our expansion by requiring $0\le k_{i11}+\cdots+k_{iJN_i}\le R$. As we assume that $x_{ijt,p}\ge\delta$ and that we are using translated Gamma distributions, we must bound

$$\sum_{\substack{k_{i11},\dots,k_{iJN_i}\\ k_{i11}+\cdots+k_{iJN_i}>R}}\prod_{p=1}^{P}e^{-\epsilon\delta\sum_{j=1}^{J}\sum_{t=1}^{N_i}k_{ijt}}. \tag{70}$$

We use the notation of Section 2.3.1. For any r, if each $k_{ijt}\ge 0$, then Lemmas C.1 and C.2 immediately yield

Theorem C.4. *We have*

$$K_i(r) \;=\; \#\{\vec{k}:k_{i11}+\cdots+k_{iJN_i}=r\} \;=\; \binom{r+JN_i-1}{JN_i-1}. \tag{71}$$

Thus $K_i(r)\le(r+JN_i-1)^{JN_i-1}/(JN_i-1)!$, *which implies that* $K_i(r)$ *grows at most polynomially. If* $x_{ijt,p}\ge 1$ *then* $K_i(x,r,\pm)$ *grows at most polynomially.*

We conclude with some arguments and techniques that are specific to having the exponential decay from the translated Gamma distributions. These exploit improved bounds for summing $K_i(r)$ for r in various ranges. We bound

$$\sum_{\substack{k_{i11},\dots,k_{iJN_i}\\ k_{i11}+\cdots+k_{iJN_i}>R}}\prod_{p=1}^{P}e^{-\epsilon\delta\sum_{j=1}^{J}\sum_{t=1}^{N_i}k_{ijt}} \;=\; \sum_{r=R+1}^{\infty}\binom{r+JN_i-1}{JN_i-1}e^{-\epsilon\delta Pr}. \tag{72}$$

By Lemma C.3 we have

$$\#\{\vec{k}:0\le k_{i11}+\cdots+k_{iJN_i}\le R\} \;=\; \binom{R+JN_i}{JN_i}. \tag{73}$$

Remark C.5. *The number of k-tuples with* $\sum_j\sum_t k_{ijt}\le R$ *is* $\binom{R+JN_i}{JN_i}$. *If we want the approximation from looking at just these terms to be good, we need the sum in (72) to be small. In this case, we initially need to evaluate* $\binom{R+JN_i}{JN_i}$ *terms, which leads to R values to store. In subsequent evaluations (note this encompasses not only calculating H_i but possibly also its partial derivatives required for Newton's Method) we only have to read in R values, an enormous savings. The more varied the data $x_{ijt,p}$ is, however, the more tuples of dot products must be stored.*

To obtain a feel for these sizes, we tabulate the number of terms arising from different values of R,J and N_i. Note it is the product JN_i that matters, not the values of J and N_i separately.

R	JN_i	$\binom{R+JN_i-1}{JN_i-1}$
5	20	$10^{4.6}$
7	20	$10^{5.8}$
9	20	$10^{6.8}$
5	30	$10^{5.4}$
7	30	$10^{6.9}$
9	30	$10^{8.2}$
5	40	$10^{6.0}$
7	40	$10^{7.7}$
9	40	$10^{9.2}$

The largest term in the expansion of H_i in Theorem 2.2 is when all $k_{ijt} = 0$, giving $+1$. When the k-sum is small (say of size s), we find terms of size $e^{-\epsilon\delta Ps}$. We have the following trivial estimate:

$$\binom{r+JN_i-1}{JN_i-1} \ \le\ (1+1)^{r+JN_i-1} \ =\ 2^{JN_i-1}e^{r\log 2}. \tag{74}$$

Assume $\epsilon\delta P > \log 2$. Then the sum in (72) is bounded by

$$\begin{aligned}\sum_{r=R+1}^{\infty} 2^{JN_i-1}e^{r\log 2}e^{-\epsilon\delta Pr} \quad &\approx\quad 2^{JN_i-1}\int_R^\infty e^{-(\epsilon\delta P-\log 2)r}dr \\ &\approx\quad \frac{2^{JN_i-1}e^{-(\epsilon\delta P-\log 2)\log R}}{\epsilon\delta P-\log 2}.\end{aligned} \tag{75}$$

If $(\epsilon\delta P - \log 2)\log R > JN_i\log 2$, the above is small. Unfortunately, it might not be small compared to the contributions from terms with a small k-sum (of size s); those contribute on the order of $e^{-\epsilon\delta Ps}$.

We perform a more delicate analysis by using dyadic decomposition, breaking the sum over $r \ge R+1$ into blocks such as $2^mR \le r \le 2^{m+1}R$, and using Lemma C.3 in each block. As the choice function $\binom{r+JN_i-1}{JN_i-1}$ is monotonically increasing in r, we find

$$\begin{aligned}\sum_{r=R+1}^{\infty}\binom{r+JN_i-1}{JN_i-1}e^{-\epsilon\delta Pr} \quad &<\quad \sum_{m=0}^{\infty}\sum_{r=2^mR}^{2^{m+1}R}\binom{r+JN_i-1}{JN_i-1}e^{-\epsilon\delta Pr} \\ &<\quad \sum_{m=0}^{\infty}\binom{2^{m+1}R+JN_i}{JN_i}e^{-\epsilon\delta P2^mR} \\ &<\quad \sum_{m=0}^{\infty}2^{JN_i}e^{2^{m+1}R\log 2}e^{-\epsilon\delta P2^mR} \\ &=\quad 2e^{-((\epsilon\delta P-2\log 2)R-JN_i\log 2)}.\end{aligned} \tag{76}$$

This is small if $(\epsilon\delta P - 2\log 2)R > JN_i\log 2$, allowing us to replace the $\log R$ in $(\epsilon\delta P - \log 2)\log R > JN_i\log 2$ with R.

A slightly better savings is attainable by using instead

$$\sum_{r=2^mR}^{2^{m+1}R}\binom{r+JN_i-1}{JN_i-1} \ =\ \binom{2^{m+1}R+JN_i}{JN_i} - \binom{2^mR-1+JN_i}{JN_i} \tag{77}$$

and using polynomial (rather than exponential) bounds. The main term is bounded by

$$\frac{(2^{m+1}R+JN_i)^{JN_i}}{(JN_i)!} \ <\ \begin{cases}(2JN_i)^{JN_i}/(JN_i)! & \text{if } 2^{m+1}R \le JN_i \\ (2^{m+2}R)^{JN_i}/(JN_i)! & \text{if } 2^{m+1}R > JN_i\end{cases} \tag{78}$$

In order to deduce which of the many possible expansions is best, and what size data sets are manageable, one needs to have explicit values for ϵ, δ and P; one can also try to exploit the cancellation from the $(-1)^{\vec{k}\cdot\vec{1}}$ and the denominator factors.

D APPLYING NEWTON'S METHOD TO THE MARGINAL POSTERIOR

Newton's Method yields a sequence of points $\vec{x}_k$ such that $f(\vec{x}_k)$ converges to a local maximum of f. If g_k and H_k are the gradient and Hessian of f at $\vec{x}_k$, then $\vec{x}_{k+1} = \vec{x}_k + \vec{p}_k$, where $\vec{p}_k$ satisfies the *linear* equation $H_k\vec{p}_k = -\vec{g}_k$.

For our problem, $\Omega = (\vec{b}, \vec{n})$. As the function we want to maximize is a product of terms, we maximize $\log f(\vec{b}, \vec{n})$, as the logarithm converts the product in (16) to a sum. To maximize

$$\log f(\vec{b}, \vec{n}) \quad = \quad \log \prod_i \mathrm{H}_i(\vec{b}, \vec{n}) \;=\; \sum_i \log \mathrm{H}_i(\vec{b}, \vec{n}) \tag{79}$$

we need the gradient and the Hessian as in standard applications of Newton's method. The gradient is

$$\nabla \log f(\vec{b}, \vec{n}) \;=\; \frac{\nabla f(\vec{b}, \vec{n})}{f(\vec{b}, \vec{n})} \quad = \quad \sum_i \frac{\nabla \mathrm{H}_i(\vec{b}, \vec{n})}{\mathrm{H}_i(\vec{b}, \vec{n})}, \tag{80}$$

and the entries of the Hessian are

$$\frac{\partial}{\partial x} \nabla \log f(\vec{b}, \vec{n}) \;=\; \sum_i \left[\frac{\frac{\partial}{\partial x} \nabla \mathrm{H}_i(\vec{b}, \vec{n})}{\mathrm{H}_i(\vec{b}, \vec{n})} - \frac{\nabla \left[\mathrm{H}_i(\vec{b}, \vec{n}) \right] \;\cdot\; \frac{\partial}{\partial x} \mathrm{H}_i(\vec{b}, \vec{n})}{\mathrm{H}_i^2(\vec{b}, \vec{n})} \right], \tag{81}$$

where $\frac{\partial}{\partial x} = \frac{\partial}{\partial b_p}$ or $\frac{\partial}{\partial x} = \frac{\partial}{\partial n_p}$.

Straightforward differentiation gives the partial derivatives. The advantage of using a Gamma distribution is the ease of differentiating and evaluating these partials. We give exact, infinite expansions; in practice, one truncates these expressions, and the same Diophantine calculations and computational savings for H_i also hold for these derivatives. Let

$$\begin{aligned} B(\vec{b}, \vec{n}, K(i)) \quad &= \quad \prod_{p=1}^{P} (1 + b_p K_{i,p})^{-n_p} \\ \vec{k} \cdot \vec{1} \quad &= \quad k_{i11} + \cdots + k_{iJN_i}. \end{aligned} \tag{82}$$

Lemma D.1 (First Derivative Expansions)**.**

$$\begin{aligned} \frac{\partial H_i(\vec{b}, \vec{n})}{\partial b_p} \quad &= \quad - \sum_{k_{i11}=0}^{\infty} \cdots \sum_{k_{iJN_i}=0}^{\infty} (-1)^{\vec{k}\cdot\vec{1}} \frac{K_{i,p} n_p}{1 + b_p K_{i,p}} B(\vec{b}, \vec{n}, K(i)) \\ \frac{\partial H_i(\vec{b}, \vec{n})}{\partial n_p} \quad &= \quad - \sum_{k_{i11}=0}^{\infty} \cdots \sum_{k_{iJN_i}=0}^{\infty} (-1)^{\vec{k}\cdot\vec{1}} \log(1 + b_p K_{i,p}) \cdot B(\vec{b}, \vec{n}, K(i)). \end{aligned} \tag{83}$$

As $b_p, K_{i,p}$ are non-negative, the logarithms are well defined above.

Lemma D.2 (Second Derivative Expansions)**.** *In the expansions below, $p \neq q$.*

$$\begin{aligned}
\frac{\partial^2 H_i(\vec{b},\vec{n})}{\partial b_p^2} &= \sum_{k_{i11}=0}^{\infty}\cdots\sum_{k_{iJN_i}=0}^{\infty}(-1)^{\vec{k}\cdot\vec{1}}\frac{K_{i,p}^2 n_p(1+n_p)}{(1+b_pK_{i,p})^2}B(\vec{b},\vec{n},K(i)).\\
\frac{\partial^2 H_i(\vec{b},\vec{n})}{\partial n_p^2} &= \sum_{k_{i11}=0}^{\infty}\cdots\sum_{k_{iJN_i}=0}^{\infty}(-1)^{\vec{k}\cdot\vec{1}}\log^2\left(1+b_pK_{i,p}\right)\cdot B(\vec{b},\vec{n},K(i)).\\
\frac{\partial^2 H_i}{\partial n_p\partial b_p} &= \sum_{k_{i11}=0}^{\infty}\cdots\sum_{k_{iJN_i}=0}^{\infty}(-1)^{\vec{k}\cdot\vec{1}}\left[\frac{K_{i,p}n_p}{1+b_pK_{i,p}}\cdot\log(1+b_pK_{i,p})\right.\\
&\qquad\left.-\frac{K_{i,p}}{1+b_pK_{i,p}}\right]\times B(\vec{b},\vec{n},K(i))\\
\frac{\partial^2 H_i(\vec{b},\vec{n})}{\partial b_p\partial n_q} &= \sum_{k_{i11}=0}^{\infty}\cdots\sum_{k_{iJN_i}=0}^{\infty}(-1)^{\vec{k}\cdot\vec{1}}\frac{K_{i,p}n_p\log(1+b_qK(i,q))}{1+b_pK_{i,p}}B(\vec{b},\vec{n},K(i))\\
\frac{\partial^2 H_i(\vec{b},\vec{n})}{\partial b_p\partial b_q} &= \sum_{k_{i11}=0}^{\infty}\cdots\sum_{k_{iJN_i}=0}^{\infty}(-1)^{\vec{k}\cdot\vec{1}}\frac{K_{i,p}n_p}{1+b_pK_{i,p}}\frac{K(i,q)n_q}{1+b_qK(i,q)}B(\vec{b},\vec{n},K(i)).\\
\frac{\partial^2 H_i}{\partial n_p\partial n_q} &= \sum_{k_{i11}=0}^{\infty}\cdots\sum_{k_{iJN_i}=0}^{\infty}(-1)^{\vec{k}\cdot\vec{1}}\log\left(1+b_pK_{i,p}\right)\log\left(1+b_qK(i,q)\right)B(\vec{b},\vec{n},K(i)).
\end{aligned} \tag{84}$$

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