Number Theory and Random Matrix Theory

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Random Matrix Theory and Number Theory Background

Random Matrix Ensembles

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1N} & a_{2N} & a_{3N} & \cdots & a_{NN} \end{pmatrix} = A^T, \quad a_{ij} = a_{ji}$$

Fix p, define

$$\text{Prob}(A) = \prod_{1 \le i \le j \le N} p(a_{ij}).$$

This means

$$\text{Prob}\left(A : a_{ij} \in [\alpha_{ij}, \beta_{ij}]\right) = \prod_{1 \le i \le j \le N} \int_{x_{ij}=\alpha_{ij}}^{\beta_{ij}} p(x_{ij}) dx_{ij}.$$

Want to understand eigenvalues of A.

Eigenvalue Distribution

$\delta(x - x_0)$ is a unit point mass at x_0:
$\int f(x)\delta(x - x_0)dx = f(x_0)$.

To each A, attach a probability measure:

$$\begin{aligned}
\mu_{A,N}(x) &= \frac{1}{N}\sum_{i=1}^{N}\delta\left(x - \frac{\lambda_i(A)}{2\sqrt{N}}\right) \\
\int_a^b \mu_{A,N}(x)dx &= \frac{\#\left\{\lambda_i : \frac{\lambda_i(A)}{2\sqrt{N}} \in [a,b]\right\}}{N} \\
\text{k}^{\text{th}}\text{ moment} &= \frac{\sum_{i=1}^{N}\lambda_i(A)^k}{2^k N^{\frac{k}{2}+1}} = \frac{\mathrm{Trace}(A^k)}{2^k N^{\frac{k}{2}+1}}.
\end{aligned}$$

Random Matrix Theory: Eigenvalue Trace Formula

Want to understand the eigenvalues of A, but it is the matrix elements that are chosen randomly and independently.

Eigenvalue Trace Lemma

Let A be an $N \times N$ matrix with eigenvalues $\lambda_i(A)$. Then

$$\text{Trace}(A^k) \;=\; \sum_{n=1}^{N} \lambda_i(A)^k,$$

where

$$\text{Trace}(A^k) \;=\; \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1}.$$

Riemann Zeta Function

$$\zeta(s) \;=\; \sum_{n=1}^{\infty} \frac{1}{n^s} \;=\; \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Functional Equation:

$$\xi(s) \;=\; \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) \;=\; \xi(1-s).$$

Riemann Hypothesis (RH):

All non-trivial zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

Observation: Spacings b/w zeros appear same as b/w eigenvalues of Complex Hermitian matrices $\overline{A}^T = A$.

General L-functions

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} = \prod_{p \text{ prime}} L_p(s,f)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Functional Equation:

$$\Lambda(s,f) = \Lambda_\infty(s,f)L(s,f) = \Lambda(1-s,f).$$

Generalized Riemann Hypothesis (GRH):

All non-trivial zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2}+i\gamma$.

Observation: Spacings b/w zeros appear same as b/w eigenvalues of Complex Hermitian matrices $\overline{A}^T = A$.

Measures of Spacings: 1-Level Density and Families

$\phi(x)$ even Schwartz function whose Fourier Transform is compactly supported.

1-level density

$$D_f(\phi) = \sum_j \phi\left(L_f \gamma_{j;f}\right)$$

1. Individual zeros contribute in limit.
2. Most of contribution is from low zeros.
3. Average over similar curves (family).

Katz-Sarnak Conjecture

For a ‘nice’ family of L-functions, the n-level density depends only on a symmetry group attached to the family.

Number Theory: Explicit Formula: Example

Cuspidal Newforms: Let $\mathcal{F}$ be a family of cupsidal newforms (say weight k, prime level N and possibly split by sign) $L(s,f)=\sum_n \lambda_f(n)/n^s$. Then

$$\begin{aligned}\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\sum_{\gamma_f}\phi\left(\frac{\log R}{2\pi}\gamma_f\right) &= \widehat{\phi}(0)+\frac{1}{2}\phi(0)-\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}P(f;\phi)\\ &\quad + O\left(\frac{\log\log R}{\log R}\right)\\ P(f;\phi) &= \sum_{p\nmid N}\lambda_f(p)\widehat{\phi}\left(\frac{\log p}{\log R}\right)\frac{2\log p}{\sqrt{p}\log R}.\end{aligned}$$

Toeplitz Ensembles
(Steven Jackson and Vincent Pham)

Previous Results

$N \times N$ Toeplitz matrix:

$$\begin{pmatrix} b_0 & b_1 & b_2 & \cdots & b_{N-1} \\ b_{-1} & b_0 & b_1 & \cdots & b_{N-2} \\ b_{-2} & b_{-1} & b_0 & \cdots & b_{N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{1-N} & b_{2-N} & b_{3-N} & \cdots & b_0 \end{pmatrix}$$

Density of eigenvalues close to, but not, a Gaussian.

Previous Results

$N \times N$ Palindromic Toeplitz matrix:

$$\begin{pmatrix} b_0 & b_1 & b_2 & \cdots & b_2 & b_1 & b_0 \\ b_1 & b_0 & b_1 & \cdots & b_3 & b_2 & b_1 \\ b_2 & b_1 & b_0 & \cdots & b_4 & b_3 & b_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ b_2 & b_3 & b_4 & \cdots & b_0 & b_1 & b_2 \\ b_1 & b_2 & b_3 & \cdots & b_1 & b_0 & b_1 \\ b_0 & b_1 & b_2 & \cdots & b_2 & b_1 & b_0 \end{pmatrix}.$$

Density of eigenvalues is the Gaussian (each configuration contributes equally).

Previous Results

$N \times N$ Doubly Palindromic Toeplitz matrix (first row):

$$(b_0\ b_1\ \cdots\ b_1\ b_0\ b_0\ b_1\ \cdots\ b_1\ b_0)$$

Questions

- What is the density of eigenvalues?
- Does each configuration contribute equally?

Matching Illustrations

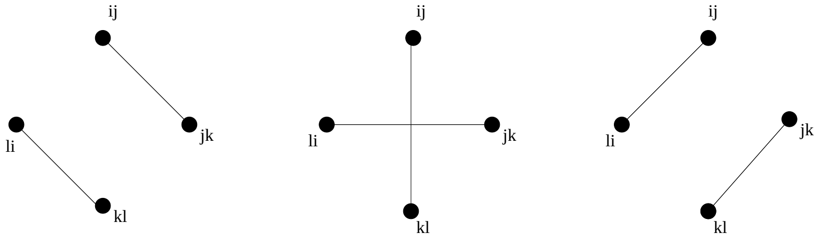

Figure: Possible Configurations for the Fourth Moment Matchings

Matching Lemma

Lemma

Given a_{ij}, let A be one of the diagonals whose entries equal a_{ij} and is on the same diagonal half of the matrix with a_{ij}. Let B be the opposite of the symmetric diagonal of A. Let b be the distance from B to the diagonal going through a_{ij}. Then the contribution from diagonals A and B to the fourth moment is the same for all configurations and equals

$$N^2(N+1-b).$$

Conjecture

The contribution from all configurations to the $2k^{\text{th}}$ moment are equal.

Matching Illustrations

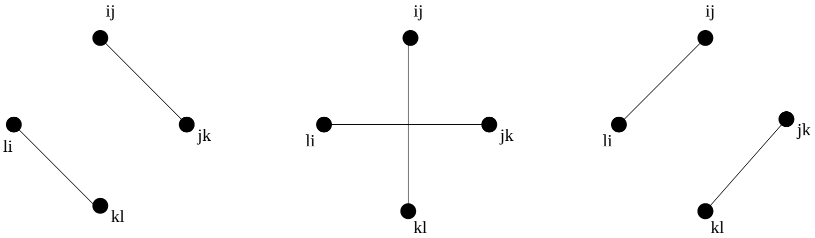

Figure: Possible Configurations for the Fourth Moment Matchings

Matching Illustrations

0	1	2	3	4	4	3	2	1	0	0	1	2	3	4	4	3	2	1	0
1	0	1	2	3	4	4	3	2	1	0	0	1	2	3	4	4	3	2	1
2	1	0	1	2	3	4	4	3	2	1	0	0	1	2	3	4	4	3	2
3	2	1	0	1	2	3	4	4	3	2	1	0	0	1	2	3	4	4	3
4	3	2	1	0	1	2	3	4	4	3	2	1	0	0	1	2	3	4	4
4	4	3	2	1	0	1	2	3	4	4	3	2	1	0	0	1	2	3	4
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0	1	2	3	4	4	3	2	1	0	0	1	2	3	4	4	3	2	1	0

Matching Illustrations

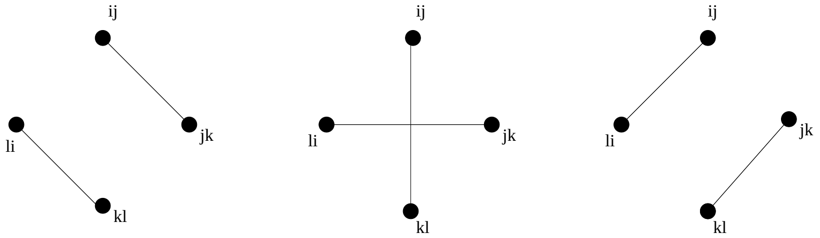

Figure: Possible Configurations for the Fourth Moment Matchings

Matching Illustrations

0	1	2	3	4	4	3	2	1	0	0	1	2	3	4	4	3	2	1	0
1	0	1	2	3	4	4	3	2	1	0	0	1	2	3	4	4	3	2	1
2	1	0	1	2	3	4	4	3	2	1	0	0	1	2	3	4	4	3	2
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1	0	0	1	2	3	4	4	3	2	1	0	1	2	3	4	4	3	2	1
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1	2	3	4	4	3	2	1	0	0	1	2	3	4	4	3	2	1	0	1
0	1	2	3	4	4	3	2	1	0	0	1	2	3	4	4	3	2	1	0

Results: Highly Palindromic Toeplitz Matrices

Theorem

For a Toeplitz matrix with 2^n palindromes, sum over 2^n similar cases to find fourth moment is

$$3 \cdot \left(\frac{2}{3}2^n + \frac{1}{3}2^{-n}\right)$$

(for doubly palindromic, equals 4.5).

Convergence of the moments

The $2k^{\text{th}}$ moment is bounded below by the Gaussian's moment, $(2k-1)!!$.

The $2k^{\text{th}}$ moment is bounded above by $(2k-1)!! \cdot (4 \cdot 2^n - 1)^{k-1}$.

Theorem

Moments grow sufficiently slowly to determine a unique probability distribution, has 'fattest' tails of any ensemble studied to date.

d-Regular Graphs
(Kesinee Ninsuwan)

Terminology

- A *d*-regular graph G is a graph where each vertex is connected to exactly d other vertices, no loops, no multiple edges, no directed edges.
- A closed walk of length n is a path $\langle v_1, v_2, \ldots, v_n \rangle$ such that $v_1 = v_n$.
- The adjacency matrix of G is the matrix $A = (a_{ij})$ where $a_{ij} = 1$ if vertices i and j are connected, 0 otherwise.

Weighted d-regular graphs

Fix a probability distribution $\mathbb{W}$.

Put a weight to each edge of G by independently drawing from $\mathbb{W}$.

The adjacency matrix of a weighted graph has entries

$$(A_w)_{ij} \;=\; \begin{cases} w_{ij}a_{ij} & \text{if } i \geq j \\ w_{ji}a_{ij} & \text{if } i > j. \end{cases}$$

Needed Results

Theorem (McKay)

As $N \to \infty$, the limiting probability density for the eigenvalues of unweighted d-regular graph G converges to Kesten's measure

$$f_d(x) = \begin{cases} \frac{d}{2\pi(d^2-x^2)}\sqrt{4(d-1)-x^2} & \text{if } |x| \le 2\sqrt{d-1} \\ 0 & \text{otherwise.} \end{cases}$$

Normalizing the eigenvalues by $2\sqrt{d-1}$ and letting $d \to \infty$, we see the probability density converges to the semicircle distribution

$$C(x) = \begin{cases} \frac{2}{\pi}\sqrt{1-x^2} & \text{if } |x| \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

New Results

Theorem

Let d grow slower than N^r for any $r > 0$ (for example, $d = \log N$ works). Normalizing the eigenvalues by $2\sqrt{d-1}$, as $N \to \infty$ the $2k^{\text{th}}$ moment of normalized eigenvalues of weighted graphs tends to the semi-circle.

Proof

- Counting weighted closed paths; as $d \to \infty$ only the path where each edge is traversed twice contributes:

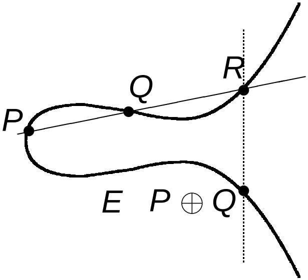
$$\sum_{\ell_1+\ell_2+\ldots+\ell_r=k} \alpha_{\ell_1,\ell_2,\ldots,\ell_r}(d)\sigma_{2\ell_1}\sigma_{2\ell_2}\cdots\sigma_{2\ell_r},$$

where $\alpha_{\ell_1,\ell_2,\ldots,\ell_r}(d)$ is a polynomial in d of degree r.

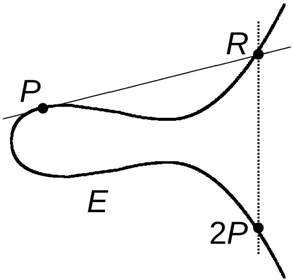
Towards an 'average' version
of the Birch and Swinnerton-Dyer Conjecture
(John Goes)

Elliptic curves: Introduction

Consider $y^2 = x^3 + ax + b$; $a, b \in \mathbb{Z}$.



Addition of distinct points P and Q



Adding a point P to itself

Elliptic curve: L-functions

$E: y^2 = x^3 + ax + b$, associate L-function

$$L(s,E) \ = \ \sum_{n=1}^{\infty} \frac{a_E(n)}{n^s} \ = \ \prod_{p \ \text{prime}} L_{p;E}(p^{-s}),$$

where

$$a_E(p) = p - \#\{(x,y) \in (\mathbb{Z}/p\mathbb{Z})^2 : y^2 \equiv x^3 + ax + b \bmod p\}.$$

Birch and Swinnerton-Dyer Conjecture: Rank of group of rational solutions equals order of vanishing of $L(s,E)$ at $s = 1/2$.

1-Level Density

For a family of elliptic curves $\mathcal{E}$ of rank r, we have

$$\frac{1}{|\mathcal{F}_{\mathcal{R}}|} \sum_{E \in \mathcal{F}_R} \phi\left(\gamma_{j,E} \frac{\log N_E}{2\pi}\right) = \left(r + \frac{1}{2}\right) \phi(0) + \widehat{\phi}(0) + \text{small}$$

if $\widehat{\phi}(x)$ is zero for $|x| \geq \sigma_{\mathcal{E}}$.

Want $\sigma_{\mathcal{E}}$ to be large, in practice can only prove results for $\sigma_{\mathcal{E}}$ small.

Question: how many zeros 'near' the central point?

Previous Results

Mestre: elliptic curve of conductor N has a zero with imaginary part at most $\frac{B}{\log \log N}$.

Expect the relevant scale to study zeros near central point to be $1 / \log N_E$.

Goal: bound (from above and below) number of zeros in a neighborhood of size $1 / \log N_E$ near the central point in a family.

New results for families (as conductors tend to infinity)

Theorem

Let $t_0 = 1/2\pi\sqrt{\sigma_{\mathcal{E}}}$. The average number of normalized zeros in $[-t_0, t_0]$ is bounded below by

$$r + \frac{1}{2} + \frac{\widehat{\phi}(0)}{\phi(0)} + \text{small},$$

and is bounded above by

$$r + \frac{1}{2} + \frac{(r + 1/2)(\psi(0) - \psi(t_0)) + \widehat{\psi}(0)}{\psi(t_0)} + \text{small}.$$

Number Field L-Functions
(Ryan Peckner)

Previous Work

Fouvry and Iwaniec investigated L-functions attached to number fields of form $\mathbb{Q}(\sqrt{-d})$.

Theorem (Fouvry and Iwaniec)

Let f be an even Schwartz function whose Fourier transform is supported in $(-1, 1)$. Then the 1-level density for the ideal class L-functions is

$$\hat{f}(0) - \frac{1}{2} f(0).$$

(i.e., the 1-level density agrees with Symplectic matrices).

New Results

Theorem

Let $\{K_\Delta\}$ be a sequence of number fields ordered by (absolute value of) discriminant, such that

- h_{K_Δ} is prime for each K_Δ in the sequence.
- There exists $c > 0$ such that $\log h_{K_\Delta} \sim c \log \Delta$ as $\Delta \to \infty$.
- $N_\Delta := [K_\Delta : \mathbb{Q}]$ is independent of Δ, say $N_\Delta = N$.

Let f be an even Schwartz function whose Fourier transform is supported in $(-1, 1)$. Then the 1-level density for the ideal class L-functions is

$$\hat{f}(0)$$

(i.e., 1-level density agrees with Unitary matrices).

L-functions Ratios Conjecture
(David Montague)

Intro

A recipe that gives conjectured values for

$$\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\frac{L(\frac{1}{2}+\alpha,f)}{L(\frac{1}{2}+\gamma,f)}.$$

This gives a heuristic for studying properties of L-functions.

Believed to be accurate up to $O(|\mathcal{F}|^{-1/2+\epsilon})$.

Will use to calculate the 1-level density for $\mathcal{F}$.

The Recipe

- Expand the numerator using the approximate functional equation:

$$L(s) = \sum_{n \le x} \frac{a_n}{n^s} + \epsilon X_L(s) \sum_{m \le y} \frac{a_m}{m^{1-s}} + R;$$

  ignore the error term R.

- Expand the denominator using its Dirichlet series:

$$\frac{1}{L(s,f)} = \sum_{h=1}^{\infty} \frac{\mu_f(h)}{h^s}.$$

- Execute the sum over $\mathcal{F}$ through the use of an averaging formula, keeping only the main (diagonal) terms. Ignore the error.

The Recipe

- Extend the n, m sums to infinity.
- Differentiate the sum wrt α, set $\alpha = \gamma = r$, giving a conjectured value for $\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}} \frac{L'(\frac{1}{2}+r,f)}{L(\frac{1}{2}+r,f)}$.
- Perform a contour integral to determine the 1-level density.

Main Results: Test for family $\mathcal{F} = H_k^{\pm}(N)$

This family is an important test: the non-diagonal terms that are dropped contribute to the main term!

Theorem: Ratios Conjecture Prediction

With $\chi(s) = \prod_p \left(1 + \frac{1}{(p-1)p^s}\right)$, the 1-level density is

$$\sum_p \frac{2\log p}{p\log R}\widehat{\phi}\left(\frac{2\log p}{\log R}\right)$$
$$\mp 2\lim_{\epsilon\downarrow 0}\int_{-\infty}^{\infty} X_L\left(\frac{1}{2}+2\pi i x\right)\chi(\epsilon+4\pi i x)\phi(t\log R)dt$$
$$-\int_{-\infty}^{\infty}\frac{X_L'}{X_L}\left(\frac{1}{2}+2\pi i t\right)\phi(t\log R)dt + O(N^{-1/2+\epsilon}),$$

Main Results: Test for family $\mathcal{F} = H_k^{\pm}(N)$

This family is an important test: the non-diagonal terms that are dropped contribute to the main term!

Theorem: Agreement with Number Theory

Assume GRH for $\zeta(s)$, Dirichlet L-functions, and $L(s, f)$. For ϕ such that $\text{supp}(\widehat{\phi}) \subset (-1, 1)$, the 1-level density agrees with the ratios conjecture prediction up to $O(N^{-1/2+\epsilon})$, and get agreement up to a power savings in N if $\text{supp}(\widehat{\phi}) \subset (-2, 2)$.