

A Symplectic Test of the L -Functions Ratios Conjecture.

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History

- Farmer (1993): Considered

$$\int_0^T \frac{\zeta(s + \alpha)\zeta(1 - s + \beta)}{\zeta(s + \gamma)\zeta(1 - s + \delta)} dt,$$

conjectured (for appropriate values)

$$T \frac{(\alpha + \delta)(\beta + \gamma)}{(\alpha + \beta)(\gamma + \delta)} - T^{1-\alpha-\beta} \frac{(\delta - \beta)(\gamma - \alpha)}{(\alpha + \beta)(\gamma + \delta)}.$$

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- Conrey-Farmer-Zirnbauer (2007): conjecture formulas for averages of products of L -functions over families:

$$R_{\mathcal{F}} = \sum_{f \in \mathcal{F}} \omega_f \frac{L\left(\frac{1}{2} + \alpha, f\right)}{L\left(\frac{1}{2} + \gamma, f\right)}.$$

Uses of the Ratios Conjecture

- **Applications:**
 - ◇ n -level correlations and densities;
 - ◇ mollifiers;
 - ◇ moments;
 - ◇ vanishing at the central point;
- **Advantages:**
 - ◇ RMT models often add arithmetic ad hoc;
 - ◇ predicts lower order terms, often to square-root level.

Inputs for 1-level density

- **Approximate Functional Equation:**

$$L(s, f) = \sum_{m \leq x} \frac{a_m}{m^s} + \epsilon \mathbb{X}_L(s) \sum_{n \leq y} \frac{a_n}{n^{1-s}};$$

- ◇ ϵ sign of the functional equation,
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- **Explicit Formula:** g Schwartz test function,

$$\sum_{f \in \mathcal{F}} \omega_f \sum_{\gamma} g\left(\gamma \frac{\log N_f}{2\pi}\right) = \frac{1}{2\pi i} \int_{(c)} - \int_{(1-c)} R'_{\mathcal{F}}(\dots) g(\dots)$$

$$\diamond R'_{\mathcal{F}}(r) = \left. \frac{\partial}{\partial \alpha} R_{\mathcal{F}}(\alpha, \gamma) \right|_{\alpha=\gamma=r}.$$

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$$\frac{1}{L(s, f)} = \sum_h \frac{\mu_f(h)}{h^s},$$

where $\mu_f(h)$ is the multiplicative function equaling 1 for $h = 1$, $-\lambda_f(p)$ if $n = p$, $\chi_0(p)$ if $h = p^2$ and 0 otherwise.

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- Execute the sum over \mathcal{F} , keeping only main (diagonal) terms.
- Extend the m and n sums to infinity (complete the products).
- Differentiate with respect to the parameters.

Main Results

Symplectic Families

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 - ◇ even (resp., odd) if $d > 0$ (resp., $d < 0$).

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Will study following families:

- ◇ even fundamental discriminants at most X ;
- ◇ $\{8d : 0 < d \leq X, d \text{ an odd, positive square-free fundamental discriminant}\}$.

Prediction from Ratios Conjecture

$$\begin{aligned} \frac{1}{X^*} \sum_{d \leq X} \sum_{\gamma_d} g\left(\gamma_d \frac{\log X}{2\pi}\right) &= \frac{1}{X^* \log X} \int_{-\infty}^{\infty} g(\tau) \sum_{d \leq X} \left[\log \frac{d}{\pi} + \frac{1}{2} \frac{\Gamma'}{\Gamma} \left(\frac{1}{4} \pm \frac{i\pi\tau}{\log X} \right) \right] d\tau \\ &+ \frac{2}{X^* \log X} \sum_{d \leq X} \int_{-\infty}^{\infty} g(\tau) \left[\frac{\zeta'}{\zeta} \left(1 + \frac{4\pi i\tau}{\log X} \right) + A'_D \left(\frac{2\pi i\tau}{\log X}; \frac{2\pi i\tau}{\log X} \right) \right. \\ &\left. - e^{-2\pi i\tau \log(d/\pi)/\log X} \frac{\Gamma\left(\frac{1}{4} - \frac{\pi i\tau}{\log X}\right)}{\Gamma\left(\frac{1}{4} + \frac{\pi i\tau}{\log X}\right)} \zeta \left(1 - \frac{4\pi i\tau}{\log X} \right) A_D \left(-\frac{2\pi i\tau}{\log X}; \frac{2\pi i\tau}{\log X} \right) \right] d\tau + O(X^{-\frac{1}{2}+\epsilon}), \end{aligned}$$

with

$$\begin{aligned} A_D(-r, r) &= \prod_p \left(1 - \frac{1}{(p+1)p^{1-2r}} - \frac{1}{p+1} \right) \cdot \left(1 - \frac{1}{p} \right)^{-1} \\ A'_D(r; r) &= \sum_p \frac{\log p}{(p+1)(p^{1+2r} - 1)}. \end{aligned}$$

Prediction from Ratios Conjecture

Main term is

$$\frac{1}{X^*} \sum_{d \leq X} \sum_{\gamma_d} g\left(\gamma_d \frac{\log X}{2\pi}\right) = \int_{-\infty}^{\infty} g(x) \left(1 - \frac{\sin(2\pi x)}{2\pi x}\right) dx + O\left(\frac{1}{\log X}\right),$$

which is the 1-level density for the scaling limit of $\mathrm{USp}(2N)$. If $\mathrm{supp}(\hat{g}) \subset (-1, 1)$, then the integral of $g(x)$ against $-\sin(2\pi x)/2\pi x$ is $-g(0)/2$.

Prediction from Ratios Conjecture

Assuming RH for $\zeta(s)$, for $\text{supp}(\widehat{g}) \subset (-\sigma, \sigma) \subset (-1, 1)$:

$$\begin{aligned} & \frac{-2}{X^* \log X} \sum_{d \leq X} \int_{-\infty}^{\infty} g(\tau) e^{-2\pi i \tau \frac{\log(d/\pi)}{\log X}} \frac{\Gamma\left(\frac{1}{4} - \frac{\pi i \tau}{\log X}\right)}{\Gamma\left(\frac{1}{4} + \frac{\pi i \tau}{\log X}\right)} \zeta\left(1 - \frac{4\pi i \tau}{\log X}\right) A_D\left(-\frac{2\pi i \tau}{\log X}; \frac{2\pi i \tau}{\log X}\right) d\tau \\ &= -\frac{g(0)}{2} + O\left(X^{-\frac{3}{4}(1-\sigma)+\epsilon}\right); \end{aligned}$$

the error term may be absorbed into the $O(X^{-1/2+\epsilon})$ error if $\sigma < 1/3$.

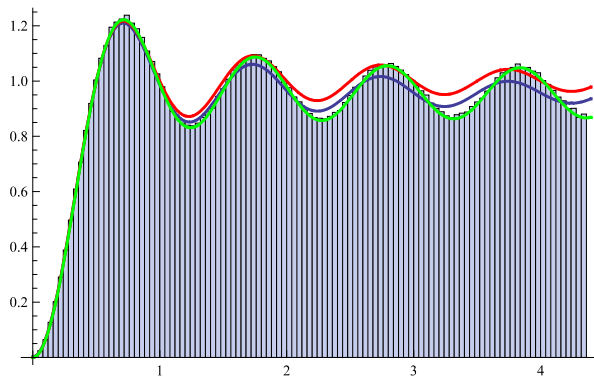
Main Results

Theorem (M– '07)

Let $\text{supp}(\widehat{g}) \subset (-\sigma, \sigma)$, assume RH for $\zeta(s)$. 1-Level Density agrees with prediction from Ratios Conjecture

- *up to $O(X^{-(1-\sigma)/2+\epsilon})$ for the family of quadratic Dirichlet characters with even fundamental discriminants at most X ;*
- *up to $O(X^{-1/2} + X^{-(1-\frac{3}{2}\sigma)+\epsilon} + X^{-\frac{3}{4}(1-\sigma)+\epsilon})$ for our sub-family. If $\sigma < 1/3$ then agrees up to $O(X^{-1/2+\epsilon})$.*

Numerics (J. Stopple): 1,003,083 negative fundamental discriminants $-d \in [10^{12}, 10^{12} + 3.3 \cdot 10^6]$



Histogram of normalized zeros ($\gamma \leq 1$, about 4 million).

- ◇ Red: main term.
- ◇ Blue: includes $O(1/\log X)$ terms.
- ◇ Green: all lower order terms.

Sketch of Proofs

Ratios Calculation

Hardest piece to analyze is

$$R(g; X) = -\frac{2}{X^* \log X} \sum_{d \leq X} \int_{-\infty}^{\infty} g(\tau) e^{-2\pi i \tau \frac{\log(d/\pi)}{\log X}} \frac{\Gamma\left(\frac{1}{4} - \frac{\pi i \tau}{\log X}\right)}{\Gamma\left(\frac{1}{4} + \frac{\pi i \tau}{\log X}\right)} \cdot \zeta\left(1 - \frac{4\pi i \tau}{\log X}\right) A_D\left(-\frac{2\pi i \tau}{\log X}; \frac{2\pi i \tau}{\log X}\right) d\tau,$$

$$A_D(-r, r) = \prod_p \left(1 - \frac{1}{(p+1)p^{1-2r}} - \frac{1}{p+1}\right) \cdot \left(1 - \frac{1}{p}\right)^{-1}.$$

Proof: shift contours, keep track of poles of ratios of Γ and zeta functions.

Ratios Calculation: Weaker result for $\text{supp}(\widehat{g}) \subset (-1, 1)$.

- d -sum is $X^* e^{-2\pi i \left(1 - \frac{\log \pi}{\log X}\right) \tau} \left(1 - \frac{2\pi i \tau}{\log X}\right)^{-1} + O(X^{1/2})$;

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- decay of g restricts τ -sum to $|\tau| \leq \log X$, Taylor expand everything but g : small error term and

$$\begin{aligned} & \int_{|\tau| \leq \log X} g(\tau) \sum_{n=-1}^N \frac{a_n}{\log^n X} (2\pi i\tau)^n e^{-2\pi i(1 - \frac{\log \pi}{\log X})\tau} d\tau \\ &= \sum_{n=-1}^N \frac{a_n}{\log^n X} \int_{|\tau| \leq \log X} (2\pi i\tau)^n g(\tau) e^{-2\pi i(1 - \frac{\log \pi}{\log X})\tau} d\tau; \end{aligned}$$

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- from decay of g can extend the τ -integral to \mathbb{R} (essential that N is fixed and finite!), for $n \geq 0$ get the Fourier transform of $g^{(n)}$ (the n^{th} derivative of g) at $1 - \frac{\pi}{\log X}$, vanishes if $\text{supp}(\widehat{g}) \subset (-1, 1)$.

Number Theory Sums

$$\begin{aligned} S_{\text{even}} &= -\frac{2}{X^*} \sum_{d \leq X} \sum_{\ell=1}^{\infty} \sum_p \frac{\chi_d(p)^2 \log p}{p^\ell \log X} \widehat{g} \left(2 \frac{\log p^\ell}{\log X} \right) \\ S_{\text{odd}} &= -\frac{2}{X^*} \sum_{d \leq X} \sum_{\ell=0}^{\infty} \sum_p \frac{\chi_d(p) \log p}{p^{(2\ell+1)/2} \log X} \widehat{g} \left(\frac{\log p^{2\ell+1}}{\log X} \right). \end{aligned}$$

Number Theory Sums

Lemma

Let $\text{supp}(\hat{g}) \subset (-\sigma, \sigma) \subset (-1, 1)$. Then

$$\begin{aligned}
 S_{\text{even}} &= -\frac{g(0)}{2} + \frac{2}{\log X} \int_{-\infty}^{\infty} g(\tau) \frac{\zeta'}{\zeta} \left(1 + \frac{4\pi i\tau}{\log X}\right) d\tau \\
 &\quad + \frac{2}{\log X} \int_{-\infty}^{\infty} g(\tau) A'_D \left(\frac{2\pi i\tau}{\log X}; \frac{2\pi i\tau}{\log X}\right) + O(X^{-\frac{1}{2}+\epsilon}) \\
 S_{\text{odd}} &= O(X^{-\frac{1-\sigma}{2}} \log^6 X).
 \end{aligned}$$

If instead we consider the family of characters χ_{8d} for odd, positive square-free $d \in (0, X)$ (d a fundamental discriminant), then

$$S_{\text{odd}} = O(X^{-1/2+\epsilon} + X^{-(1-\frac{3}{2}\sigma)+\epsilon}).$$

Analysis of S_{even}

$\chi_d(p)^2 = 1$ except when $p|d$. Replace $\chi_d(p)^2$ with 1, and subtract off the contribution from when $p|d$:

$$\begin{aligned} S_{\text{even}} &= -2 \sum_{\ell=1}^{\infty} \sum_p \frac{\log p}{p^\ell \log X} \widehat{g} \left(2 \frac{\log p^\ell}{\log X} \right) \\ &\quad + \frac{2}{X^*} \sum_{d \leq X} \sum_{\ell=1}^{\infty} \sum_{p|d} \frac{\log p}{p^\ell \log X} \widehat{g} \left(2 \frac{\log p^\ell}{\log X} \right) \\ &= S_{\text{even};1} + S_{\text{even};2}. \end{aligned}$$

Lemma (Perron's Formula)

$$S_{\text{even};1} = -\frac{g(0)}{2} + \frac{2}{\log X} \int_{-\infty}^{\infty} g(\tau) \frac{\zeta'}{\zeta} \left(1 + \frac{4\pi i\tau}{\log X} \right) d\tau.$$

Analysis of S_{even} : $S_{\text{even};2}$

This piece gives us $\int g(\tau) A'_D(-\dots, \dots)$.

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- ◇ For $p < X^{1/2}$: $\sum_{d \leq X, p|d} 1 = \frac{X^*}{p+1} + O(X^{1/2})$.

Analysis of S_{even} : $S_{\text{even};2}$

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- Main ideas:
 - ◇ Restrict to $p \leq X^{1/2}$.
 - ◇ For $p < X^{1/2}$: $\sum_{d \leq X, p|d} 1 = \frac{X^*}{p+1} + O(X^{1/2})$.
 - ◇ Use Fourier Transform to expand \widehat{g} .

Analysis of S_{odd}

$$S_{\text{odd}} = -\frac{2}{X^*} \sum_{\ell=0}^{\infty} \sum_p \frac{\log p}{p^{(2\ell+1)/2} \log X} \widehat{g} \left(\frac{\log p^{2\ell+1}}{\log X} \right) \sum_{d \leq X} \chi_d(p).$$

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Jutila's bound

$$\sum_{\substack{1 < n \leq N \\ n \text{ non-square}}} \left| \sum_{\substack{0 < d \leq X \\ d \text{ fund. disc.}}} \chi_d(n) \right|^2 \ll NX \log^{10} N.$$

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Proof: Cauchy-Schwarz and Jutila: $p^{2\ell+1}$ non-square:

$$\left(\sum_{\ell=0}^{\infty} \sum_{p^{(2\ell+1)/2} \leq X^\sigma} \left| \sum_{d \leq X} \chi_d(p) \right|^2 \right)^{1/2} \ll X^{\frac{1+\sigma}{2}} \log^5 X.$$

Analysis of S_{odd} : Extending Support

More technical, replace Jutila's bound by applying Poisson Summation to character sums.

Lemma






Let $\text{supp}(\widehat{g}) \subset (-\sigma, \sigma) \subset (-1, 1)$. For family $\{8d : 0 < d \leq X, d \text{ an odd, positive square-free fundamental discriminant}\}$, $S_{\text{odd}} = O(X^{-\frac{1}{2}+\epsilon} + X^{-(1-\frac{3}{2}\sigma)+\epsilon})$.
In particular, if $\sigma < 1/3$ then $S_{\text{odd}} = O(X^{-1/2+\epsilon})$.






Conclusions






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- Ratios Conjecture gives detailed predictions (up to $X^{1/2+\epsilon}$).
- Number Theory agrees with predictions for suitably restricted test functions.
- Numerics quite good.







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