

# Tests of the $L$ -Functions Ratios Conjecture

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## History

- Farmer (1993): Considered

$$\int_0^T \frac{\zeta(s + \alpha)\zeta(1 - s + \beta)}{\zeta(s + \gamma)\zeta(1 - s + \delta)} dt,$$

conjectured (for appropriate values)

$$T \frac{(\alpha + \delta)(\beta + \gamma)}{(\alpha + \beta)(\gamma + \delta)} - T^{1-\alpha-\beta} \frac{(\delta - \beta)(\gamma - \alpha)}{(\alpha + \beta)(\gamma + \delta)}.$$

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- Conrey-Farmer-Zirnbauer (2007): conjecture formulas for averages of products of  $L$ -functions over families:

$$R_{\mathcal{F}} = \sum_{f \in \mathcal{F}} \omega_f \frac{L\left(\frac{1}{2} + \alpha, f\right)}{L\left(\frac{1}{2} + \gamma, f\right)}.$$

## Uses of the Ratios Conjecture

- **Applications:**
  - ◇  $n$ -level correlations and densities;
  - ◇ mollifiers;
  - ◇ moments;
  - ◇ vanishing at the central point.
- **Advantages:**
  - ◇ RMT models often add arithmetic ad hoc;
  - ◇ Predicts lower order terms to square-root level;
  - ◇ Fast computations.

## Inputs for 1-level density

- **Approximate Functional Equation:**

$$L(s, f) = \sum_{m \leq x} \frac{a_m}{m^s} + \epsilon \mathbb{X}_L(s) \sum_{n \leq y} \frac{a_n}{n^{1-s}} + \text{Error};$$

- ◇  $\epsilon$  sign of the functional equation,
- ◇  $\mathbb{X}_L(s)$  ratio of  $\Gamma$ -factors from functional equation.

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- **Explicit Formula:**  $g$  Schwartz test function,

$$\sum_{f \in \mathcal{F}} \omega_f \sum_{\gamma} g\left(\gamma \frac{\log N_f}{2\pi}\right) = \frac{1}{2\pi i} \int_{(c)-(1-c)} R'_{\mathcal{F}}(\dots) g(\dots)$$

$$\diamond R'_{\mathcal{F}}(r) = \left. \frac{\partial}{\partial \alpha} R_{\mathcal{F}}(\alpha, \gamma) \right|_{\alpha=\gamma=r}.$$

## Procedure (Recipe)

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$$\frac{1}{L(s, f)} = \sum_h \frac{\mu_f(h)}{h^s},$$

where  $\mu_f(h)$  is the multiplicative function equaling 1 for  $h = 1$ ,  $-\lambda_f(p)$  if  $n = p$ ,  $\chi_0(p)$  if  $h = p^2$  and 0 otherwise.

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- Differentiate with respect to the parameters.

## Symplectic Results

## Symplectic Families

- Fundamental discriminants:  $d$  square-free and 1 modulo 4, or  $d/4$  square-free and 2 or 3 modulo 4.
- Associated character  $\chi_d$ :
  - ◇  $\chi_d(-1) = 1$  say  $d$  even;
  - ◇  $\chi_d(-1) = -1$  say  $d$  odd.
  - ◇ even (resp., odd) if  $d > 0$  (resp.,  $d < 0$ ).

### Will study following families:

- ◇ even fundamental discriminants at most  $X$ ;
- ◇  $\{8d : 0 < d \leq X, d \text{ an odd, positive square-free fundamental discriminant}\}$ .

## Prediction from Ratios Conjecture

$$\begin{aligned} \frac{1}{X^*} \sum_{d \leq X} \sum_{\gamma_d} g\left(\gamma_d \frac{\log X}{2\pi}\right) &= \frac{1}{X^* \log X} \int_{-\infty}^{\infty} g(\tau) \sum_{d \leq X} \left[ \log \frac{d}{\pi} + \frac{1}{2} \frac{\Gamma'}{\Gamma} \left( \frac{1}{4} \pm \frac{i\pi\tau}{\log X} \right) \right] d\tau \\ &+ \frac{2}{X^* \log X} \sum_{d \leq X} \int_{-\infty}^{\infty} g(\tau) \left[ \frac{\zeta'}{\zeta} \left( 1 + \frac{4\pi i\tau}{\log X} \right) + A'_D \left( \frac{2\pi i\tau}{\log X}; \frac{2\pi i\tau}{\log X} \right) \right. \\ &\left. - e^{-2\pi i\tau \log(d/\pi)/\log X} \frac{\Gamma\left(\frac{1}{4} - \frac{\pi i\tau}{\log X}\right)}{\Gamma\left(\frac{1}{4} + \frac{\pi i\tau}{\log X}\right)} \zeta \left( 1 - \frac{4\pi i\tau}{\log X} \right) A_D \left( -\frac{2\pi i\tau}{\log X}; \frac{2\pi i\tau}{\log X} \right) \right] d\tau + o(X^{-\frac{1}{2}+\epsilon}), \end{aligned}$$

with

$$\begin{aligned} A_D(-r, r) &= \prod_p \left( 1 - \frac{1}{(p+1)p^{1-2r}} - \frac{1}{p+1} \right) \cdot \left( 1 - \frac{1}{p} \right)^{-1} \\ A'_D(r; r) &= \sum_p \frac{\log p}{(p+1)(p^{1+2r} - 1)}. \end{aligned}$$

**Proof:** Contour shifts,  $A_D(-r; r) = \zeta(2)/\zeta(2-2r)$ .

## Prediction from Ratios Conjecture (cont)

Main term is

$$\frac{1}{X^*} \sum_{d \leq X} \sum_{\gamma_d} g\left(\gamma_d \frac{\log X}{2\pi}\right) = \int_{-\infty}^{\infty} g(x) \left(1 - \frac{\sin(2\pi x)}{2\pi x}\right) dx + O\left(\frac{1}{\log X}\right),$$

which is the 1-level density for the scaling limit of  $\mathrm{USp}(2N)$ . If  $\mathrm{supp}(\hat{g}) \subset (-1, 1)$ , then the integral of  $g(x)$  against  $-\sin(2\pi x)/2\pi x$  is  $-g(0)/2$ .

## Prediction from Ratios Conjecture (cont)

Assuming RH for  $\zeta(s)$ , for  $\text{supp}(\widehat{g}) \subset (-\sigma, \sigma) \subset (-1, 1)$ :

$$\begin{aligned} & \frac{-2}{X^* \log X} \sum_{d \leq X} \int_{-\infty}^{\infty} g(\tau) e^{-2\pi i \tau \frac{\log(d/\pi)}{\log X}} \frac{\Gamma\left(\frac{1}{4} - \frac{\pi i \tau}{\log X}\right)}{\Gamma\left(\frac{1}{4} + \frac{\pi i \tau}{\log X}\right)} \zeta\left(1 - \frac{4\pi i \tau}{\log X}\right) A_D\left(-\frac{2\pi i \tau}{\log X}; \frac{2\pi i \tau}{\log X}\right) d\tau \\ &= -\frac{g(0)}{2} + O\left(X^{-\frac{3}{4}(1-\sigma)+\epsilon}\right); \end{aligned}$$

error term absorbed into  $O(X^{-1/2+\epsilon})$  if  $\sigma < 1/3$ .

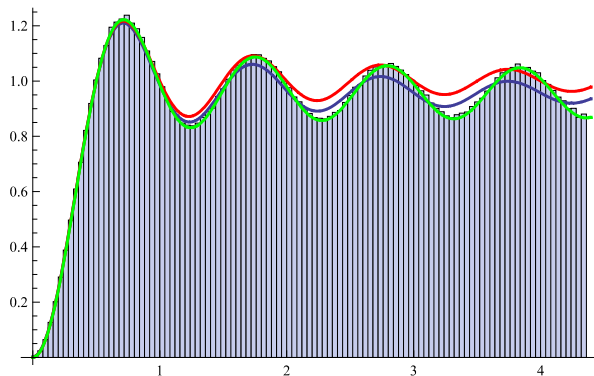
## Main Results

### Theorem (M- '07)

Let  $\text{supp}(\widehat{g}) \subset (-\sigma, \sigma)$ , assume RH for  $\zeta(s)$ . 1-Level Density agrees with prediction from Ratios Conjecture

- up to  $O(X^{-(1-\sigma)/2+\epsilon})$  for the family of quadratic Dirichlet characters with even fundamental discriminants at most  $X$ ;
- up to  $O(X^{-1/2} + X^{-(1-\frac{3}{2}\sigma)+\epsilon} + X^{-\frac{3}{4}(1-\sigma)+\epsilon})$  for our sub-family. If  $\sigma < 1/3$  then agrees up to  $O(X^{-1/2+\epsilon})$ .

# Numerics (J. Stopple): 1,003,083 negative fundamental discriminants $-d \in [10^{12}, 10^{12} + 3.3 \cdot 10^6]$



Histogram of normalized zeros ( $\gamma \leq 1$ , about 4 million).

- ◇ Red: main term.
- ◇ Blue: includes  $O(1/\log X)$  terms.
- ◇ Green: all lower order terms.

## Sketch of Symplectic Proofs

## Ratios Calculation

Hardest piece to analyze is

$$R(g; X) = -\frac{2}{X^* \log X} \sum_{d \leq X} \int_{-\infty}^{\infty} g(\tau) e^{-2\pi i \tau \frac{\log(d/\pi)}{\log X}} \frac{\Gamma\left(\frac{1}{4} - \frac{\pi i \tau}{\log X}\right)}{\Gamma\left(\frac{1}{4} + \frac{\pi i \tau}{\log X}\right)} \cdot \zeta\left(1 - \frac{4\pi i \tau}{\log X}\right) A_D\left(-\frac{2\pi i \tau}{\log X}; \frac{2\pi i \tau}{\log X}\right) d\tau,$$

$$A_D(-r, r) = \prod_p \left(1 - \frac{1}{(p+1)p^{1-2r}} - \frac{1}{p+1}\right) \cdot \left(1 - \frac{1}{p}\right)^{-1}.$$

Proof: shift contours, keep track of poles of ratios of  $\Gamma$  and zeta functions,  $A_D(-r; r) = \zeta(2)/\zeta(2-2r)$ .

## Ratios Calculation: Weaker result for $\text{supp}(\widehat{g}) \subset (-1, 1)$ .

- $d$ -sum is  $X^* e^{-2\pi i \left(1 - \frac{\log \pi}{\log X}\right) \tau} \left(1 - \frac{2\pi i \tau}{\log X}\right)^{-1} + O(X^{1/2})$ ;

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- decay of  $g$  restricts  $\tau$ -sum to  $|\tau| \leq \log X$ , Taylor expand everything but  $g$ : small error term and

$$\int_{|\tau| \leq \log X} g(\tau) \sum_{n=-1}^N \frac{a_n}{\log^n X} (2\pi i\tau)^n e^{-2\pi i(1 - \frac{\log \pi}{\log X})\tau} d\tau$$

$$= \sum_{n=-1}^N \frac{a_n}{\log^n X} \int_{|\tau| \leq \log X} (2\pi i\tau)^n g(\tau) e^{-2\pi i(1 - \frac{\log \pi}{\log X})\tau} d\tau;$$

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- decay of  $g$  restricts  $\tau$ -sum to  $|\tau| \leq \log X$ , Taylor expand everything but  $g$ : small error term and

$$\begin{aligned} & \int_{|\tau| \leq \log X} g(\tau) \sum_{n=-1}^N \frac{a_n}{\log^n X} (2\pi i\tau)^n e^{-2\pi i(1 - \frac{\log \pi}{\log X})\tau} d\tau \\ &= \sum_{n=-1}^N \frac{a_n}{\log^n X} \int_{|\tau| \leq \log X} (2\pi i\tau)^n g(\tau) e^{-2\pi i(1 - \frac{\log \pi}{\log X})\tau} d\tau; \end{aligned}$$

- from decay of  $g$  can extend the  $\tau$ -integral to  $\mathbb{R}$  (essential that  $N$  is fixed and finite!), for  $n \geq 0$  get the Fourier transform of  $g^{(n)}$  (the  $n^{\text{th}}$  derivative of  $g$ ) at  $1 - \frac{\pi}{\log X}$ , vanishes if  $\text{supp}(\widehat{g}) \subset (-1, 1)$ .

## Number Theory Sums

$$S_{\text{even}} = -\frac{2}{X^*} \sum_{d \leq X} \sum_{\ell=1}^{\infty} \sum_p \frac{\chi_d(p)^2 \log p}{p^\ell \log X} \widehat{g} \left( 2 \frac{\log p^\ell}{\log X} \right)$$

$$S_{\text{odd}} = -\frac{2}{X^*} \sum_{d \leq X} \sum_{\ell=0}^{\infty} \sum_p \frac{\chi_d(p) \log p}{p^{(2\ell+1)/2} \log X} \widehat{g} \left( \frac{\log p^{2\ell+1}}{\log X} \right).$$

## Number Theory Sums

### Lemma

Let  $\text{supp}(\widehat{g}) \subset (-\sigma, \sigma) \subset (-1, 1)$ . Then

$$\begin{aligned}
 S_{\text{even}} &= -\frac{g(0)}{2} + \frac{2}{\log X} \int_{-\infty}^{\infty} g(\tau) \frac{\zeta'}{\zeta} \left(1 + \frac{4\pi i\tau}{\log X}\right) d\tau \\
 &\quad + \frac{2}{\log X} \int_{-\infty}^{\infty} g(\tau) A'_D \left(\frac{2\pi i\tau}{\log X}; \frac{2\pi i\tau}{\log X}\right) + O(X^{-\frac{1}{2}+\epsilon}) \\
 S_{\text{odd}} &= O(X^{-\frac{1-\sigma}{2}} \log^6 X).
 \end{aligned}$$

If instead we consider the family of characters  $\chi_{8d}$  for odd, positive square-free  $d \in (0, X)$  ( $d$  a fundamental discriminant), then

$$S_{\text{odd}} = O(X^{-1/2+\epsilon} + X^{-(1-\frac{3}{2}\sigma)+\epsilon}).$$

## Analysis of $S_{\text{even}}$

$\chi_d(p)^2 = 1$  except when  $p|d$ . Replace  $\chi_d(p)^2$  with 1, and subtract off the contribution from when  $p|d$ :

$$\begin{aligned} S_{\text{even}} &= -2 \sum_{\ell=1}^{\infty} \sum_p \frac{\log p}{p^\ell \log X} \widehat{g} \left( 2 \frac{\log p^\ell}{\log X} \right) \\ &\quad + \frac{2}{X^*} \sum_{d \leq X} \sum_{\ell=1}^{\infty} \sum_{p|d} \frac{\log p}{p^\ell \log X} \widehat{g} \left( 2 \frac{\log p^\ell}{\log X} \right) \\ &= S_{\text{even};1} + S_{\text{even};2}. \end{aligned}$$

### Lemma (Perron's Formula)

$$S_{\text{even};1} = -\frac{g(0)}{2} + \frac{2}{\log X} \int_{-\infty}^{\infty} g(\tau) \frac{\zeta'}{\zeta} \left( 1 + \frac{4\pi i\tau}{\log X} \right) d\tau.$$

## Analysis of $S_{\text{even}}$ : $S_{\text{even};2}$

This piece gives us  $\int g(\tau) A'_D(-\dots, \dots)$ .

- Main ideas:
  - ◇ Restrict to  $p \leq X^{1/2}$ .
  - ◇ For  $p < X^{1/2}$ :  $\sum_{d \leq X, p|d} 1 = \frac{X^*}{p+1} + O(X^{1/2})$ .
  - ◇ Use Fourier Transform to expand  $\widehat{g}$ .

## Analysis of $S_{\text{odd}}$

$$S_{\text{odd}} = -\frac{2}{X^*} \sum_{\ell=0}^{\infty} \sum_p \frac{\log p}{p^{(2\ell+1)/2} \log X} \widehat{g} \left( \frac{\log p^{2\ell+1}}{\log X} \right) \sum_{d \leq X} \chi_d(p).$$

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### Jutila's bound

$$\sum_{\substack{1 < n \leq N \\ n \text{ non-square}}} \left| \sum_{\substack{0 < d \leq X \\ d \text{ fund. disc.}}} \chi_d(n) \right|^2 \ll NX \log^{10} N.$$

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Proof: Cauchy-Schwarz and Jutila:  $p^{2\ell+1}$  non-square:

$$\left( \sum_{\ell=0}^{\infty} \sum_{p^{(2\ell+1)/2} \leq X^\sigma} \left| \sum_{d \leq X} \chi_d(p) \right|^2 \right)^{1/2} \ll X^{\frac{1+\sigma}{2}} \log^5 X.$$

## Analysis of $S_{\text{odd}}$ : Extending Support

More technical, replace Jutila's bound by applying Poisson Summation to character sums (Gao's thesis, Michigan 2005).

### Lemma

Let  $\text{supp}(\widehat{g}) \subset (-\sigma, \sigma) \subset (-1, 1)$ . For family  $\{8d : 0 < d \leq X, d \text{ an odd, positive square-free fundamental discriminant}\}$ ,  $S_{\text{odd}} = O(X^{-\frac{1}{2}+\epsilon} + X^{-(1-\frac{3}{2}\sigma)+\epsilon})$ .  
In particular, if  $\sigma < 1/3$  then  $S_{\text{odd}} = O(X^{-1/2+\epsilon})$ .

Orthogonal Results  
(joint with David Montague)

## Background

Study  $L(s, f) = \sum \lambda_f(n)n^{-s}$  with  $f$  ranging over cuspidal newforms of weight  $k$  and prime level  $N \rightarrow \infty$ .

Iwaniec-Luo-Sarnak calculated 1-level density if  $\text{supp}(\widehat{\phi}) \subset (-2, 2)$ .

Key ingredient: averaging  $\lambda_f(n)$ 's over family by the Petersson formula.

**Note:** Use harmonic weights and assume level  $N$  prime to facilitate using Petersson formula.

## Petersson Formula

Let

$$\Delta_{k,N}(m, n) = \sum_{f \in \mathcal{B}_k(N)} \omega_f(N) \lambda_f(m) \lambda_f(n).$$

We have

$$\Delta_{k,N}(m, n) = \delta(m, n) + 2\pi i^k \sum_{c \equiv 0 \pmod{N}} \frac{S(m, n; c)}{c} J_{k-1} \left( \frac{4\pi \sqrt{mn}}{c} \right)$$

where  $\delta(m, n)$  is the Kronecker symbol

$$S(m, n; c) = \sum_{d \pmod{c}}^* \exp \left( 2\pi i \frac{md + n\bar{d}}{c} \right)$$

is the classical Kloosterman sum ( $d\bar{d} \equiv 1 \pmod{c}$ ), and  $J_{k-1}(x)$  is a Bessel function.

## Consequences of the Petersson Formula

$$\Delta_{k,N}(m, n) = \delta(m, n) + 2\pi i^k \sum_{c \equiv 0 \pmod{N}} \frac{S(m, n; c)}{c} J_{k-1} \left( \frac{4\pi \sqrt{mn}}{c} \right)$$

The Bessel-Kloosterman piece contributes an error term if  $\sigma < 1$  and a main term otherwise.

The ‘diagonal’ piece does not include the Bessel-Kloosterman term, which we know contributes!

Possible danger: Ratios Conjecture says only to keep diagonal or main terms, and dropping a smaller contribution which becomes quite large!

## Main Results: Test for family $\mathcal{F} = H_k^\pm(N)$

This family is an important test: the non-diagonal terms that are dropped contribute to the main term!

### Theorem: Ratios Conjecture Prediction (M-Montague)

With  $\chi(s) = \prod_p \left(1 + \frac{1}{(p-1)p^s}\right)$ , the 1-level density is

$$\sum_p \frac{2 \log p}{p \log R} \widehat{\phi} \left( \frac{2 \log p}{\log R} \right)$$

$$\mp 2 \lim_{\epsilon \downarrow 0} \int_{-\infty}^{\infty} \mathbb{X}_L \left( \frac{1}{2} + 2\pi i x \right) \chi(\epsilon + 4\pi i x) \phi(t \log R) dt$$

$$- \int_{-\infty}^{\infty} \frac{\mathbb{X}'_L}{\mathbb{X}_L} \left( \frac{1}{2} + 2\pi i t \right) \phi(t \log R) dt + O(N^{-1/2+\epsilon}).$$

## Main Results: Test for family $\mathcal{F} = H_k^\pm(N)$

This family is an important test: the non-diagonal terms that are dropped contribute to the main term!

### Theorem: Agreement with Number Theory (M-Montague)

Assume GRH for  $\zeta(s)$ , Dirichlet  $L$ -functions, and  $L(s, f)$ . For  $\phi$  such that  $\text{supp}(\widehat{\phi}) \subset (-1, 1)$ , the 1-level density agrees with the ratios conjecture prediction up to  $O(N^{-1/2+\epsilon})$ , and get agreement up to a power savings in  $N$  if  $\text{supp}(\widehat{\phi}) \subset (-2, 2)$ .

## Key fact

## Theorem:

For  $\Re(\alpha), \Re(\gamma) > 0$ , the Ratios Conjecture predicts that

$$\begin{aligned}
 \mathcal{R}_{\pm}(N) &:= \sum_{f \in H_k^{\pm}(N)} \omega_f^{\pm}(N) \frac{L(\frac{1}{2} + \alpha, f)}{L(\frac{1}{2} + \gamma, f)} \\
 &= \prod_p \left( 1 - \frac{1}{p^{1+\alpha+\gamma}} + \frac{1}{p^{1+2\gamma}} \right) \pm \mathbb{X}_L \left( \frac{1}{2} + \alpha \right) \\
 &\quad \cdot \frac{1}{\zeta(1 - \alpha + \gamma)} \prod_p \left( 1 + \frac{p^{1-\alpha+\gamma}}{p^{1+2\gamma}(p^{1-\alpha+\gamma} - 1)} \right) \\
 &\quad + O(N^{-1/2+\epsilon}).
 \end{aligned}$$

## Proof (cont)

**Proof:**  $\mathcal{R}_{\pm}(N)$  equals

$$\sum_{f \in H_k^*(N)} (1 \pm \epsilon_f) \omega_f^*(N) \left( \sum_{h=1}^{\infty} \frac{\mu_f(h)}{h^{\frac{1}{2} + \gamma}} \right) \left[ \sum_{m \leq x} \frac{\lambda_f(m)}{m^{\frac{1}{2} + \alpha}} + \epsilon_f \mathbb{X}_L \left( \frac{1}{2} + \alpha \right) \sum_{n \leq y} \frac{\lambda_f(n)}{n^{\frac{1}{2} - \alpha}} \right].$$

Expanding, find  $\mathcal{R}_{\pm}(N)$  is

$$\begin{aligned} & \sum_{f \in H_k^*(N)} \omega_f^*(N) \sum_{h=1}^{\infty} \frac{\mu_f(h)}{h^{\frac{1}{2} + \gamma}} \left[ \sum_{m \leq x} \frac{\lambda_f(m)}{m^{\frac{1}{2} + \alpha}} + \epsilon_f \mathbb{X}_L \left( \frac{1}{2} + \alpha \right) \sum_{n \leq y} \frac{\lambda_f(n)}{n^{\frac{1}{2} - \alpha}} \right] \\ & \pm \sum_{f \in H_k^*(N)} \omega_f^*(N) \sum_{h=1}^{\infty} \frac{\mu_f(h)}{h^{\frac{1}{2} + \gamma}} \left[ \epsilon_f \sum_{m \leq x} \frac{\lambda_f(m)}{m^{\frac{1}{2} + \alpha}} + \mathbb{X}_L \left( \frac{1}{2} + \alpha \right) \sum_{n \leq y} \frac{\lambda_f(n)}{n^{\frac{1}{2} - \alpha}} \right]. \end{aligned}$$

Terms involving  $\epsilon_f$  are negligible and may be dropped (part of Ratios Conjecture, but can prove small).

## Proof (cont)

Left with

$$S_1 := \sum_{f \in H_k^*(N)} \omega_f^*(N) \sum_{h=1}^{\infty} \frac{\mu_f(h)}{h^{\frac{1}{2}+\gamma}} \sum_{m \leq x} \frac{\lambda_f(m)}{m^{\frac{1}{2}+\alpha}}$$

$$S_2 := \pm \sum_{f \in H_k^*(N)} \omega_f^*(N) \sum_{h=1}^{\infty} \frac{\mu_f(h)}{h^{\frac{1}{2}+\gamma}} \mathbb{X}_L \left( \frac{1}{2} + \alpha \right) \sum_{n \leq y} \frac{\lambda_f(n)}{n^{\frac{1}{2}-\alpha}}.$$

Analysis yields

$$S_1 = \prod_p \left( 1 - \frac{1}{p^{1+\alpha+\gamma}} + \frac{1}{p^{1+2\gamma}} \right) + O(N^{-1/2+\epsilon})$$

$$S_2 = \pm \mathbb{X}_L \left( \frac{1}{2} + \alpha \right) \frac{1}{\zeta(1-\alpha+\gamma)} \prod_p \left( 1 + \frac{p^{1-\alpha+\gamma}}{p^{1+2\gamma}(p^{1-\alpha+\gamma}-1)} \right) + O(N^{-1/2+\epsilon}).$$

## Proof (cont)

Differentiating yields

$$\sum_{f \in H_k^\pm(N)} \omega_f^\pm(N) \frac{L'(\frac{1}{2} + r, f)}{L(\frac{1}{2} + r, f)} = \sum_p \left( \frac{\log p}{p^{1+2r}} \right) \mp \mathbb{X}_L \left( \frac{1}{2} + r \right) \chi(2r) + O(N^{-1/2+\epsilon}),$$

where  $\chi(s)$  is defined as

$$\chi(s) := \prod_p \left( 1 + \frac{1}{(p-1)p^s} \right).$$

Quadratic Twist Families (work in progress)  
(joint with Duc Khiem Huynh, Ralph Morrison)

## Families

- 1 Studying quadratic twists of a fixed elliptic curve (with Duc Khiem Huynh);
- 2 Studying quadratic twists of the  $\tau$  function (with Ralph Morrison).

Second family easier (all primes are good).

Difficulty: analyzing product over prime piece.

## Predictions

### Number Theory Predictions:

$$\begin{aligned}
 & \sum_{d \in \mathcal{F}(X)} \sum_{\gamma_d} g(\gamma_d) \\
 = & \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\nu) \left( \sum_{d \in \mathcal{F}(X)} \left[ 2 \log \left( \frac{d}{2\pi} \right) + \frac{\Gamma'}{\Gamma} (6 + i\nu) + \frac{\Gamma'}{\Gamma} (6 - i\nu) \right] \right. \\
 & \left. + 2 \left( - \sum_p \sum_{k=1}^{\infty} \frac{(\alpha_p^{2k} + \bar{\alpha}_p^{2k}) \log p}{p^{k(1+2i\nu)}} + \sum_p \frac{\log p}{(p+1)} \sum_{k=1}^{\infty} \frac{(\alpha_p^{2k} + \bar{\alpha}_p^{2k})}{p^{k(1+2i\nu)}} \right) \right) d\nu \\
 & + O(X^{1/2} \log \log X).
 \end{aligned}$$

# Predictions

## Ratios' Prediction:

$$\begin{aligned}
 & \sum_{d \in \mathcal{F}(X)} \sum_{\gamma_d} g(\gamma_d) \\
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\nu) \left( \sum_{d \in \mathcal{F}(X)} \left[ 2 \log \left( \frac{d}{2\pi} \right) + \frac{\Gamma'}{\Gamma} (6 + i\nu) + \frac{\Gamma'}{\Gamma} (6 - i\nu) \right] \right. \\
 & \quad + 2 \left( -\frac{\zeta'}{\zeta} (1 + 2i\nu) + \frac{L'_{\Delta}}{L_{\Delta}}(\text{sym}^2, 1 + 2i\nu) + B'_{\Delta}(i\nu; i\nu) \right. \\
 & \quad \left. \left. - \left( \frac{d}{2\pi} \right)^{-2it} \frac{\Gamma(6 - i\nu)}{\Gamma(6 + i\nu)} \frac{\zeta(1 + 2i\nu) L_{\Delta}(\text{sym}^2, 1 - 2i\nu)}{L_{\Delta}(\text{sym}^2, 1)} B_{\Delta}(-i\nu, i\nu) \right) \right) d\nu \\
 & \quad + O(X^{1/2+\varepsilon}).
 \end{aligned}$$

## Matching terms (cont)

Easier piece to analyze:

$$-\frac{\zeta'}{\zeta}(1+2i\nu) + \frac{L'_{\Delta}}{L_{\Delta}}(\text{sym}^2, 1+2i\nu) = -\sum_p \sum_{k=1}^{\infty} \frac{(\alpha_p^{2k} + \bar{\alpha}_p^{2k}) \log p}{p^{k(1+2i\nu)}}.$$

## Matching terms (cont)

Harder piece to analyze:

$$B_{\Delta}(\alpha; \gamma) = \prod_p \left( 1 + \frac{p}{p+1} \left( \sum_{m=1}^{\infty} \frac{\tau^*(p^{2m})}{p^{m(1+2\alpha)}} - \frac{\tau^*(p)}{p^{1+\alpha+\gamma}} \sum_{m=0}^{\infty} \frac{\tau^*(p^{2m+1})}{p^{m(1+2\alpha)}} + \frac{1}{p^{1+2\gamma}} \sum_{m=0}^{\infty} \frac{\tau^*(p^{2m})}{p^{m(1+2\alpha)}} \right) \right) \\ \times \frac{\left( 1 - \frac{\tau^*(p^2)}{p^{1+2\alpha}} + \frac{\tau^*(p^2)}{p^{2+4\alpha}} - \frac{1}{p^{3+6\alpha}} \right) \left( 1 - \frac{1}{p^{1+2\gamma}} \right)}{\left( 1 - \frac{\tau^*(p^2)}{p^{1+\alpha+\gamma}} + \frac{\tau^*(p^2)}{p^{2+2\alpha+2\gamma}} - \frac{1}{p^{3+3\alpha+3\gamma}} \right) \left( 1 - \frac{1}{p^{1+\alpha+\gamma}} \right)}$$

Differentiating with respect to  $\alpha$  and evaluating at  $\alpha = \gamma = i\nu$ , we have

$$B'_{\Delta}(i\nu; i\nu) \\ = \sum_p \log p \left( \frac{1}{p+1} \sum_{m=1}^{\infty} \frac{\alpha_p^{2m} + \bar{\alpha}_p^{2m}}{p^{m(1+2i\nu)}} - \sum_{m=1}^{\infty} \frac{\alpha_p^{2m} + \bar{\alpha}_p^{2m}}{p^{m(1+2i\nu)}} + \frac{\frac{\tau^*(p^2)}{p^{1+2i\nu}} - \frac{2 \cdot \tau^*(p^2)}{p^{2+4i\nu}} + \frac{3}{p^{3+6i\nu}}}{1 - \frac{\tau^*(p^2)}{p^{1+2i\nu}} + \frac{\tau^*(p^2)}{p^{2+4i\nu}} - \frac{1}{p^{3+6i\nu}}} + \frac{1}{1 - p^{1+2i\nu}} \right) \\ = \sum_p \log p \left( \frac{1}{p+1} \sum_{m=1}^{\infty} \frac{\alpha_p^{2m} + \bar{\alpha}_p^{2m}}{p^{m(1+2i\nu)}} + 0 \right) \\ = \sum_p \frac{\log p}{p+1} \sum_{m=1}^{\infty} \frac{\alpha_p^{2m} + \bar{\alpha}_p^{2m}}{p^{m(1+2i\nu)}}$$

# Status

Need to show the following is small:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} g(\nu) \left( \sum_{d \in \mathcal{F}(X)} \left( \frac{d}{2\pi} \right)^{-2it} \frac{\Gamma(6 - i\nu)}{\Gamma(6 + i\nu)} \frac{\zeta(1 + 2i\nu) L_{\Delta}(\text{sym}^2, 1 - 2i\nu)}{L_{\Delta}(\text{sym}^2, 1)} B_{\Delta}(-i\nu, i\nu) \right) d\nu$$

We have:






$$B_{\Delta}(-i\nu, i\nu) = \prod_p \left( 1 + \frac{p}{p+1} \left( \sum_{m=1}^{\infty} \frac{\tau^*(p^{2m})}{p^{m(1-2i\nu)}} - \frac{\tau^*(p)}{p} \sum_{m=0}^{\infty} \frac{\tau^*(p^{2m+1})}{p^{m(1-2i\nu)}} + \frac{1}{p^{1+2i\nu}} \sum_{m=0}^{\infty} \frac{\tau^*(p^{2m})}{p^{m(1-2i\nu)}} \right) \right) \\ \times \frac{\left( 1 - \frac{\tau^*(p^2)}{p^{1-2i\nu}} + \frac{\tau^*(p^2)}{p^{2-4i\nu}} - \frac{1}{p^{3-6i\nu}} \right) \left( 1 - \frac{1}{p^{1+2i\nu}} \right)}{\left( 1 - \frac{\tau^*(p^2)}{p} + \frac{\tau^*(p^2)}{p^2} - \frac{1}{p^3} \right) \left( 1 - \frac{1}{p} \right)}$$






Conclusions






## Conclusions






- Ratios Conjecture gives detailed predictions (up to  $X^{1/2+\epsilon}$ ).
- Number Theory agrees with predictions for suitably restricted test functions.
- Numerics quite good.
- Similar results other families.
  - ◇ All Dirichlet characters: SMALL '09.
  - ◇ Quadratic twists of  $\tau$ -function and a fixed elliptic curve: D. K. Huynh, S. J. Miller and R. Morrison.







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