

How low can we go? Lower order terms in CLTs from Benford's Law to Elliptic Curves

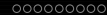
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NES MAA Fall 2009 Meeting, November 21, 2009

Outline

- Review of Central Limit Theorem results.
- Applications to Benford's Law.
- Applications to Elliptic Curves.



Central Limit Theorems

General Statement

Central Limit Theorem

Let X_1, \dots, X_N be **nice** iidrv with mean μ and variance σ^2 .
Then

$$Y_N = \frac{X_1 + \dots + X_N}{N}, \quad \lim_{N \rightarrow \infty} \frac{Y_N - \mathbb{E}[Y_N]}{\text{StDev}(Y_N)} \rightarrow N(0, 1).$$

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$$k^{\text{th}} \text{ centered moment} = \int_{-\infty}^{\infty} (x - \mu)^k f(x) dx.$$

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- What is nice? Finite higher moments:

$$k^{\text{th}} \text{ centered moment} = \int_{-\infty}^{\infty} (x - \mu)^k f(x) dx.$$

- Speed of convergence controlled by higher moments (especially third).

Overview of Benford's Law

Benford's Law: Newcomb (1881), Benford (1938)

Statement

For many data sets, probability of observing a first digit of d base B is $\log_B \left(\frac{d+1}{d} \right)$; base 10 about 30% are 1s.

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- Not all data sets satisfy Benford's Law.
 - ◇ Long street $[1, L]$: $L = 199$ versus $L = 999$.
 - ◇ Oscillates between $1/9$ and $5/9$ with first digit 1.
 - ◇ **Many streets of different sizes: close to Benford.**

Examples

- recurrence relations
- special functions (such as $n!$)
- iterates of power, exponential, rational maps
- products of random variables
- L -functions, characteristic polynomials
- iterates of the $3x + 1$ map
- differences of order statistics
- hydrology and financial data
- many hierarchical Bayesian models

Applications

- analyzing round-off errors
- determining the optimal way to store numbers
- detecting tax and image fraud, and data integrity

Mantissas

Mantissa: $x = M_{10}(x) \cdot 10^k$, k integer.



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Key observation: $\log_{10}(x) = \log_{10}(\tilde{x}) \pmod{1}$ if and only if x and \tilde{x} have the same leading digits. Thus often study $y = \log_{10} x$.

Equidistribution and Benford's Law

Equidistribution

$\{y_n\}_{n=1}^{\infty}$ is equidistributed modulo 1 if probability $y_n \bmod 1 \in [a, b]$ tends to $b - a$:

$$\frac{\#\{n \leq N : y_n \bmod 1 \in [a, b]\}}{N} \rightarrow b - a.$$

Equidistribution and Benford's Law

Equidistribution

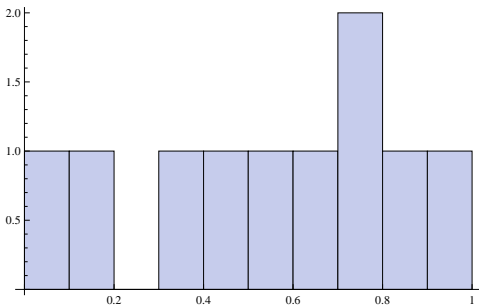
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Theorem: Kronecker, Weyl

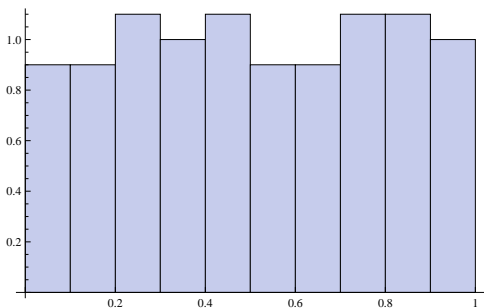
$\beta \notin \mathbb{Q}$, $n\beta$ is equidistributed mod 1.

Example of Equidistribution: $n\sqrt{\pi} \bmod 1$



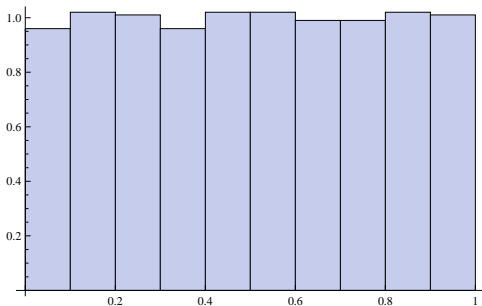
$n\sqrt{\pi} \bmod 1$ for $n \leq 10$

Example of Equidistribution: $n\sqrt{\pi} \bmod 1$



$n\sqrt{\pi} \bmod 1$ for $n \leq 100$

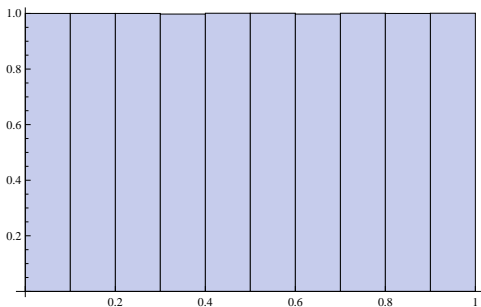
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Example of Equidistribution: $n\sqrt{\pi} \bmod 1$



$n\sqrt{\pi} \bmod 1$ for $n \leq 10,000$

Logarithms and Benford's Law

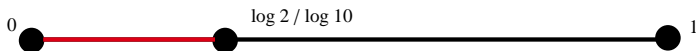
Fundamental Equivalence

Data set $\{x_i\}$ is Benford base B if $\{y_i\}$ is equidistributed mod 1, where $y_i = \log_B x_i$.

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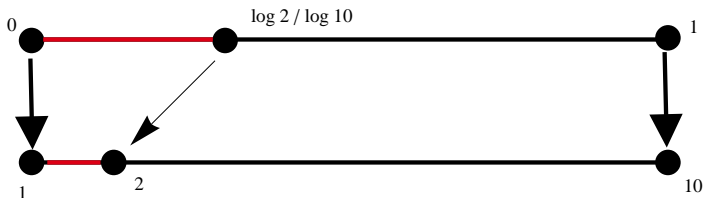
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Benford's Law and the
 $3x + 1$ Problem

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- $7 \rightarrow_1 11 \rightarrow_1 17 \rightarrow_2 13 \rightarrow_3 5 \rightarrow_4 1 \rightarrow_2 1$,
 2-path $(1, 1)$, 5-path $(1, 1, 2, 3, 4)$.
m-path: (k_1, \dots, k_m) .

Structure Theorem: Sinai, Kontorovich-Sinai

Theorem (Sinai, Kontorovich-Sinai)

k_i -values are i.i.d.r.v. (geometric, $1/2$):

$$\mathbb{P} \left(\frac{\log_2 \left[\frac{x_m}{\left(\frac{3}{4}\right)^m x_0} \right]}{\sqrt{2m}} \leq a \right) = \mathbb{P} \left(\frac{S_m - 2m}{\sqrt{2m}} \leq a \right)$$

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$3x + 1$ and Benford

Theorem (Kontorovich and M–, 2005)

As $m \rightarrow \infty$, $x_m/(3/4)^m x_0$ is Benford.

Theorem (Lagarias-Soundararajan 2006)

$X \geq 2^N$, for all but at most $c(B)N^{-1/36}$ X initial seeds the distribution of the first N iterates of the $3x + 1$ map are within $2N^{-1/36}$ of the Benford probabilities.

Sketch of the proof

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- Failed Proof: lattices, bad errors.
- CLT: $(S_m - 2m)/\sqrt{2m} \rightarrow N(0, 1)$.
- Quantified Equidistribution: $I_\ell = \{\ell M, \dots, (\ell + 1)M - 1\}$,
 $M \ll m^{1/2}$
 $\log_B 2$ of irrationality type $\kappa < \infty$:

$$\#\{k \in I_\ell : k \log_B 2 \bmod 1 \in [a, b]\} = M(b - a) + O(M^r),$$

$$r = 1 + \epsilon - 1/\kappa < 1.$$

Irrationality Type

Irrationality type

α has irrationality type κ if κ is the supremum of all γ with

$$\liminf_{q \rightarrow \infty} q^{\gamma+1} \min_p \left| \alpha - \frac{p}{q} \right| = 0.$$

- Algebraic irrationals: type 1 (Roth's Thm).
- Theory of Linear Forms: $\log_B 2$ of finite type.

Linear Forms

Theorem (Baker)

$\alpha_1, \dots, \alpha_n$ algebraic numbers height $A_j \geq 4$, $\beta_1, \dots, \beta_n \in \mathbb{Q}$ with height at most $B \geq 4$,

$$\Lambda = \beta_1 \log \alpha_1 + \dots + \beta_n \log \alpha_n.$$

If $\Lambda \neq 0$ then $|\Lambda| > B^{-C\Omega \log \Omega'}$, with $d = [\mathbb{Q}(\alpha_i, \beta_j) : \mathbb{Q}]$,
 $C = (16nd)^{200n}$, $\Omega = \prod_j \log A_j$, $\Omega' = \Omega / \log A_n$.

Gives $\log_{10} 2$ of finite type, with $\kappa < 1.2 \cdot 10^{602}$:

$$|\log_{10} 2 - p/q| = |q \log 2 - p \log 10| / q \log 10.$$

$3x + 1$ Data: random 10,000 digit number, $2^k \parallel 3x + 1$

80,514 iterations ($((4/3)^n = a_0$ predicts 80,319);
 $\chi^2 = 13.5$ (5% 15.5).

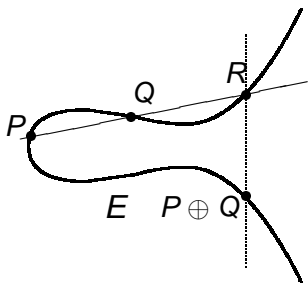
| Digit | Number | Observed | Benford |
|-------|--------|----------|---------|
| 1 | 24251 | 0.301 | 0.301 |
| 2 | 14156 | 0.176 | 0.176 |
| 3 | 10227 | 0.127 | 0.125 |
| 4 | 7931 | 0.099 | 0.097 |
| 5 | 6359 | 0.079 | 0.079 |
| 6 | 5372 | 0.067 | 0.067 |
| 7 | 4476 | 0.056 | 0.058 |
| 8 | 4092 | 0.051 | 0.051 |
| 9 | 3650 | 0.045 | 0.046 |

Elliptic Curves

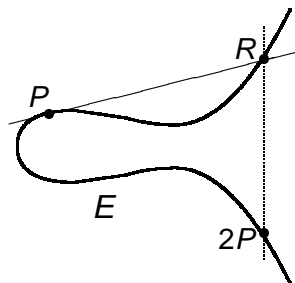
Mordell-Weil Group

Elliptic curve $y^2 = x^3 + ax + b$ with rational solutions

$P = (x_1, y_1)$ and $Q = (x_2, y_2)$ and connecting line $y = mx + b$.



Addition of distinct points P and Q



Adding a point P to itself

$$E(\mathbb{Q}) \approx E(\mathbb{Q})_{\text{tors}} \oplus \mathbb{Z}^r$$

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \operatorname{Re}(s) > 1.$$

Functional Equation:

$$\xi(s) = \Gamma\left(\frac{s}{2}\right) \pi^{-\frac{s}{2}} \zeta(s) = \xi(1-s).$$

Riemann Hypothesis (RH):

All non-trivial zeros have $\operatorname{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

General L -functions

$$L(s, f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} = \prod_{p \text{ prime}} L_p(s, f)^{-1}, \quad \operatorname{Re}(s) > 1.$$

Functional Equation:

$$\Lambda(s, f) = \Lambda_{\infty}(s, f)L(s, f) = \Lambda(1 - s, f).$$

Generalized Riemann Hypothesis (GRH):

All non-trivial zeros have $\operatorname{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

Elliptic curve L -function

$E : y^2 = x^3 + ax + b$, associate L -function

$$L(s, E) = \sum_{n=1}^{\infty} \frac{a_E(n)}{n^s} = \prod_{p \text{ prime}} L_E(p^{-s}),$$

where

$$a_E(p) = p - \#\{(x, y) \in (\mathbb{Z}/p\mathbb{Z})^2 : y^2 \equiv x^3 + ax + b \pmod{p}\}.$$

Birch and Swinnerton-Dyer Conjecture

Rank of group of rational solutions equals order of vanishing of $L(s, E)$ at $s = 1/2$.



1-Level Density

L -function $L(s, f)$: by RH non-trivial zeros $\frac{1}{2} + i\gamma_{f,j}$.

N_f : analytic conductor.

$\varphi(x)$: compactly supported even Schwartz function.

$$D_{1,f}(\varphi) = \sum_j \varphi\left(\frac{\log N_f}{2\pi} \gamma_{f,j}\right)$$

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Katz-Sarnak Conjecture:

$$\begin{aligned} D_{1,\mathcal{F}}(\varphi) &= \lim_{N \rightarrow \infty} \frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} D_{1,f}(\varphi) = \int \varphi(\mathbf{x}) \rho_{G(\mathcal{F})}(\mathbf{x}) d\mathbf{x} \\ &= \int \widehat{\varphi}(u) \widehat{\rho}_{G(\mathcal{F})}(u) du. \end{aligned}$$

Explicit Formula (Contour Integration)

$$-\frac{\zeta'(s)}{\zeta(s)} = -\frac{d}{ds} \log \zeta(s) = -\frac{d}{ds} \log \prod_p (1 - p^{-s})^{-1}$$

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Contour Integration:

$$\int -\frac{\zeta'(s)}{\zeta(s)} \frac{x^s}{s} ds \quad \text{vs} \quad \sum_p \log p \int \left(\frac{x}{p}\right)^s \frac{ds}{s}.$$

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Contour Integration (see Fourier Transform arising):

$$\int -\frac{\zeta'(s)}{\zeta(s)} \phi(s) ds \quad \text{vs} \quad \sum_p \log p \int \phi(s) e^{-\sigma \log p} e^{-it \log p} ds.$$

Interplay b/w zeros and coefficients

1-Level Expansion

$$\begin{aligned}
 D_{1,\mathcal{F}}(\phi) &= \frac{1}{|\mathcal{F}|} \sum_{E \in \mathcal{F}} \sum_j \phi \left(\frac{\log N_E}{2\pi} \gamma_E^{(j)} \right) \\
 &= -\frac{2}{|\mathcal{F}|} \sum_{E \in \mathcal{F}} \sum_p \frac{\log p}{\log N_E} \frac{1}{p} \hat{\phi} \left(\frac{\log p}{\log N_E} \right) a_E(p) \\
 &\quad - \frac{2}{|\mathcal{F}|} \sum_{E \in \mathcal{F}} \sum_p \frac{\log p}{\log N_E} \frac{1}{p^2} \hat{\phi} \left(2 \frac{\log p}{\log N_E} \right) a_E^2(p) \\
 &\quad + \hat{\phi}(0) + \phi(0) + o \left(\frac{\log \log N_E}{\log N_E} \right)
 \end{aligned}$$

Inputs

One-parameter family:

$$\mathcal{E} : y^2 = x^3 + A(T)x + B(T), \quad A(T), B(T) \in \mathbb{Z}[T],$$

take $t \in \mathbb{Z}$.

Let

$$A_{r,\mathcal{F}}(p) = \sum_{t(p)} a_t^r(p), \quad r = 1 \text{ or } 2.$$

For many families

$$(1) : A_{1,\mathcal{F}}(p) = -rp + O(1)$$

$$(2) : A_{2,\mathcal{F}}(p) = p^2 + O(p^{3/2})$$

Main Term

Theorem: M- '04

For small support, one-param family of rank r over $\mathbb{Q}(T)$:

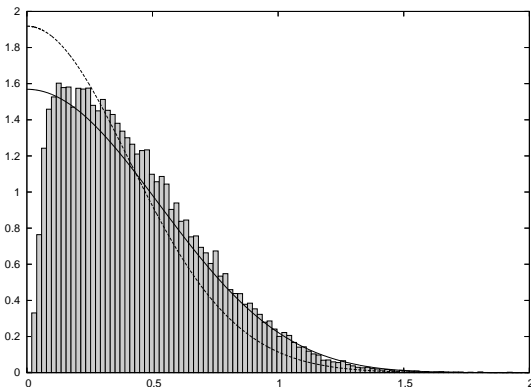
$$\lim_{N \rightarrow \infty} \frac{1}{|\mathcal{F}_N|} \sum_{E_t \in \mathcal{F}_N} \sum_j \varphi \left(\frac{\log N_{E_t}}{2\pi} \gamma_{E_t, j} \right) = \int \varphi(\mathbf{x}) \rho_{\mathcal{G}}(\mathbf{x}) d\mathbf{x} + r\varphi(0)$$

where

$$\mathcal{G} = \begin{cases} \text{SO} & \text{if half odd} \\ \text{SO}(\text{even}) & \text{if all even} \\ \text{SO}(\text{odd}) & \text{if all odd} \end{cases}$$

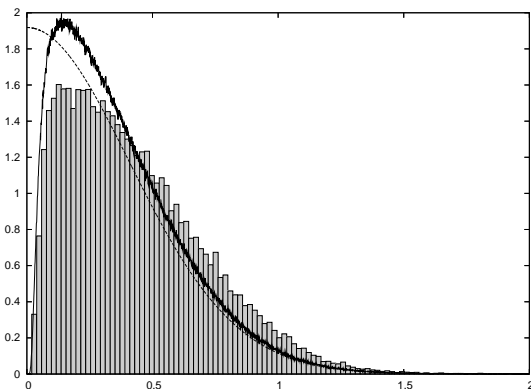
Confirm Katz-Sarnak for main term

Modeling lowest zero (data & calculations from Duc Khiem Huynh)



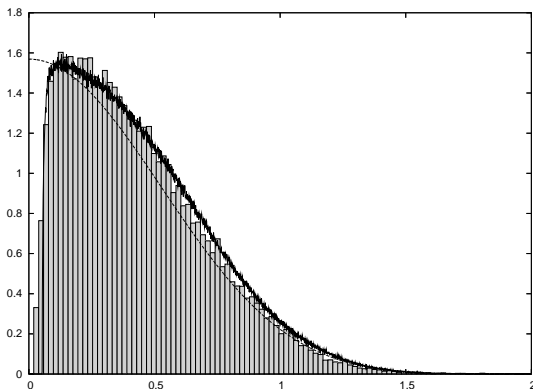
Lowest zero $L_{E_{11}}(s, \chi_d)$, $0 < d < 400,000$ (bar chart), lowest eigenvalue $SO(2N)$ w' N_{eff} (solid), standard N_0 (dashed).

Modeling lowest zero (data & calculations from Duc Khiem Huynh)



Lowest zero $L_{E_{11}}(s, \chi_d)$, $0 < d < 400,000$ (bar chart), lowest eigenvalue $SO(2N)$ w' $N_0 = 12$ (solid) w' discretisation and w' standard $N_0 = 12.26$ (dashed) w/o discretisation.

Modeling lowest zero (data & calculations from Duc Khiem Huynh)



Lowest zero $L_{E_{11}}(s, \chi_d)$, $0 < d < 400,000$ (bar chart), lowest eigenvalue $SO(2N)$ w' $N_{\text{eff}} = 2$ (solid) w' discretisation and w' $N_{\text{eff}} = 2.32$ (dashed) w/o discretisation.

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