

# On the probability that random graphs are Ramanujan

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Slides and paper available at  
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Expanders and Ramanujan Graphs:  
Construction and Applications  
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## Conjectures

## Expanders and Eigenvalues

- Expanding Constant

$$h(G) := \inf \left\{ \frac{|\partial U|}{\min(|U|, |V \setminus U|)} : U \subset V, |U| > 0 \right\}$$

- $\{G_m\}$  family of expanders if  $\exists \epsilon$  with  $h(G_m) \geq \epsilon$  and  $|G_m| \rightarrow \infty$ .

- Cheeger-Buser Inequalities

$$\frac{d - \lambda_2(G)}{2} \leq h(G) \leq 2\sqrt{2d(d - \lambda_2(G))}$$

- Applications: sparse ( $|E|$  grows at most linearly with  $|V|$ ), highly connected.
  - ◇ communication network theory:
    - superconcentrators, nonblocking networks
  - ◇ coding theory, cryptography.

## Known and conjectured results for $\lambda_2$

- (Alon-Boppana, Burger, Serre)  $\{G_m\}$  family of finite connected  $d$ -regular graphs,  $\lim_{m \rightarrow \infty} |G_m| = \infty$ :

$$\liminf_{m \rightarrow \infty} \lambda_2(G_m) \geq 2\sqrt{d-1}$$

- As  $|G| \rightarrow \infty$ , for  $d \geq 3$  and any  $\epsilon > 0$ , "most"  $d$ -regular graphs  $G$  have

$$\lambda_2(G) \leq 2\sqrt{d-1} + \epsilon$$

(conjectured by Alon, proved for many families by Friedman).

## Questions

For a family of  $d$ -regular graphs:

- What is the *distribution* of  $\lambda_2$ ?
- What *percent* of the graphs are Ramanujan?

$\lambda(G) = \max(\lambda_+(G), \lambda_-(G))$ , where  $\lambda_{\pm}(G)$  are largest non-trivial positive (negative) eigenvalues. If bipartite  $\lambda_-(G) = -\lambda_+(G)$ . If connected  $\lambda_2(G) = \lambda_+(G)$ .

## Families Investigated ( $N$ even)

- $\mathcal{CI}_{N,d}$ :  $d$ -regular connected graphs generated by choosing  $d$  perfect matchings.
- $\mathcal{SCI}_{N,d}$ : subset of  $\mathcal{CI}_{N,d}$  that are simple.
- $\mathcal{CB}_{N,d}$ :  $d$ -regular connected bipartite graphs generated by choosing  $d$  permutations.
- $\mathcal{SCB}_{N,d}$ : subset of  $\mathcal{CB}_{N,d}$  that are simple.

## Tracy-Widom Distribution

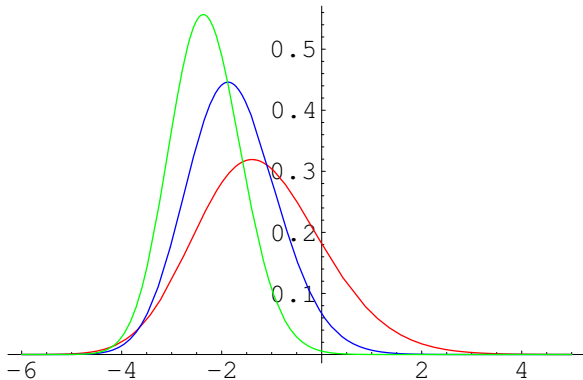
Limiting distribution of the normalized largest eigenvalues for ensembles of matrices: GOE ( $\beta = 1$ ), GUE ( $\beta = 2$ ), GSE ( $\beta = 4$ )

### Applications

- Length of largest increasing subsequence of random permutations.
- Largest principle component of covariances matrices.
- Young tableaux, random tilings, queuing theory, superconductors....

## Tracy-Widom Plots

Plots of the three Tracy-Widom distributions:  $f_1(s)$  is red,  $f_2(s)$  is blue and  $f_4(s)$  is green.



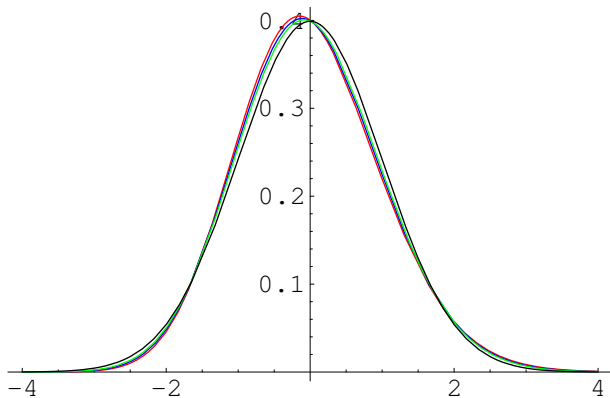
## Tracy-Widom Distributions

Parameters for the Tracy-Widom distributions.  $F_\beta$  is the cumulative distribution function for  $f_\beta$ , and  $F_\beta(\mu_\beta)$  is the mass of  $f_\beta$  to the left of its mean.

	Mean $\mu$	Std Dev $\sigma$	$F_\beta(\mu_\beta)$
TW( $\beta = 1$ )	-1.21	1.268	0.5197
TW( $\beta = 2$ )	-1.77	0.902	0.5150
TW( $\beta = 4$ )	-2.31	0.720	0.5111
Std Normal	0.00	1.000	0.5000

## Normalized Tracy-Widom Plots

Plots normalized to have mean 0 and variance 1:  $f_1^{\text{norm}}(s)$  is red,  $f_2^{\text{norm}}(s)$  is blue,  $f_4^{\text{norm}}(s)$  is green, standard normal is black.



## Conjectures

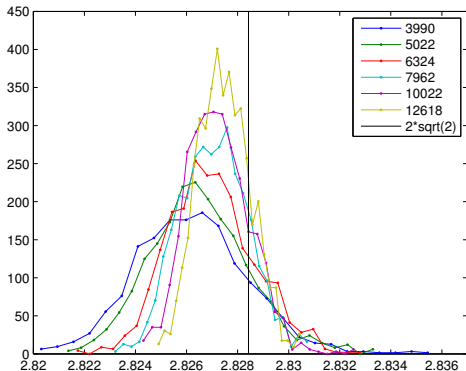
### Conjectures

- The distribution of  $\lambda_{\pm}(G)$  converges to the  $\beta = 1$  Tracy-Widom distribution as  $N \rightarrow \infty$  in all studied families.
- For non-bipartite families,  $\lambda_{\pm}(G)$  are independent.
- The percent of the graphs that are Ramanujan approaches 52% as  $N \rightarrow \infty$  (resp., 27%) in bipartite (resp., non-bipartite) families.

Evidence weaker for  $\mathcal{CB}_{N,d}$  ( $d$ -regular connected bipartite graphs, not necessarily simple).

## Distribution of $\lambda_+(G)$

Distribution of  $\lambda_+(G)$  for 1000 graphs randomly chosen from  $\mathcal{CI}_{N,3}$  for various  $N$  (vertical line is  $2\sqrt{2}$ ).



## Statistical evidence for conjectures

- Well-modeled by Tracy-Widom with  $\beta = 1$ .
- Means approach  $2\sqrt{d-1}$  according to power law.
- Variance approach 0 according to power law.
- Comparing the exponents of the power laws, see the number of standard deviations that  $2\sqrt{d-1}$  falls to the right of the mean goes to 0 as  $N \rightarrow \infty$ .
- $\lambda_{\pm}(G)$  appear independent in non-bipartite families.
- As  $N \rightarrow \infty$  the probability that a graph is Ramanujan is the mass of the Tracy-Widom distribution to the left of its mean (52%) if bipartite (27% otherwise).

## Power law exponents of means and standard deviations

- Means:  $\mu_{\mathcal{F}_{N,d}} \approx 2\sqrt{d-1} - c_{\mu,N,d} N^{m(\mathcal{F}_{N,d})}$
- Standard Deviations:  $\sigma_{\mathcal{F}_{N,d}} \approx c_{\sigma,N,d} N^{s(\mathcal{F}_{N,d})}$
- Thus  $2\sqrt{d-1} \approx \mu_{\mathcal{F}_{N,d}} + \frac{c_{\mu,N,d}}{c_{\sigma,N,d}} N^{m(\mathcal{F}_{N,d})-s(\mathcal{F}_{N,d})} \sigma_{\mathcal{F}_{N,d}}$

### Ramanujan Threshold

As  $N \rightarrow \infty$ , if  $m(\mathcal{F}_{N,d}) < s(\mathcal{F}_{N,d})$  then  $2\sqrt{d-1}$  falls zero standard deviations to the right of the mean.

## 3-Regular Graphs

## Experiments: Comparisons with Tracy-Widom Distribution

- Each set is 1000 random 3-regular graphs from  $\mathcal{CI}_{N,3}$  normalized to have mean 0 and variance 1.
- 19 degrees of freedom, critical values 30.1435 ( $\alpha = .05$ ) and 36.1908 ( $\alpha = .01$ ).
- Only showing subset of data.

$\chi^2$ -Tests of  $\lambda_+(G)$  for  $\mathcal{CI}_{N,3}$  versus Tracy-Widom DistributionsCritical values: 30.1 ( $\alpha = .05$ ), 36.2 ( $\alpha = .01$ ).

$N$	$TW_1^{\text{norm}}$	$TW_2^{\text{norm}}$	$TW_4^{\text{norm}}$	$N(0, 1)$
26	52.4	43.7	36.8	30.3
100	72.1	41.3	28.9	13.2
796	3.7	4.9	7.0	19.3
3168	17.4	19.6	24.0	61.3
6324	20.8	19.8	21.4	28.6
12618	9.9	9.3	10.6	17.2
20000	37.4	41.1	41.4	71.2
mean (all)	32.5	27.2	24.9	49.1
median (all)	20.0	19.1	18.0	25.2
mean (last 10)	22.3	24.9	29.1	66.7
median (last 10)	21.2	21.8	22.2	64.5

$\chi^2$ -Tests of  $\lambda_+(G)$  against  $\beta = 1$  Tracy-WidomCritical values: 30.1 ( $\alpha = .05$ ), 36.2 ( $\alpha = .01$ ).

$N$	$CI_{N,3}$	$SCI_{N,3}$	$CB_{N,3}$	$SCB_{N,3}$
26	52.4	111.6	142.7	14.3
100	72.1	19.8	23.4	18.5
796	3.7	14.9	20.9	19.6
3168	17.4	22.2	70.6	25.4
12618	9.9	13.1	36.9	13.7
20000	37.4	14.9	27.4	12.1
mean (all)	32	21	78	19
standard deviation (all)	42	18	180	7
mean (last 10)	22	17	44	17
standard deviation (last 10)	8	5	37	8
mean (last 5)	22	17	32	14
standard deviation (last 5)	10	4	23	1

## Experiment: Mass to the left of the mean for $\lambda_+(G)$

- Each set of 1000 3-regular graphs.
- mass to the left of the mean of the Tracy-Widom distributions:
  - ◇ 0.519652 ( $\beta = 1$ )
  - ◇ 0.515016 ( $\beta = 2$ )
  - ◇ 0.511072 ( $\beta = 4$ )
  - ◇ 0.500000 (standard normal).
- two-sided z-test: critical thresholds: 1.96 (for  $\alpha = .05$ ) and 2.575 (for  $\alpha = .01$ ).

## Experiment: Mass to the left of the mean for $\mathcal{CI}_{N,3}$

Critical values: 1.96 ( $\alpha = .05$ ), 2.575 ( $\alpha = .01$ ).

$N$	Obs mass	$Z_{TW,1}$	$Z_{TW,2}$	$Z_{TW,4}$	$Z_{StdNorm}$
26	0.483	-2.320	-2.026	-1.776	-1.075
100	0.489	-1.940	-1.646	-1.396	-0.696
796	0.521	0.085	0.379	0.628	1.328
6324	0.523	0.212	0.505	0.755	1.455
20000	0.526	0.402	0.695	0.944	1.644
$\mu$ (last 10)	0.518	0.473	0.531	0.655	1.202
$\tilde{\mu}$ (last 10)	0.523	0.411	0.537	0.755	1.455
$\mu$ (last 5)	0.517	0.591	0.532	0.630	1.050
$\tilde{\mu}$ (last 5)	0.515	0.421	0.695	0.700	0.949

## Experiment: Mass left of mean: 3-Regular, sets of 100,000

Discarded: Matlab's algorithm didn't converge.  
Critical values: 1.96 ( $\alpha = .05$ ), 2.575 ( $\alpha = .01$ ).

$CI_{N,3}$	$Z_{TW,1}$	$Z_{TW,2}$	$Z_{TW,4}$	$Z_{StdNorm}$	Discarded
1002	0.239	3.173	5.667	12.668	0
2000	-0.128	2.806	5.300	12.301	0
5022	1.265	4.198	6.692	13.693	0
10022	0.391	3.324	5.819	12.820	0
40000	2.334	5.267	7.761	14.762	0

$SCI_{N,3}$	$Z_{TW,1}$	$Z_{TW,2}$	$Z_{TW,4}$	$Z_{StdNorm}$	Discarded
1002	-1.451	1.483	3.978	10.979	0
2000	-0.457	2.477	4.971	11.972	0
5022	-0.042	2.891	5.386	12.387	1

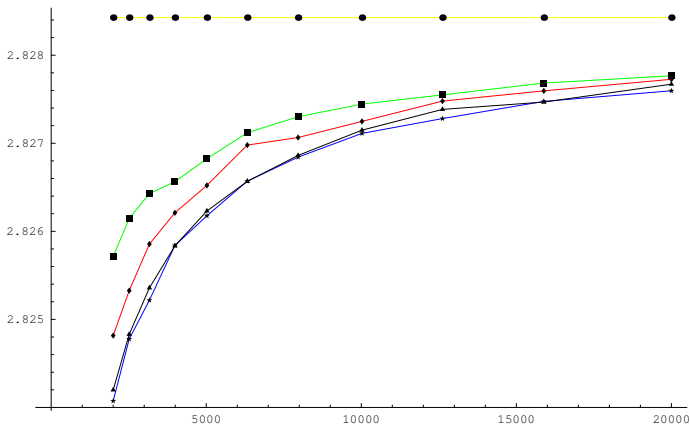
## Experiment: Mass left of mean: 3-Regular, sets of 100,000

Critical values: 1.96 ( $\alpha = .05$ ), 2.575 ( $\alpha = .01$ ).

$CB_{N,3}$	$Z_{TW,1}$	$Z_{TW,2}$	$Z_{TW,4}$	$Z_{StdNorm}$	Discarded
1002	3.151	6.083	8.577	15.577	0
2000	3.787	6.719	9.213	16.213	1
5022	3.563	6.495	8.989	15.989	4
10022	2.049	4.982	7.476	14.477	0
12618	3.701	6.634	9.127	16.128	0
15886	2.999	5.931	8.425	15.426	0
20000	2.106	5.039	7.533	14.534	0
40000	1.853	4.786	7.280	14.281	0
$SCB_{N,3}$	$Z_{TW,1}$	$Z_{TW,2}$	$Z_{TW,4}$	$Z_{StdNorm}$	Discarded
1002	-1.963	0.971	3.465	10.467	0
2000	-0.767	2.167	4.661	11.663	2
5022	-0.064	2.869	5.364	12.365	4

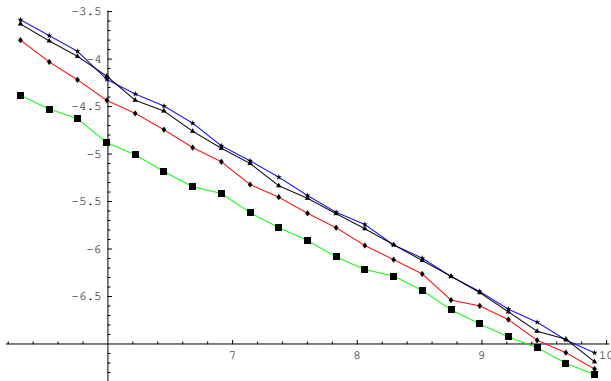
## 3-regular graphs: Sample means of $\lambda_+(G)$

Sets of 1000 random 3-regular graphs.  $\mathcal{CI}_{N,3}$  is red,  $\mathcal{SCI}_{N,3}$  is blue,  $\mathcal{CB}_{N,3}$  is green,  $\mathcal{SCB}_{N,3}$  is black; the solid yellow line is  $2\sqrt{2} \approx 2.8284$ .



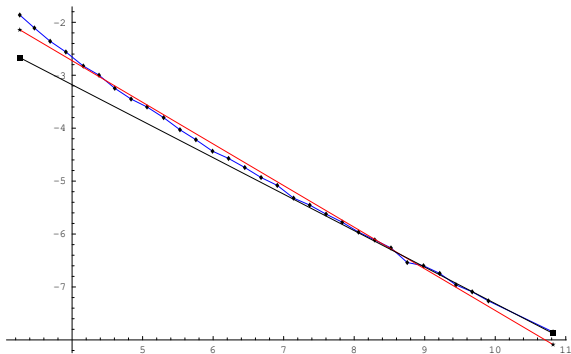
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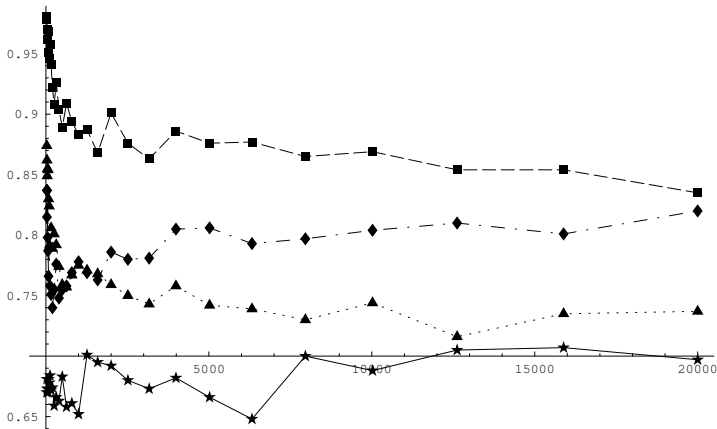
## 3-regular graphs: best fit means of $\lambda_+(G)$

Logarithm of the mean on  $\log(c_{\mu, N, 3} N^{m(CI_{N, 3})})$  on  $N$ . Blue: data; red: best fit (all); black: best fit (last 10).



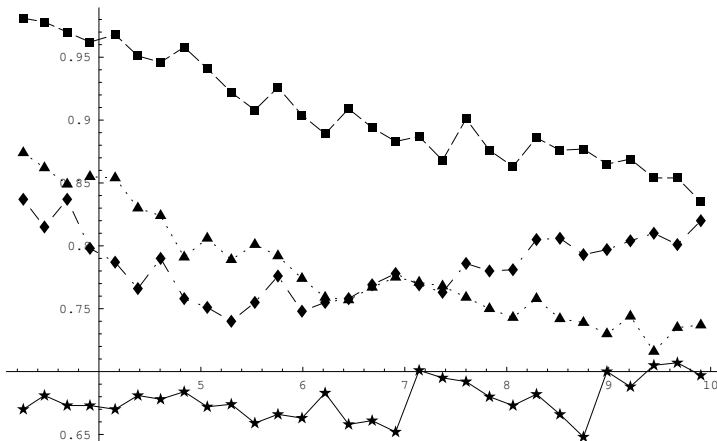
## 3-regular graphs: percent Ramanujan

Each set is 1000 random 3-regular graphs with  $N$  vertices.  $CI_{N,3}$  are stars,  $SCI_{N,3}$  are diamonds,  $CB_{N,3}$  are triangles,  $SCB_{N,3}$  are boxes.



## 3-regular graphs: percent Ramanujan

Each set is 1000 random 3-regular graphs with  $N$  vertices.  $CI_{N,3}$  are stars,  $SCI_{N,3}$  are diamonds,  $CB_{N,3}$  are triangles,  $SCB_{N,3}$  are boxes.



## Best-fit exponents ( $d = 3$ ) for $\lambda_+(G)$

First table means  $m(\mathcal{F})$ , second std devs  $s(\mathcal{F})$ .

**Bold entries:  $m(\mathcal{F}) > s(\mathcal{F})$ .**

$N$	$CI_{N,3}$	$SCI_{N,3}$	$CB_{N,3}$	$SCB_{N,3}$
{26, ..., 20000}	-0.795	-0.828	-0.723	-0.833
{80, ..., 20000}	-0.761	-0.790	-0.671	-0.789
{252, ..., 20000}	-0.735	-0.762	-0.638	-0.761
{26, ..., 64}	-1.058	-1.105	-1.065	-1.151
{80, ..., 200}	-0.854	-0.949	-0.982	-0.968
{232, ..., 632}	-0.773	-0.840	-0.737	-0.842
{796, ..., 2000}	-0.762	-0.805	-0.649	-0.785
{2516, ..., 6324}	-0.791	-0.741	-0.579	-0.718
{7962, ..., 20000}	-0.728	-0.701	-0.584	-0.757
$N$	$CI_{N,3}$	$SCI_{N,3}$	$CB_{N,3}$	$SCB_{N,3}$
{26, ..., 20000}	-0.713	-0.725	-0.709	-0.729
{80, ..., 20000}	-0.693	-0.703	<b>-0.697</b>	-0.706
{252, ..., 20000}	-0.679	-0.691	<b>-0.688</b>	-0.696
{26, ..., 64}	-0.863	-0.918	-0.794	-0.957
{80, ..., 200}	-0.694	-0.752	-0.719	-0.750
{232, ..., 632}	-0.718	-0.716	-0.714	-0.734
{796, ..., 2000}	-0.602	-0.648	<b>-0.705</b>	-0.763
{2516, ..., 6324}	-0.614	-0.668	<b>-0.770</b>	-0.688
{7962, ..., 20000}	-0.543	<b>-0.716</b>	<b>-0.671</b>	-0.648







# Best-fit exponents ( $d = 3$ ) for $\lambda_+(G)$







$$2\sqrt{d-1} \approx \mu_{\mathcal{F}_{N,d}} + \frac{c_{\mu,N,d}}{c_{\sigma,N,d}} N^{m(\mathcal{F}_{N,d})-s(\mathcal{F}_{N,d})} \sigma_{\mathcal{F}_{N,d}}$$






$m(\mathcal{F}_{N,d}) - s(\mathcal{F}_{N,d})$ , **Bold entries  $m(\mathcal{F}) > s(\mathcal{F})$ .**






$N$	$CI_{N,3}$	$SCI_{N,3}$	$CB_{N,3}$	$SCB_{N,3}$
{26, ..., 20000}	-0.082	-0.103	-0.014	-0.104
{80, ..., 20000}	-0.068	-0.087	<b>0.026</b>	-0.083
{252, ..., 20000}	-0.056	-0.071	<b>0.050</b>	-0.065
{26, ..., 64}	-0.195	-0.187	-0.271	-0.194
{80, ..., 200}	-0.160	-0.197	-0.263	-0.218
{232, ..., 632}	-0.055	-0.124	-0.023	-0.108
{796, ..., 2000}	-0.160	-0.157	<b>0.056</b>	-0.022
{2516, ..., 6324}	-0.177	-0.073	<b>0.191</b>	-0.030
{7962, ..., 20000}	-0.185	<b>0.015</b>	<b>0.087</b>	-0.109







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