

# The Pythagorean Won-Loss Theorem: An introduction to modeling

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Bennington College, December 5, 2008

## Acknowledgements



Sal Baxamusa, Phil Birnbaum, Ray Ciccolella, Steve Johnston, Michelle Manes, Russ Mann, students of Math 162 and Math 197 at Brown.



Dedicated to my great uncle Newt Bromberg (a lifetime Red Sox fan who promised me that I would live to see a World Series Championship in Boston).



Chris Long and the San Diego Padres.

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- Introduce some of the techniques of modeling.
- Discuss the mathematics behind the models and model testing.
- Show how advanced theory enters in simple problems.
- Further avenues for research for students.

## Numerical Observation: Pythagorean Won-Loss Formula

### Parameters

- $RS_{\text{obs}}$ : average number of runs scored per game;
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### James' Won-Loss Formula (NUMERICAL Observation)

$$\text{Won} - \text{Loss Percentage} = \frac{RS_{\text{obs}}^{\gamma}}{RS_{\text{obs}}^{\gamma} + RA_{\text{obs}}^{\gamma}}$$

$\gamma$  originally taken as 2, numerical studies show best  $\gamma$  is about 1.82.

## Applications of the Pythagorean Won-Loss Formula

- **Extrapolation:** use half-way through season to predict a team's performance.
- **Evaluation:** see if consistently over-perform or under-perform.
- **Advantage:** Other statistics / formulas (run-differential per game); this is easy to use, depends only on two simple numbers for a team.

## Probability Review

- **Probability density:**

- ◇  $p(x) \geq 0$ ;

- ◇  $\int_{-\infty}^{\infty} p(x) dx = 1$ ;

- ◇  $X$  random variable with density  $p(x)$ :

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- **Variance**  $\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 p(x)dx.$
- **Independence:** two random variables are independent if knowledge of one does not give knowledge of the other.

## Modeling the Real World

### Guidelines for Modeling:

- Model should capture key features of the system;
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In general these are conflicting goals.

How should we try and model baseball games?

## Modeling the Real World (cont)

### Possible Model:

- Runs Scored and Runs Allowed independent random variables;
- $f_{RS}(x)$ ,  $g_{RA}(y)$ : probability density functions for runs scored (allowed).

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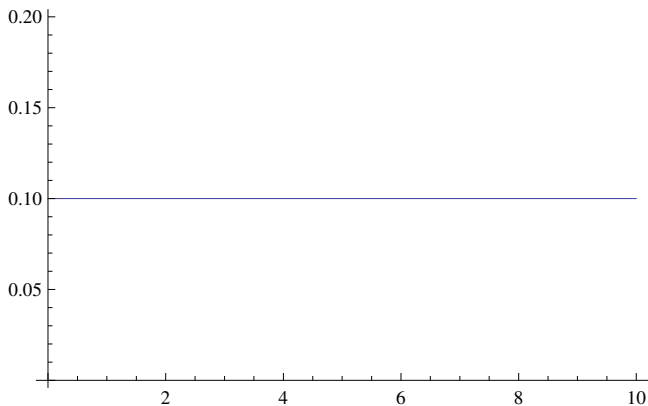
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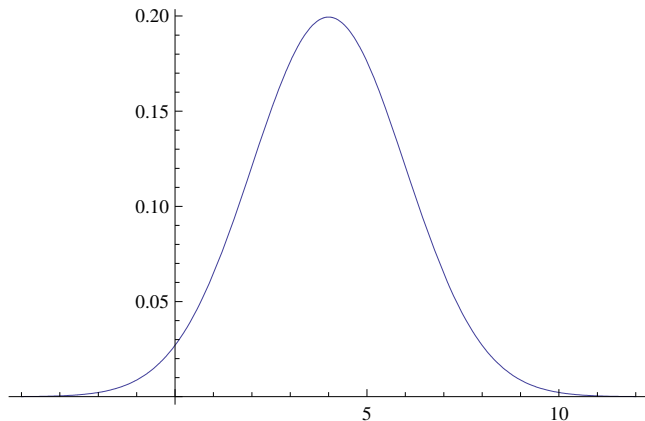
- Can the integral (or sum) be completed in closed form?
- Are the runs scored and allowed independent random variables?
- What are  $f_{RS}$  and  $g_{RA}$ ?

## Choices for $f_{RS}$ and $g_{RA}$



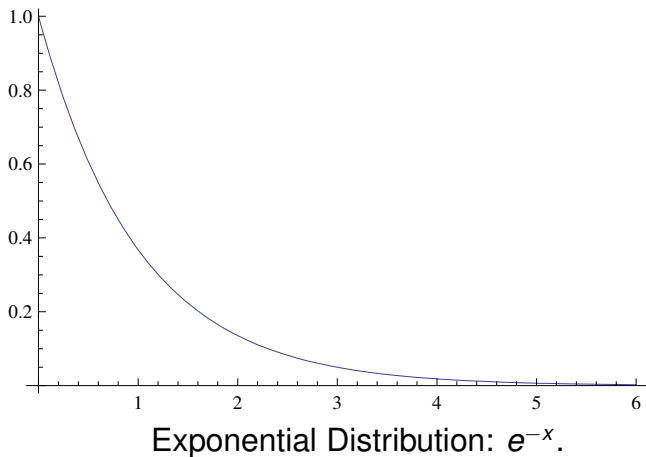
Uniform Distribution on  $[0, 10]$ .

## Choices for $f_{RS}$ and $g_{RA}$



Normal Distribution: mean 4, standard deviation 2.

## Choices for $f_{RS}$ and $g_{RA}$



## Three Parameter Weibull

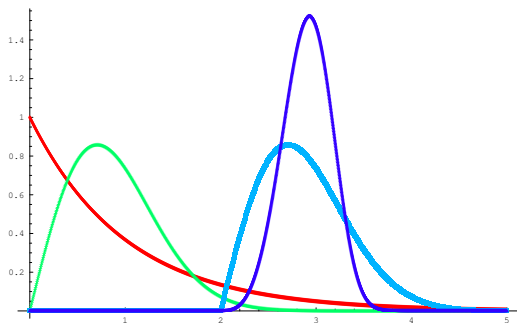
Weibull distribution:

$$f(x; \alpha, \beta, \gamma) = \begin{cases} \frac{\gamma}{\alpha} \left(\frac{x-\beta}{\alpha}\right)^{\gamma-1} e^{-((x-\beta)/\alpha)^\gamma} & \text{if } x \geq \beta \\ 0 & \text{otherwise.} \end{cases}$$

- $\alpha$ : scale (variance: meters versus centimeters);
- $\beta$ : origin (mean: translation, zero point);
- $\gamma$ : shape (behavior near  $\beta$  and at infinity).

Various values give different shapes, but can we find  $\alpha, \beta, \gamma$  such that it fits observed data? Is the Weibull theoretically tractable?

## Weibull Plots: Parameters $(\alpha, \beta, \gamma)$



Red:  $(1, 0, 1)$  (exponential); Green:  $(1, 0, 2)$ ; Cyan:  $(1, 2, 2)$ ;  
Blue:  $(1, 2, 4)$

## Gamma Distribution

- For  $s \in \mathbb{C}$  with the real part of  $s$  greater than 0, define the  $\Gamma$ -function:

$$\Gamma(s) = \int_0^{\infty} e^{-u} u^{s-1} du = \int_0^{\infty} e^{-u} u^s \frac{du}{u}.$$

- Generalizes factorial function:  $\Gamma(n) = (n-1)!$  for  $n \geq 1$  an integer.

## Weibull Integrations

$$\mu_{\alpha,\beta,\gamma} = \int_{\beta}^{\infty} x \cdot \frac{\gamma}{\alpha} \left( \frac{x - \beta}{\alpha} \right)^{\gamma-1} e^{-((x-\beta)/\alpha)^{\gamma}} dx$$

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## Pythagorean Won-Loss Formula

### Theorem (Pythagorean Won-Loss Formula)

*Let the runs scored and allowed per game be two independent random variables drawn from Weibull distributions  $(\alpha_{RS}, \beta, \gamma)$  and  $(\alpha_{RA}, \beta, \gamma)$ ;  $\alpha_{RS}$  and  $\alpha_{RA}$  are chosen so that the means are RS and RA. If  $\gamma > 0$  then*

$$\text{Won-Loss Percentage}(\text{RS}, \text{RA}, \beta, \gamma) = \frac{(\text{RS} - \beta)^\gamma}{(\text{RS} - \beta)^\gamma + (\text{RA} - \beta)^\gamma}$$

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In baseball take  $\beta = -1/2$  (from runs must be integers).  
RS -  $\beta$  estimates average runs scored, RA -  $\beta$  estimates average runs allowed.

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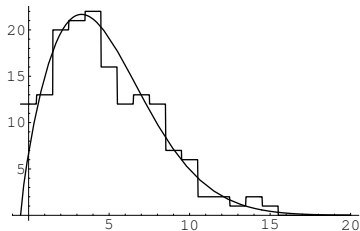
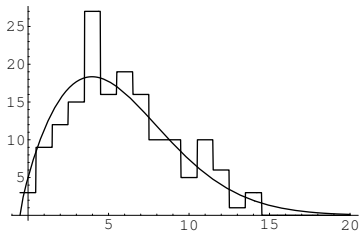
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Find the values of  $(\alpha_{\text{RS}}, \alpha_{\text{RA}}, \gamma)$  that minimize

$$\sum_{k=1}^{\#\text{Bins}} (\text{RS}_{\text{obs}}(k) - \#\text{Games} \cdot A(\alpha_{\text{RS}}, -1/2, \gamma, k))^2 \\ + \sum_{k=1}^{\#\text{Bins}} (\text{RA}_{\text{obs}}(k) - \#\text{Games} \cdot A(\alpha_{\text{RA}}, -1/2, \gamma, k))^2 .$$

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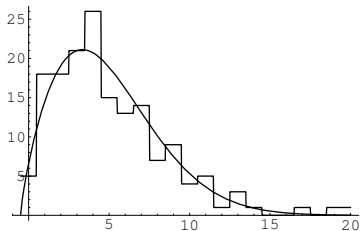
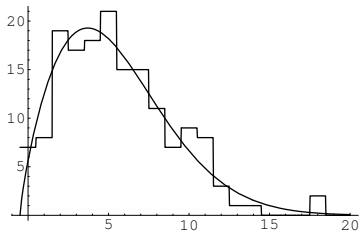
Plots of RS (predicted vs observed) and RA (predicted vs observed) for the Boston Red Sox



Using as bins  $[-.5, .5] \cup [.5, 1.5] \cup \dots \cup [7.5, 8.5] \cup [8.5, 9.5] \cup [9.5, 11.5] \cup [11.5, \infty)$ .

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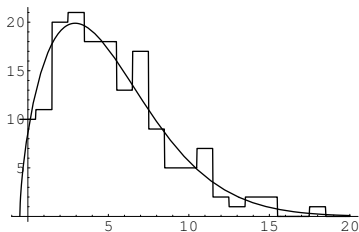
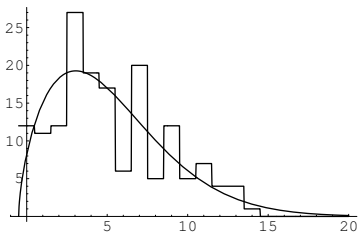
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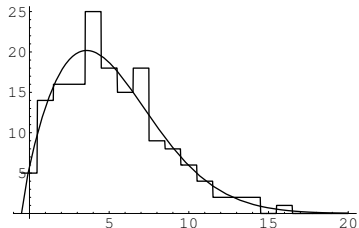
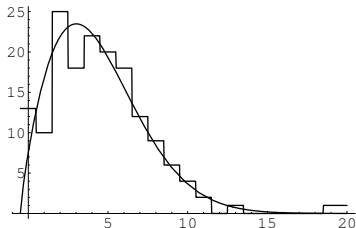
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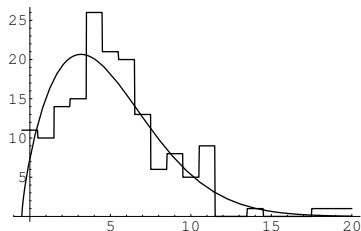
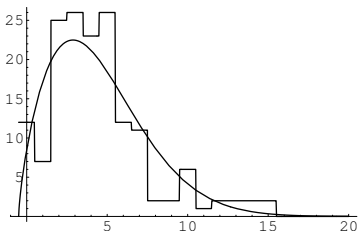
Plots of RS (predicted vs observed) and RA (predicted vs observed) for the Tampa Bay Devil Rays



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Plots of RS (predicted vs observed) and RA (predicted vs observed) for the Toronto Blue Jays



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  - ◇  $\chi^2 = \sum_i \frac{(\text{Obs}(i) - Np_i)^2}{Np_i}$ .
  - ◇ Smaller  $\chi^2$ , more likely correct model.

## Bonferroni Adjustments

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About 95% have  $499,000 \leq \# \text{Heads} \leq 501,000$ .

Consider  $N$  independent experiments of flipping a fair coin 1,000,000 times. *What is the probability that at least one of set doesn't have  $499,000 \leq \# \text{Heads} \leq 501,000$ ?*

| <b>N</b> | <b>Probability</b> |
|----------|--------------------|
| 5        | 22.62              |
| 14       | 51.23              |
| 50       | 92.31              |

See unlikely events happen as  $N$  increases!

## Data Analysis: $\chi^2$ Tests

| Team                 | RS+RA $\chi^2$ : 20 d.f. | Indep $\chi^2$ : 109 d.f. |
|----------------------|--------------------------|---------------------------|
| Boston Red Sox       | 15.63                    | 83.19                     |
| New York Yankees     | 12.60                    | 129.13                    |
| Baltimore Orioles    | 29.11                    | 116.88                    |
| Tampa Bay Devil Rays | 13.67                    | 111.08                    |
| Toronto Blue Jays    | 41.18                    | 100.11                    |
| Minnesota Twins      | 17.46                    | 97.93                     |
| Chicago White Sox    | 22.51                    | 153.07                    |
| Cleveland Indians    | 17.88                    | 107.14                    |
| Detroit Tigers       | 12.50                    | 131.27                    |
| Kansas City Royals   | 28.18                    | 111.45                    |
| Los Angeles Angels   | 23.19                    | 125.13                    |
| Oakland Athletics    | 30.22                    | 133.72                    |
| Texas Rangers        | 16.57                    | 111.96                    |
| Seattle Mariners     | 21.57                    | 141.00                    |

20 d.f.: 31.41 (at the 95% level) and 37.57 (at the 99% level).

109 d.f.: 134.4 (at the 95% level) and 146.3 (at the 99% level).

Bonferroni Adjustment:

20 d.f.: 41.14 (at the 95% level) and 46.38 (at the 99% level).

109 d.f.: 152.9 (at the 95% level) and 162.2 (at the 99% level).

## Data Analysis: Structural Zeros

- For independence of runs scored and allowed, use bins  $[0, 1) \cup [1, 2) \cup [2, 3) \cup \dots \cup [8, 9) \cup [9, 10) \cup [10, 11) \cup [11, \infty)$ .
- Have an  $r \times c$  contingency table with **structural zeros** (runs scored and allowed per game are never equal).
- (Essentially)  $O_{r,r} = 0$  for all  $r$ , use an iterative fitting procedure to obtain maximum likelihood estimators for  $E_{r,c}$  (expected frequency of cell  $(r, c)$  assuming that, given runs scored and allowed are distinct, the runs scored and allowed are independent).

## Testing the Model: Data from Method of Maximum Likelihood

| Team                 | Obs Wins | Pred Wins | ObsPerc | PredPerc | GamesDiff | $\gamma$ |
|----------------------|----------|-----------|---------|----------|-----------|----------|
| Boston Red Sox       | 98       | 93.0      | 0.605   | 0.574    | 5.03      | 1.82     |
| New York Yankees     | 101      | 87.5      | 0.623   | 0.540    | 13.49     | 1.78     |
| Baltimore Orioles    | 78       | 83.1      | 0.481   | 0.513    | -5.08     | 1.66     |
| Tampa Bay Devil Rays | 70       | 69.6      | 0.435   | 0.432    | 0.38      | 1.83     |
| Toronto Blue Jays    | 67       | 74.6      | 0.416   | 0.464    | -7.65     | 1.97     |
| Minnesota Twins      | 92       | 84.7      | 0.568   | 0.523    | 7.31      | 1.79     |
| Chicago White Sox    | 83       | 85.3      | 0.512   | 0.527    | -2.33     | 1.73     |
| Cleveland Indians    | 80       | 80.0      | 0.494   | 0.494    | 0.        | 1.79     |
| Detroit Tigers       | 72       | 80.0      | 0.444   | 0.494    | -8.02     | 1.78     |
| Kansas City Royals   | 58       | 68.7      | 0.358   | 0.424    | -10.65    | 1.76     |
| Los Angeles Angels   | 92       | 87.5      | 0.568   | 0.540    | 4.53      | 1.71     |
| Oakland Athletics    | 91       | 84.0      | 0.562   | 0.519    | 6.99      | 1.76     |
| Texas Rangers        | 89       | 87.3      | 0.549   | 0.539    | 1.71      | 1.90     |
| Seattle Mariners     | 63       | 70.7      | 0.389   | 0.436    | -7.66     | 1.78     |

$\gamma$ : mean = 1.74, standard deviation = .06, median = 1.76;  
close to numerically observed value of 1.82.

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- Find parameters such that Weibulls are good fits;
- Runs scored and allowed per game are statistically independent;
- Pythagorean Won-Loss Formula is a consequence of our model;
- Best  $\gamma$  (both close to observed best 1.82):
  - ◇ Method of Least Squares: 1.79;
  - ◇ Method of Maximum Likelihood: 1.74.

## Future Work

- **Micro-analysis:** runs scored and allowed are not entirely independent (big lead, close game), run production smaller for inter-league games in NL parks, et cetera.
- **Other sports:** Does the same model work? How does  $\gamma$  depend on the sport?
- **Closed forms:** Are there other probability distributions that give integrals which can be determined in closed form?
- **Valuing Runs:** Pythagorean formula used to value players (10 runs equals 1 win); better model leads to better team.

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- **Improving Pythagorean Model**
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- **Pitch data**
  - ◇ Advantage of seeing pitcher
  - ◇ Pitch location data

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## Appendix I: Proof of the Pythagorean Won-Loss Formula

Let  $X$  and  $Y$  be independent random variables with Weibull distributions  $(\alpha_{RS}, \beta, \gamma)$  and  $(\alpha_{RA}, \beta, \gamma)$  respectively. To have means of  $RS - \beta$  and  $RA - \beta$  our calculations for the means imply

$$\alpha_{RS} = \frac{RS - \beta}{\Gamma(1 + \gamma^{-1})}, \quad \alpha_{RA} = \frac{RA - \beta}{\Gamma(1 + \gamma^{-1})}.$$

We need only calculate the probability that  $X$  exceeds  $Y$ . We use the integral of a probability density is 1.

## Appendix I: Proof of the Pythagorean Won-Loss Formula (cont)

$$\begin{aligned}
 \text{Prob}(X > Y) &= \int_{x=\beta}^{\infty} \int_{y=\beta}^x f(x; \alpha_{RS}, \beta, \gamma) f(y; \alpha_{RA}, \beta, \gamma) dy dx \\
 &= \int_{\beta}^{\infty} \int_{\beta}^x \frac{\gamma}{\alpha_{RS}} \left( \frac{x-\beta}{\alpha_{RS}} \right)^{\gamma-1} e^{-\left(\frac{x-\beta}{\alpha_{RS}}\right)^{\gamma}} \frac{\gamma}{\alpha_{RA}} \left( \frac{y-\beta}{\alpha_{RA}} \right)^{\gamma-1} e^{-\left(\frac{y-\beta}{\alpha_{RA}}\right)^{\gamma}} dy dx \\
 &= \int_{x=0}^{\infty} \frac{\gamma}{\alpha_{RS}} \left( \frac{x}{\alpha_{RS}} \right)^{\gamma-1} e^{-\left(\frac{x}{\alpha_{RS}}\right)^{\gamma}} \left[ \int_{y=0}^x \frac{\gamma}{\alpha_{RA}} \left( \frac{y}{\alpha_{RA}} \right)^{\gamma-1} e^{-\left(\frac{y}{\alpha_{RA}}\right)^{\gamma}} dy \right] dx \\
 &= \int_{x=0}^{\infty} \frac{\gamma}{\alpha_{RS}} \left( \frac{x}{\alpha_{RS}} \right)^{\gamma-1} e^{-(x/\alpha_{RS})^{\gamma}} \left[ 1 - e^{-(x/\alpha_{RA})^{\gamma}} \right] dx \\
 &= 1 - \int_{x=0}^{\infty} \frac{\gamma}{\alpha_{RS}} \left( \frac{x}{\alpha_{RS}} \right)^{\gamma-1} e^{-(x/\alpha)^{\gamma}} dx,
 \end{aligned}$$

where we have set

$$\frac{1}{\alpha^{\gamma}} = \frac{1}{\alpha_{RS}^{\gamma}} + \frac{1}{\alpha_{RA}^{\gamma}} = \frac{\alpha_{RS}^{\gamma} + \alpha_{RA}^{\gamma}}{\alpha_{RS}^{\gamma} \alpha_{RA}^{\gamma}}.$$

## Appendix I: Proof of the Pythagorean Won-Loss Formula (cont)

$$\begin{aligned}
 \text{Prob}(X > Y) &= 1 - \frac{\alpha^\gamma}{\alpha_{RS}^\gamma} \int_0^\infty \frac{\gamma}{\alpha} \left(\frac{x}{\alpha}\right)^{\gamma-1} e^{-(x/\alpha)^\gamma} dx \\
 &= 1 - \frac{\alpha^\gamma}{\alpha_{RS}^\gamma} \\
 &= 1 - \frac{1}{\alpha_{RS}^\gamma} \frac{\alpha_{RS}^\gamma \alpha_{RA}^\gamma}{\alpha_{RS}^\gamma + \alpha_{RA}^\gamma} \\
 &= \frac{\alpha_{RS}^\gamma}{\alpha_{RS}^\gamma + \alpha_{RA}^\gamma}.
 \end{aligned}$$

We substitute the relations for  $\alpha_{RS}$  and  $\alpha_{RA}$  and find that

$$\text{Prob}(X > Y) = \frac{(\text{RS} - \beta)^\gamma}{(\text{RS} - \beta)^\gamma + (\text{RA} - \beta)^\gamma}.$$

Note  $\text{RS} - \beta$  estimates  $\text{RS}_{\text{obs}}$ ,  $\text{RA} - \beta$  estimates  $\text{RA}_{\text{obs}}$ .

## Appendix II: Best Fit Weibulls and Structural Zeros

The fits *look* good, but are they? Do  $\chi^2$ -tests:

- Let  $\text{Bin}(k)$  denote the  $k^{\text{th}}$  bin.
- $O_{r,c}$ : the observed number of games where the team's runs scored is in  $\text{Bin}(r)$  and the runs allowed are in  $\text{Bin}(c)$ .
- $E_{r,c} = \frac{\sum_{c'} O_{r,c'} \cdot \sum_{r'} O_{r',c}}{\# \text{Games}}$  is the expected frequency of cell  $(r, c)$ .
- Then

$$\sum_{r=1}^{\# \text{Rows}} \sum_{c=1}^{\# \text{Columns}} \frac{(O_{r,c} - E_{r,c})^2}{E_{r,c}}$$

is a  $\chi^2$  distribution with  $(\# \text{Rows} - 1)(\# \text{Columns} - 1)$  degrees of freedom.

## Appendix II: Best Fit Weibulls and Structural Zeros (cont)

For independence of runs scored and allowed, use bins

$$[0, 1) \cup [1, 2) \cup [2, 3) \cup \dots \cup [8, 9) \cup [9, 10) \cup [10, 11) \cup [11, \infty).$$

Have an  $r \times c$  contingency table (with  $r = c = 12$ ); however, there are *structural zeros* (runs scored and allowed per game can never be equal).

(Essentially)  $O_{r,r} = 0$  for all  $r$ . We use the iterative fitting procedure to obtain maximum likelihood estimators for the  $E_{r,c}$ , the expected frequency of cell  $(r, c)$  under the assumption that, given that the runs scored and allowed are distinct, the runs scored and allowed are independent.

For  $1 \leq r, c \leq 12$ , let  $E_{r,c}^{(0)} = 1$  if  $r \neq c$  and 0 if  $r = c$ . Set

$$X_{r,+} = \sum_{c=1}^{12} O_{r,c}, \quad X_{+,c} = \sum_{r=1}^{12} O_{r,c}.$$

Then

$$E_{r,c}^{(\ell)} = \begin{cases} E_{r,c}^{(\ell-1)} X_{r,+} / \sum_{c=1}^{12} E_{r,c}^{(\ell-1)} & \text{if } \ell \text{ is odd} \\ E_{r,c}^{(\ell-1)} X_{+,c} / \sum_{r=1}^{12} E_{r,c}^{(\ell-1)} & \text{if } \ell \text{ is even,} \end{cases}$$

and

$$E_{r,c} = \lim_{\ell \rightarrow \infty} E_{r,c}^{(\ell)};$$

the iterations converge very quickly. (If we had a complete two-dimensional contingency table, then the iteration reduces to the standard values, namely  $E_{r,c} = \sum_{c'} O_{r,c'} \cdot \sum_{r'} O_{r',c} / \# \text{Games}$ .) Note

$$\sum_{r=1}^{12} \sum_{\substack{c=1 \\ c \neq r}}^{12} \frac{(O_{r,c} - E_{r,c})^2}{E_{r,c}}$$

## Appendix III: Central Limit Theorem

Convolution of  $f$  and  $g$ :

$$h(y) = (f * g)(y) = \int_{\mathbb{R}} f(x)g(y-x)dx = \int_{\mathbb{R}} f(x-y)g(x)dx.$$

$X_1$  and  $X_2$  independent random variables with probability density  $p$ .

$$\text{Prob}(X_i \in [x, x + \Delta x]) = \int_x^{x+\Delta x} p(t)dt \approx p(x)\Delta x.$$

$$\text{Prob}(X_1 + X_2 \in [x, x + \Delta x]) = \int_{x_1=-\infty}^{\infty} \int_{x_2=x-x_1}^{x+\Delta x-x_1} p(x_1)p(x_2)dx_2dx_1.$$

As  $\Delta x \rightarrow 0$  we obtain the convolution of  $p$  with itself:

$$\text{Prob}(X_1 + X_2 \in [a, b]) = \int_a^b (p * p)(z)dz.$$

Exercise to show non-negative and integrates to 1.

## Appendix III: Statement of Central Limit Theorem

- For simplicity, assume  $p$  has mean zero, variance one, finite third moment and is of sufficiently rapid decay so that all convolution integrals that arise converge:  $p$  an infinitely differentiable function satisfying

$$\int_{-\infty}^{\infty} xp(x)dx = 0, \quad \int_{-\infty}^{\infty} x^2 p(x)dx = 1, \quad \int_{-\infty}^{\infty} |x|^3 p(x)dx < \infty.$$

- Assume  $X_1, X_2, \dots$  are independent identically distributed random variables drawn from  $p$ .
- Define  $S_N = \sum_{i=1}^N X_i$ .
- Standard Gaussian (mean zero, variance one) is  $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ .

**Central Limit Theorem** Let  $X_i, S_N$  be as above and assume the third moment of each  $X_i$  is finite. Then  $S_N/\sqrt{N}$  converges in probability to the standard Gaussian:

$$\lim_{N \rightarrow \infty} \text{Prob} \left( \frac{S_N}{\sqrt{N}} \in [a, b] \right) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2} dx.$$

## Appendix III: Proof of the Central Limit Theorem

- The Fourier transform of  $p$  is

$$\hat{p}(y) = \int_{-\infty}^{\infty} p(x) e^{-2\pi ixy} dx.$$

- Derivative of  $\hat{g}$  is the Fourier transform of  $2\pi ixg(x)$ ; differentiation (hard) is converted to multiplication (easy).

$$\hat{g}'(y) = \int_{-\infty}^{\infty} 2\pi ix \cdot g(x) e^{-2\pi ixy} dx.$$

If  $g$  is a probability density,  $\hat{g}'(0) = 2\pi i\mathbb{E}[x]$  and  $\hat{g}''(0) = -4\pi^2\mathbb{E}[x^2]$ .

- Natural to use the Fourier transform to analyze probability distributions. The mean and variance are simple multiples of the derivatives of  $\hat{p}$  at zero:  $\hat{p}'(0) = 0$ ,  $\hat{p}''(0) = -4\pi^2$ .
- We Taylor expand  $\hat{p}$  (need technical conditions on  $p$ ):

$$\hat{p}(y) = 1 + \frac{p''(0)}{2} y^2 + \dots = 1 - 2\pi^2 y^2 + O(y^3).$$

Near the origin, the above shows  $\hat{p}$  looks like a concave down parabola.

## Appendix III: Proof of the Central Limit Theorem (cont)

- $\text{Prob}(X_1 + \dots + X_N \in [a, b]) = \int_a^b (p * \dots * p)(z) dz.$
- The Fourier transform converts convolution to multiplication. If  $\text{FT}[f](y)$  denotes the Fourier transform of  $f$  evaluated at  $y$ :

$$\text{FT}[p * \dots * p](y) = \hat{p}(y) \cdot \dots \cdot \hat{p}(y).$$

- Do not want the distribution of  $X_1 + \dots + X_N = x$ , but rather

$$S_N = \frac{X_1 + \dots + X_N}{\sqrt{N}} = x.$$

- If  $B(x) = A(cx)$  for some fixed  $c \neq 0$ , then  $\hat{B}(y) = \frac{1}{c} \hat{A}\left(\frac{y}{c}\right).$

- $\text{Prob}\left(\frac{X_1 + \dots + X_N}{\sqrt{N}} = x\right) = (\sqrt{N}p * \dots * \sqrt{N}p)(x\sqrt{N}).$

- $\text{FT}\left[(\sqrt{N}p * \dots * \sqrt{N}p)(x\sqrt{N})\right](y) = \left[\hat{p}\left(\frac{y}{\sqrt{N}}\right)\right]^N.$

## Appendix III: Proof of the Central Limit Theorem (cont)

- Can find the Fourier transform of the distribution of  $S_N$ :

$$\left[ \widehat{p} \left( \frac{y}{\sqrt{N}} \right) \right]^N.$$

- Take the limit as  $N \rightarrow \infty$  for **fixed**  $y$ .
- Know  $\widehat{p}(y) = 1 - 2\pi^2 y^2 + O(y^3)$ . Thus study

$$\left[ 1 - \frac{2\pi^2 y^2}{N} + O \left( \frac{y^3}{N^{3/2}} \right) \right]^N.$$

- For any **fixed**  $y$ ,

$$\lim_{N \rightarrow \infty} \left[ 1 - \frac{2\pi^2 y^2}{N} + O \left( \frac{y^3}{N^{3/2}} \right) \right]^N = e^{-2\pi^2 y^2}.$$

- Fourier transform of  $e^{-2\pi^2 y^2}$  at  $x$  is  $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ .

## Appendix III: Proof of the Central Limit Theorem (cont)

We have shown:

- the Fourier transform of the distribution of  $S_N$  converges to  $e^{-2\pi y^2}$ ;
- the Fourier transform of  $e^{-2\pi y^2}$  is  $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ .

Therefore the distribution of  $S_N$  equalling  $x$  converges to  $\frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ .

We need complex analysis to justify this conclusion. Must be careful: Consider

$$g(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

All the Taylor coefficients about  $x = 0$  are zero, but the function is not identically zero in a neighborhood of  $x = 0$ .

## Appendix IV: Best Fit Weibulls from Method of Maximum Likelihood

The likelihood function depends on:  $\alpha_{RS}, \alpha_{RA}, \beta = -.5, \gamma$ .

Let  $A(\alpha, -.5, \gamma, k)$  denote the area in Bin( $k$ ) of the Weibull with parameters  $\alpha, -.5, \gamma$ . The sample likelihood function

$L(\alpha_{RS}, \alpha_{RA}, -.5, \gamma)$  is

$$\begin{aligned} & \binom{\#Games}{RS_{obs}(1), \dots, RS_{obs}(\#Bins)} \prod_{k=1}^{\#Bins} A(\alpha_{RS}, -.5, \gamma, k)^{RS_{obs}(k)} \\ & \cdot \binom{\#Games}{RA_{obs}(1), \dots, RA_{obs}(\#Bins)} \prod_{k=1}^{\#Bins} A(\alpha_{RA}, -.5, \gamma, k)^{RA_{obs}(k)}. \end{aligned}$$

For each team we find the values of the parameters  $\alpha_{RS}, \alpha_{RA}$  and  $\gamma$  that maximize the likelihood. Computationally, it is equivalent to maximize the logarithm of the likelihood, and we may ignore the multinomial coefficients as they are independent of the parameters.