A Symplectic Test of the L-Functions Ratios Conjecture

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L-functions

- Riemann zeta function:

$$\zeta(s) \ = \ \sum_{n=1}^{\infty} \frac{1}{n^s} \ = \ \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

- Dirichlet L-functions:

$$L(s, \chi_d) \ = \ \sum_{n=1}^{\infty} \frac{\chi_d(n)}{n^s} \ = \ \prod_p \left(1 - \frac{\chi_d(p)}{p^s}\right)^{-1}.$$

- Elliptic curve L-functions: build up with data related to number of solutions modulo p.

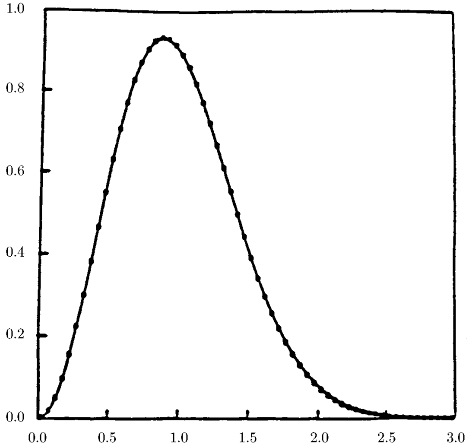
Properties of zeros of *L*-functions

- infinitude of primes, primes in arithmetic progression.
- Chebyshev's bias: $\pi_{3,4}(x) \geq \pi_{1,4}(x)$ 'most' of the time.
- Birch and Swinnerton-Dyer conjecture.
- Goldfeld, Gross-Zagier: bound for $h(D)$ from L-functions with many central point zeros.
- Even better estimates for $h(D)$ if a positive percentage of zeros of $\zeta(s)$ are at most $1/2 - \epsilon$ of the average spacing to the next zero.

Distribution of zeros

- $\zeta(s) \neq 0$ for $\mathfrak{Re}(s) = 1$: $\pi(x)$, $\pi_{a,q}(x)$.
- GRH: error terms.
- GSH: Chebyshev's bias.
- Analytic rank, adjacent spacings: $h(D)$.

Measures of Spacings: *n*-Level Correlations



n-level correlation: $\{\alpha_j\}$ an increasing sequence of numbers, $B \subset \mathbb{R}^{n-1}$ a compact box:

Measures of Spacings: *n*-Level Density and Families

Let g_i be even Schwartz functions whose Fourier Transform is compactly supported, $L(s,f)$ an L-function with zeros $\frac{1}{2} + i\gamma_f$ and conductor Q_f:

$$D_{n,f}(g) = \sum_{\substack{j_1,\dots,j_n \\ j_i \neq \pm j_k}} g_1\left(\gamma_{f,j_1}\frac{\log Q_f}{2\pi}\right)\cdots g_n\left(\gamma_{f,j_n}\frac{\log Q_f}{2\pi}\right)$$

- Properties of *n*-level density:
  - ◇ Individual zeros contribute in limit
  - ◇ Most of contribution is from low zeros
  - ◇ Average over similar *L*-functions (family)
- To any geometric family, Katz-Sarnak predict the *n*-level density depends only on a symmetry group (a classical compact group) attached to the family.

n-Level Density

n-level density: $\mathcal{F} = \cup \mathcal{F}_N$ a family of L-functions ordered by conductors, g_k an even Schwartz function:

$$D_{n,\mathcal{F}}(g) = \lim_{N\to\infty} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_N} \sum_{\substack{j_1,\dots,j_n \\ j_i \neq \pm j_k}} g_1\left(\frac{\log Q_f}{2\pi}\gamma_{j_1;f}\right)\cdots g_n\left(\frac{\log Q_f}{2\pi}\gamma_{j_n;f}\right)$$

As $N \to \infty$, 1-level density converges to

$$\int g(x) W_{1,\mathcal{G}(\mathcal{F})}(x)dx \quad = \quad \int \widehat{g}(u)\widehat{W}_{1,\mathcal{G}(\mathcal{F})}(u)du.$$

Conjecture (Katz-Sarnak)

(In the limit) Distribution of zeros near central point agrees with distribution of eigenvalues near 1 of a classical compact group.

1-Level Densities

Let $\mathcal{G}$ be one of the classical compact groups: Unitary, Symplectic, Orthogonal (or SO(even), SO(odd)).
If $\mathrm{supp}(\widehat{g}) \subset (-1, 1)$, 1-level density of $\mathcal{G}$ is

$$\widehat{g}(0) \quad - \quad c_{\mathcal{G}} \, \frac{g(0)}{2},$$

where

$$c_{\mathcal{G}} \; = \; \begin{cases} 0 & \mathcal{G} \text{ is Unitary} \\ 1 & \mathcal{G} \text{ is Symplectic} \\ -1 & \mathcal{G} \text{ is Orthogonal.} \end{cases}$$

Some Results

- **Orthogonal:**
  ◇ Iwaniec-Luo-Sarnak: 1-level density for $H_k^{\pm}(N)$, N square-free.
  ◇ Miller, Young: families of elliptic curves.
  ◇ Güloğlu: 1-level for $\{\mathrm{Sym}^r f : f \in H_k(1)\}$, r odd.
- **Symplectic:**
  ◇ Rubinstein: n-level densities for $L(s, \chi_d)$.
  ◇ Güloğlu: 1-level for $\{\mathrm{Sym}^r f : f \in H_k(1)\}$, r even.
- **Unitary:**
  ◇ Hughes-Rudnick, Miller: families of primitive Dirichlet characters.

Identifying the Symmetry Groups

- Often an analysis of the monodromy group in the function field case suggests the answer.
- All simple families studied to date are built from GL_1 or GL_2 *L*-functions.
- Tools: Explicit Formula, Orthogonality of Characters / Petersson Formula.
- How to identify symmetry group in general? One possibility is by the signs of the functional equation:
- **Folklore Conjecture:** If all signs are even and no corresponding family with odd signs, Symplectic symmetry; otherwise SO(even). (False!)

Explicit Formula

- π: cuspidal automorphic representation on GL_n.
- $Q_\pi > 0$: analytic conductor of $L(s,\pi) = \sum \lambda_\pi(n)/n^s$.
- By GRH the non-trivial zeros are $\frac{1}{2} + i\gamma_{\pi,j}$.
- Satake parameters $\{\alpha_{\pi,i}(p)\}_{i=1}^n$; $\lambda_\pi(p^\nu) = \sum_{i=1}^n \alpha_{\pi,i}(p)^\nu$.
- $L(s,\pi) = \sum_n \frac{\lambda_\pi(n)}{n^s} = \prod_p \prod_{i=1}^n (1 - \alpha_{\pi,i}(p)p^{-s})^{-1}$.

$$\sum_j g\left(\gamma_{\pi,j}\frac{\log Q_\pi}{2\pi}\right) = \widehat{g}(0) - 2\sum_{p,\nu} \widehat{g}\left(\frac{\nu \log p}{\log Q_\pi}\right)\frac{\lambda_\pi(p^\nu)\log p}{p^{\nu/2}\log Q_\pi}$$

Some Results: Rankin-Selberg Convolution of Families

Symmetry constant: $c_{\mathcal{L}} = 0$ (resp, 1 or -1) if family $\mathcal{L}$ has unitary (resp, symplectic or orthogonal) symmetry.

Rankin-Selberg convolution: Satake parameters for $\pi_{1,p} \times \pi_{2,p}$ are

$$\{\alpha_{\pi_1 \times \pi_2}(k)\}_{k=1}^{nm} = \{\alpha_{\pi_1}(i) \cdot \alpha_{\pi_2}(j)\}_{\substack{1 \le i \le n \\ 1 \le j \le m}}.$$

Theorem (Dueñez-Miller)

If $\mathcal{F}$ and $\mathcal{G}$ are *nice* families of L-functions, then $c_{\mathcal{F} \times \mathcal{G}} = c_{\mathcal{F}} \cdot c_{\mathcal{G}}$.

Ratios Conjecture

History

- Farmer (1993): Considered

$$\int_0^T \frac{\zeta(s+\alpha)\zeta(1-s+\beta)}{\zeta(s+\gamma)\zeta(1-s+\delta)}\, dt,$$

conjectured (for appropriate values)

$$T\frac{(\alpha+\delta)(\beta+\gamma)}{(\alpha+\beta)(\gamma+\delta)} - T^{1-\alpha-\beta}\frac{(\delta-\beta)(\gamma-\alpha)}{(\alpha+\beta)(\gamma+\delta)}.$$

- Conrey-Farmer-Zirnbauer (2007): conjecture formulas for averages of products of L-functions over families:

$$R_{\mathcal{F}} \;=\; \sum_{f\in\mathcal{F}} \omega_f \frac{L\left(\frac{1}{2}+\alpha, f\right)}{L\left(\frac{1}{2}+\gamma, f\right)}.$$

Uses of the Ratios Conjecture

- Applications:
  - ⋄ *n*-level correlations and densities;
  - ⋄ mollifiers;
  - ⋄ moments;
  - ⋄ vanishing at the central point;

- Advantages:
  - ⋄ RMT models often add arithmetic ad hoc;
  - ⋄ predicts lower order terms, often to square-root level.

Inputs for 1-level density

- Approximate Functional Equation:

$$L(s,f) \;=\; \sum_{m\le x}\frac{a_m}{m^s} + \epsilon\mathbb{X}_L(s)\sum_{n\le y}\frac{a_n}{n^{1-s}};$$

◇ ϵ sign of the functional equation,
◇ $\mathbb{X}_L(s)$ ratio of Γ-factors from functional equation.

- Explicit Formula: g Schwartz test function,

$$\sum_{f\in\mathcal{F}}\omega_f\sum_{\gamma} g\left(\gamma\frac{\log N_f}{2\pi}\right) \;=\; \frac{1}{2\pi i}\int_{(c)} - \int_{(1-c)} R'_{\mathcal{F}}(\cdots)g(\cdots)$$

◇ $R'_{\mathcal{F}}(r) = \frac{\partial}{\partial\alpha}R_{\mathcal{F}}(\alpha,\gamma)\Big|_{\alpha=\gamma=r}$.

Procedure

- Use approximate functional equation to expand numerator.
- Expand denominator by generalized Mobius function: cusp form

$$\frac{1}{L(s,f)} = \sum_h \frac{\mu_f(h)}{h^s},$$

where $\mu_f(h)$ is the multiplicative function equaling 1 for $h = 1$, $-\lambda_f(p)$ if $n = p$, $\chi_0(p)$ if $h = p^2$ and 0 otherwise.
- Execute the sum over $\mathcal{F}$, keeping only main (diagonal) terms.
- Extend the m and n sums to infinity (complete the products).
- Differentiate with respect to the parameters.

Main Results

Symplectic Families

- Fundamental discriminants: d square-free and 1 modulo 4, or $d/4$ square-free and 2 or 3 modulo 4.
- Associated character χ_d:
  ◇ $\chi_d(-1) = 1$ say d even;
  ◇ $\chi_d(-1) = -1$ say d odd.
  ◇ even (resp., odd) if $d > 0$ (resp., $d < 0$).

Will study following families:

◇ even fundamental discriminants at most X;
◇ $\{8d : 0 < d \leq X, d$ an odd, positive square-free fundamental discriminant$\}$.

Prediction from Ratios Conjecture

$$\frac{1}{X^*}\sum_{d\le X}\sum_{\gamma_d} g\left(\gamma_d\frac{\log X}{2\pi}\right) = \frac{1}{X^*\log X}\int_{-\infty}^{\infty} g(\tau)\sum_{d\le X}\left[\log\frac{d}{\pi}+\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}\pm\frac{i\pi\tau}{\log X}\right)\right]d\tau$$
$$+\frac{2}{X^*\log X}\sum_{d\le X}\int_{-\infty}^{\infty} g(\tau)\left[\frac{\zeta'}{\zeta}\left(1+\frac{4\pi i\tau}{\log X}\right)+A_D'\left(\frac{2\pi i\tau}{\log X};\frac{2\pi i\tau}{\log X}\right)\right.$$
$$\left.-e^{-2\pi i\tau\log(d/\pi)/\log X}\frac{\Gamma\left(\frac{1}{4}-\frac{\pi i\tau}{\log X}\right)}{\Gamma\left(\frac{1}{4}+\frac{\pi i\tau}{\log X}\right)}\zeta\left(1-\frac{4\pi i\tau}{\log X}\right)A_D\left(-\frac{2\pi i\tau}{\log X};\frac{2\pi i\tau}{\log X}\right)\right]d\tau + O(X^{-\frac{1}{2}+\epsilon}),$$

with

$$A_D(-r,r) = \prod_p\left(1-\frac{1}{(p+1)p^{1-2r}}-\frac{1}{p+1}\right)\cdot\left(1-\frac{1}{p}\right)^{-1}$$
$$A_D'(r;r) = \sum_p\frac{\log p}{(p+1)(p^{1+2r}-1)}.$$

Prediction from Ratios Conjecture

Main term is

$$\frac{1}{X^*}\sum_{d\leq X}\sum_{\gamma_d} g\left(\gamma_d \frac{\log X}{2\pi}\right) = \int_{-\infty}^{\infty} g(x)\left(1-\frac{\sin(2\pi x)}{2\pi x}\right)dx + O\left(\frac{1}{\log X}\right),$$

which is the 1-level density for the scaling limit of $\mathrm{USp}(2N)$. If $\mathrm{supp}(\widehat{g}) \subset (-1,1)$, then the integral of $g(x)$ against $-\sin(2\pi x)/2\pi x$ is $-g(0)/2$.

Prediction from Ratios Conjecture

Assuming RH for $\zeta(s)$, for $\mathrm{supp}(\widehat{g}) \subset (-\sigma, \sigma) \subset (-1, 1)$:

$$\frac{-2}{X^* \log X} \sum_{d \leq X} \int_{-\infty}^{\infty} g(\tau)\, e^{-2\pi i \tau \frac{\log(d/\pi)}{\log X}} \frac{\Gamma\left(\frac{1}{4} - \frac{\pi i \tau}{\log X}\right)}{\Gamma\left(\frac{1}{4} + \frac{\pi i \tau}{\log X}\right)} \zeta\left(1 - \frac{4\pi i \tau}{\log X}\right) A_D\left(-\frac{2\pi i \tau}{\log X}; \frac{2\pi i \tau}{\log X}\right) d\tau$$

$$= -\frac{g(0)}{2} + O(X^{-\frac{3}{4}(1-\sigma)+\epsilon});$$

the error term may be absorbed into the $O(X^{-1/2+\epsilon})$ error if $\sigma < 1/3$.

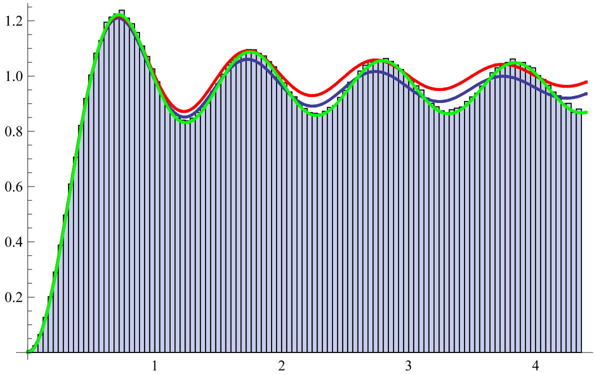
Main Results

Theorem (M– '07)

Let $\text{supp}(\widehat{g}) \subset (-\sigma, \sigma)$, *assume RH for* $\zeta(s)$. 1*-Level Density agrees with prediction from Ratios Conjecture*

- *up to* $O(X^{-(1-\sigma)/2+\epsilon})$ *for the family of quadratic Dirichlet characters with even fundamental discriminants at most* X*;*
- *up to* $O(X^{-1/2} + X^{-(1-\frac{3}{2}\sigma)+\epsilon} + X^{-\frac{3}{4}(1-\sigma)+\epsilon})$ *for our sub-family. If* $\sigma < 1/3$ *then agrees up to* $O(X^{-1/2+\epsilon})$.

Numerics (J. Stopple): 1,003,083 negative fundamental discriminants $-d \in [10^{12}, 10^{12} + 3.3 \cdot 10^6]$



Histogram of normalized zeros ($\gamma \le 1$, about 4 million).
$\diamond$ Red: main term. $\diamond$ Blue: includes $O(1/\log X)$ terms.
$\diamond$ Green: all lower order terms.

Sketch of Proofs

Ratios Calculation

Hardest piece to analyze is

$$R(g;X) = -\frac{2}{X^* \log X}\sum_{d\le X}\int_{-\infty}^{\infty} g(\tau) e^{-2\pi i\tau\frac{\log(d/\pi)}{\log X}} \frac{\Gamma\left(\frac{1}{4}-\frac{\pi i\tau}{\log X}\right)}{\Gamma\left(\frac{1}{4}+\frac{\pi i\tau}{\log X}\right)} \cdot \zeta\left(1-\frac{4\pi i\tau}{\log X}\right) A_D\left(-\frac{2\pi i\tau}{\log X};\frac{2\pi i\tau}{\log X}\right) d\tau,$$

$$A_D(-r,r) = \prod_p \left(1-\frac{1}{(p+1)p^{1-2r}}-\frac{1}{p+1}\right)\cdot\left(1-\frac{1}{p}\right)^{-1}.$$

Proof: shift contours, keep track of poles of ratios of Γ and zeta functions.

Ratios Calculation: Weaker result for $\mathrm{supp}(\widehat{g}) \subset (-1, 1)$.

- d-sum is $X^* e^{-2\pi i\left(1-\frac{\log \pi}{\log X}\right)\tau}\left(1-\frac{2\pi i\tau}{\log X}\right)^{-1} + O(X^{1/2})$;
- decay of g restricts τ-sum to $|\tau| \le \log X$, Taylor expand everything but g: small error term and

$$\int_{|\tau|\le\log X} g(\tau) \sum_{n=-1}^{N} \frac{a_n}{\log^n X}(2\pi i\tau)^n e^{-2\pi i\left(1-\frac{\log \pi}{\log X}\right)\tau} d\tau$$

$$= \sum_{n=-1}^{N} \frac{a_n}{\log^n X} \int_{|\tau|\le\log X} (2\pi i\tau)^n g(\tau) e^{-2\pi i\left(1-\frac{\log \pi}{\log X}\right)\tau} d\tau;$$

- from decay of g can extend the τ-integral to $\mathbb{R}$ (essential that N is fixed and finite!), for $n \ge 0$ get the Fourier transform of $g^{(n)}$ (the n^{th} derivative of g) at $1-\frac{\pi}{\log X}$, vanishes if $\mathrm{supp}(\widehat{g}) \subset (-1, 1)$.

Number Theory Sums

$$S_{\text{even}} = -\frac{2}{X^*}\sum_{d\le X}\sum_{\ell=1}^{\infty}\sum_{p}\frac{\chi_d(p)^2\log p}{p^\ell \log X}\,\widehat{g}\left(2\frac{\log p^\ell}{\log X}\right)$$

$$S_{\text{odd}} = -\frac{2}{X^*}\sum_{d\le X}\sum_{\ell=0}^{\infty}\sum_{p}\frac{\chi_d(p)\log p}{p^{(2\ell+1)/2}\log X}\,\widehat{g}\left(\frac{\log p^{2\ell+1}}{\log X}\right).$$

Number Theory Sums

Lemma

Let $\mathrm{supp}(\widehat{g}) \subset (-\sigma, \sigma) \subset (-1, 1)$. *Then*

$$\begin{aligned} S_{\rm even} &= -\frac{g(0)}{2} + \frac{2}{\log X}\int_{-\infty}^{\infty} g(\tau)\frac{\zeta'}{\zeta}\left(1 + \frac{4\pi i\tau}{\log X}\right) d\tau \\ &\quad + \frac{2}{\log X}\int_{-\infty}^{\infty} g(\tau) A_D'\left(\frac{2\pi i\tau}{\log X}; \frac{2\pi i\tau}{\log X}\right) + O(X^{-\frac{1}{2}+\epsilon}) \\ S_{\rm odd} &= O(X^{-\frac{1-\sigma}{2}} \log^6 X). \end{aligned}$$

If instead we consider the family of characters χ_{8d} for odd, positive square-free $d \in (0, X)$ (d a fundamental discriminant), then

$$S_{\rm odd} = O(X^{-1/2+\epsilon} + X^{-(1-\frac{3}{2}\sigma)+\epsilon}).$$

Analysis of $S_{\rm even}$

$\chi_d(p)^2 = 1$ except when $p|d$. Replace $\chi_d(p)^2$ with 1, and subtract off the contribution from when $p|d$:

$$\begin{aligned} S_{\rm even} &= -2\sum_{\ell=1}^{\infty}\sum_{p}\frac{\log p}{p^\ell \log X}\,\widehat{g}\left(2\frac{\log p^\ell}{\log X}\right) \\ &\quad +\frac{2}{X^*}\sum_{d\le X}\sum_{\ell=1}^{\infty}\sum_{p|d}\frac{\log p}{p^\ell \log X}\,\widehat{g}\left(2\frac{\log p^\ell}{\log X}\right) \\ &= S_{\rm even;1} + S_{\rm even;2}. \end{aligned}$$

Lemma (Perron's Formula)

$$S_{\rm even;1} = -\frac{g(0)}{2} + \frac{2}{\log X}\int_{-\infty}^{\infty} g(\tau)\frac{\zeta'}{\zeta}\left(1+\frac{4\pi i\tau}{\log X}\right)d\tau.$$

Analysis of S_{even}: $S_{\text{even};2}$

This piece gives us $\int g(\tau) A'_D(-\cdots,\cdots)$.

- Main ideas:
  - ◇ Restrict to $p \le X^{1/2}$.
  - ◇ For $p < X^{1/2}$: $\sum_{d \le X, p|d} 1 = \frac{X^*}{p+1} + O(X^{1/2})$.
  - ◇ Use Fourier Transform to expand $\widehat{g}$.

Analysis of S_{odd}

$$S_{\text{odd}} = -\frac{2}{X^*}\sum_{\ell=0}^{\infty}\sum_{p}\frac{\log p}{p^{(2\ell+1)/2}\log X}\,\widehat{g}\left(\frac{\log p^{2\ell+1}}{\log X}\right)\sum_{d\le X}\chi_d(p).$$

Jutila's bound

$$\sum_{\substack{1<n\le N\\ n\ \text{non-square}}}\left|\sum_{\substack{0<d\le X\\ d\ \text{fund. disc.}}}\chi_d(n)\right|^2 \ll NX\log^{10}N.$$

Proof: Cauchy-Schwarz and Jutila: $p^{2\ell+1}$ non-square:

$$\left(\sum_{\ell=0}^{\infty}\sum_{p^{(2\ell+1)/2}\le X^\sigma}\left|\sum_{d\le X}\chi_d(p)\right|^2\right)^{1/2} \ll X^{\frac{1+\sigma}{2}}\log^5 X.$$

Analysis of S_{odd}: Extending Support

More technical, replace Jutila's bound by applying Poisson Summation to character sums.

Lemma

Let $\operatorname{supp}(\widehat{g}) \subset (-\sigma, \sigma) \subset (-1, 1)$. *For family* $\{8d : \ 0 < d \leq X, \ d$ *an odd, positive square-free fundamental discriminant*$\}$, $S_{\text{odd}} = O(X^{-\frac{1}{2}+\epsilon} + X^{-(1-\frac{3}{2}\sigma)+\epsilon})$. *In particular, if* $\sigma < 1/3$ *then* $S_{\text{odd}} = O(X^{-1/2+\epsilon})$.

Conclusions

Conclusions

- Ratios Conjecture gives detailed predictions (up to $X^{1/2+\epsilon}$).
- Number Theory agrees with predictions for suitably restricted test functions.
- Numerics quite good.

Appendix

RH and the Prime Number Theorem

From $\zeta(s) = \sum n^{-s} = \prod_p (1 - p^{-s})^{-1}$, logarithmic derivative is

$$-\frac{\zeta'(s)}{\zeta(s)} = \sum \frac{\Lambda(n)}{n^s},$$

where $\Lambda(n) = \log p$ if $n = p^k$ and is 0 otherwise.

Take Mellin transform, integrate and shift contour. Find

$$\sum_{n \le x} \Lambda(n) = x - \sum_{\rho} \frac{x^\rho}{\rho},$$

where $\rho = 1/2 + i\gamma$ runs over non-trivial zeros of $\zeta(s)$.

Partial summation gives Prime Number Theorem (to first order, there are $x/\log x$ primes at most x) if $\mathfrak{Re}\rho < 1$.

The smaller max $\mathfrak{Re}(\rho)$ is, the better the error term in the Prime Number Theorem. The Riemann Hypothesis (RH) says $\mathfrak{Re}(\rho) = 1/2$.

Primes in Arithmetic Progression

To study number primes $p \equiv a \bmod q$, use

$$L(s,\chi) \;=\; \sum \frac{\chi(n)}{n^s} \;=\; \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

Key sum: $\frac{1}{\phi(q)} \sum_{\chi \bmod q} \chi(n)$ is 1 if $n \equiv 1 \bmod q$ and 0 otherwise.

Similar arguments give

$$\sum_{p \equiv a \bmod q} \frac{\log p}{p^s} \;=\; -\frac{1}{\phi(q)} \sum_{\chi \bmod q} \frac{L'(s,\chi)}{L(s,\chi)} \chi(\overline{a}) + \mathrm{Good}(s).$$

Note: To understand $\{p \equiv a \bmod q\}$ need to understand *all* $L(s,\chi)$; see benefit of studying a family.

GSH and Chebyshev's Bias

$\pi_{3,4}(x) \geq \pi_{1,4}(x)$ and $\pi_{2,3}(x) \geq \pi_{1,3}(x)$ 'most' of the time. Use analytic density:

$$\mathrm{Den}_{\mathrm{an}}(S) \;=\; \limsup \frac{1}{\log T} \int_{S \cap [2,T]} \frac{dt}{t}.$$

Have $\pi_{3,4}(x) \geq \pi_{1,4}(x)$ with analytic density .9959 (first flip at 26861); $\pi_{2,3}(x) \geq \pi_{1,3}(x)$ with analytic density .9990 (first flip $\approx 6 \cdot 10^{11}$).

Non-residues beat residues. Key ingredient Generalized Simplicity Hypothesis (GSH): the zeros of $L(s,\chi)$ are linearly independent over $\mathbb{Q}$.

Structure of zeros important: GSH used to show a flow on a torus is full (becomes equidistributed).

Class Number

Class number: measures failure of unique factorization (order of ideal class group).

Imaginary quadratic field $Q(\sqrt{D})$, fundamental discriminant $D < 0$, I group of non-zero fractional ideals, P subgroup of principal ideals, $\mathcal{H} = I/P$ class group, $h(D) = \#\mathcal{H}$ the class number. Dirichlet proved

$$L(1, \chi_D) = \frac{2\pi h(D)}{w_D\sqrt{D}},$$

where χ_D the quadratic character and $w_D = 2$ if $D < -4$, 4 if $D = -4$ and 6 if $D = -3$.

Theorem: $h(D) = 1 \Leftrightarrow -D \in \{3, 4, 7, 8, 11, 19, 43, 67, 163\}$.

Class Number and Distribution of Zeros I

Expect $\frac{\sqrt{|D|}}{\log\log|D|} \ll h(D) \ll \sqrt{|D|}\log\log|D|$. Siegel proved $h(D) > c(\epsilon)|D|^{1/2-\epsilon}$ (but ineffective).

Goldfeld, Gross-Zagier: f primitive cusp form of weight k, level N, trivial central character, suppose $m = \mathrm{ord}_{s=1/2}L(s,f)L(s,\chi_D) \geq 3$, $g = m-1$ or $m-2$ so that $(-1)^g = \omega(f)\omega(f_{\chi_D})$ (signs of fnal eqs). Then have effective bound

$$h(D) \gg (\log|D|)^{g-1} \prod_{p|D}\left(1+\frac{1}{p}\right)^{-3}\left(1+\frac{\lambda(p)\sqrt{p}}{p+1}\right)^{-1}.$$

Good result from using an elliptic curve that vanishes to order 3 at $s = 1/2$, application of many zeros at central point.

Class Number and Distribution of Zeros II

Assume a positive percent of zeros (or $cT\log T/(\log|D|)^A$) of zeros with $\gamma \le T$) of $\zeta(s)$ are at most $1/2-\epsilon$ of the average spacing from the next zero $\zeta(s)$. Then $h(D) \gg \sqrt{|D|}/(\log|D|)^B$, all constants computable.

See actual spacings between zeros are tied to number theory (have positive percent are less than half the average spacing if GUE Conjecture holds for adjacent spacings).

Instead of $1/2-\epsilon$, under RH have: .68 (Montgomery), .5179 (Montgomery-Odlyzko), .5171 (Conrey-Ghosh-Gonek), .5169 (Conrey-Iwaniec) (Montgomery says led to pair correlation conjecture by looking at gaps between zeros of $\zeta(s)$ and $h(D)$).

References

- M. V. Berry, *Semiclassical formula for the number variance of the Riemann zeros*, Nonlinearity **1** (1988), 399–407.
- M. V. Berry and J. P. Keating, *The Riemann zeros and eigenvalue asymptotics*, Siam Review **41** (1999), no. 2, 236–266.
- E. Bogomolny, O. Bohigas, P. Leboeuf and A. G. Monastra, *On the spacing distribution of the Riemann zeros: corrections to the asymptotic result*, Journal of Physics A: Mathematical and General **39** (2006), no. 34, 10743–10754.
- E. B. Bogomolny and J. P. Keating, *Gutzwiller's trace formula and spectral statistics: beyond the diagonal approximation*, Phys. Rev. Lett. **77** (1996), no. 8, 1472–1475.
- B. Conrey and D. Farmer, *Mean values of L-functions and symmetry*, Internat. Math. Res. Notices 2000, no. 17, 883–908.

- B. Conrey, D. Farmer, P. Keating, M. Rubinstein and N. Snaith, *Integral moments of L-functions*, Proc. London Math. Soc. (3) **91** (2005), no. 1, 33–104.
- J. B. Conrey, D. W. Farmer and M. R. Zirnbauer, *Autocorrelation of ratios of L-functions*, preprint. http://arxiv.org/abs/0711.0718
- J. B. Conrey, D. W. Farmer and M. R. Zirnbauer, *Howe pairs, supersymmetry, and ratios of random characteristic polynomials for the classical compact groups*, preprint. http://arxiv.org/abs/math-ph/0511024
- J. B. Conrey and H. Iwaniec, *Spacing of Zeros of Hecke L-Functions and the Class Number Problem*, Acta Arith. **103** (2002) no. 3, 259–312.
- J. B. Conrey and N. C. Snaith, *Applications of the L-functions Ratios Conjecture*, Proc. Lon. Math. Soc. **93** (2007), no 3, 594–646.

- J. B. Conrey and N. C. Snaith, *Triple correlation of the Riemann zeros*, preprint. http://arxiv.org/abs/math/0610495
- H. Davenport, *Multiplicative Number Theory, 2nd edition*, Graduate Texts in Mathematics **74**, Springer-Verlag, New York, 1980, revised by H. Montgomery.
- E. Dueñez, D. K. Huynh, J. P. Keating, S. J. Miller and N. C. Snaith, work in progress.
- E. Dueñez and S. J. Miller, *The low lying zeros of a* $GL(4)$ *and a* $GL(6)$ *family of L-functions*, Compositio Mathematica **142** (2006), no. 6, 1403–1425.
- E. Dueñez and S. J. Miller, *The effect of convolving families of L-functions on the underlying group symmetries*, preprint. http://arxiv.org/abs/math/0607688
- A. Erdélyi and F. G. Tricomi, *The asymptotic expansion of a ratio of gamma functions*, Pacific J. Math. **1** (1951), no. 1, 133–142.

- E. Fouvry and H. Iwaniec, *Low-lying zeros of dihedral L-functions*, Duke Math. J. **116** (2003), no. 2, 189-217.
- P. Gao, *N-level density of the low-lying zeros of quadratic Dirichlet L-functions*, Ph. D thesis, University of Michigan, 2005.
- A. Güloğlu, *Low-Lying Zeros of Symmetric Power L-Functions*, Internat. Math. Res. Notices 2005, no. 9, 517-550.
- G. Hardy and E. Wright, *An Introduction to the Theory of Numbers*, fifth edition, Oxford Science Publications, Clarendon Press, Oxford, 1995.
- D. Hejhal, *On the triple correlation of zeros of the zeta function*, Internat. Math. Res. Notices 1994, no. 7, 294-302.

- C. Hughes and S. J. Miller, *Low-lying zeros of L-functions with orthogonal symmtry*, Duke Math. J., **136** (2007), no. 1, 115–172.
- C. Hughes and Z. Rudnick, *Linear Statistics of Low-Lying Zeros of L-functions*, Quart. J. Math. Oxford **54** (2003), 309–333.
- D. K. Huynh, J. P. Keating and N. C. Snaith, work in progress.
- H. Iwaniec, W. Luo and P. Sarnak, *Low lying zeros of families of L-functions*, Inst. Hautes Études Sci. Publ. Math. **91**, 2000, 55–131.
- M. Jutila, *On character sums and class numbers*, Journal of Number Theory **5** (1973), 203–214.
- M. Jutila, *On mean values of Dirichlet polynomials with real characters*, Acta Arith. **27** (1975), 191–198.

M. Jutila, *On the mean value of $L(1/2, \chi)$ for real characters*, Analysis **1** (1981), no. 2, 149–161.

N. Katz and P. Sarnak, *Random Matrices, Frobenius Eigenvalues and Monodromy*, AMS Colloquium Publications **45**, AMS, Providence, 1999.

N. Katz and P. Sarnak, *Zeros of zeta functions and symmetries*, Bull. AMS **36**, 1999, $1 - 26$.

J. P. Keating, *Statistics of quantum eigenvalues and the Riemann zeros*, in Supersymmetry and Trace Formulae: Chaos and Disorder, eds. I. V. Lerner, J. P. Keating & D. E Khmelnitskii (Plenum Press), 1–15.

J. P. Keating and N. C. Snaith, *Random matrix theory and $\zeta(1/2 + it)$*, Comm. Math. Phys. **214** (2000), no. 1, 57–89.

J. P. Keating and N. C. Snaith, *Random matrix theory and L-functions at $s = 1/2$*, Comm. Math. Phys. **214** (2000), no. 1, 91–110.

J. P. Keating and N. C. Snaith, *Random matrices and L-functions*, Random matrix theory, J. Phys. A **36** (2003), no. 12, 2859–2881.

S. J. Miller, *1- and 2-level densities for families of elliptic curves: evidence for the underlying group symmetries*, Compositio Mathematica **104** (2004), 952–992.

S. J. Miller, *Variation in the number of points on elliptic curves and applications to excess rank*, C. R. Math. Rep. Acad. Sci. Canada **27** (2005), no. 4, 111–120.

S. J. Miller, *Lower order terms in the 1-level density for families of holomorphic cuspidal newforms*, preprint. http://arxiv.org/abs/0704.0924

- H. Montgomery, *The pair correlation of zeros of the zeta function*, Analytic Number Theory, Proc. Sympos. Pure Math. **24**, Amer. Math. Soc., Providence, 1973, 181 – 193.
- A. Odlyzko, *On the distribution of spacings between zeros of the zeta function*, Math. Comp. **48** (1987), no. 177, 273–308.
- A. Odlyzko, *The* 10^{22}*-nd zero of the Riemann zeta function*, Proc. Conference on Dynamical, Spectral and Arithmetic Zeta-Functions, M. van Frankenhuysen and M. L. Lapidus, eds., Amer. Math. Soc., Contemporary Math. series, 2001, http://www.research.att.com/~amo/doc/zeta.html.
- A. E. Özlük and C. Snyder, *Small zeros of quadratic L-functions*, Bull. Austral. Math. Soc. **47** (1993), no. 2, 307–319.
- A. E. Özlük and C. Snyder, *On the distribution of the nontrivial zeros of quadratic L-functions close to the real axis*, Acta Arith. **91** (1999), no. 3, 209–228.

- G. Ricotta and E. Royer, *Statistics for low-lying zeros of symmetric power L-functions in the level aspect*, preprint. http://arxiv.org/abs/math/0703760
- E. Royer, *Petits zéros de fonctions L de formes modulaires*, Acta Arith. **99** (2001), no. 2, 147-172.
- M. Rubinstein, *Low-lying zeros of L–functions and random matrix theory*, Duke Math. J. **109**, (2001), 147–181.
- M. Rubinstein, *Computational methods and experiments in analytic number theory*. Pages 407–483 in Recent Perspectives in Random Matrix Theory and Number Theory, ed. F. Mezzadri and N. C. Snaith editors, 2005.
- Z. Rudnick and P. Sarnak, *Zeros of principal L-functions and random matrix theory*, Duke Math. J. **81**, 1996, 269 – 322.
- K. Soundararajan, *Nonvanishing of quadratic Dirichlet L-functions at* $s = 1/2$, Ann. of Math. (2) **152** (2000), 447–488.

- J. Stopple, *The quadratic character experiment*, preprint. http://arxiv.org/abs/0802.4255
- M. Young, *Lower-order terms of the 1-level density of families of elliptic curves*, Internat. Math. Res. Notices 2005, no. 10, 587–633.
- M. Young, *Low-lying zeros of families of elliptic curves*, J. Amer. Math. Soc. **19** (2006), no. 1, 205–250.