

From the Manhattan Project to Number Theory

Steven J Miller
Williams College

Steven.J.Miller@williams.edu
<http://www.williams.edu/go/math/sjmilller>

Wellesley College, February 2, 2010

Acknowledgements

- Much of this is joint and current work with Eduardo Dueñez, Duc Khiem Huynh, Jon Keating and Nina Snaith.
- Computer programs written with Adam O'Brien, Jon Hsu, Leo Goldmahker, Stephen Lu and Mike Rubinstein.

Outline

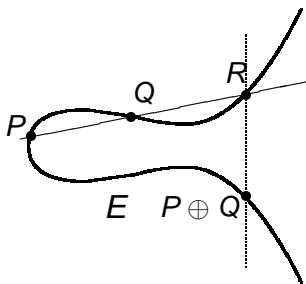
- Review elliptic curves and L -functions.
- Introduce relevant RMT ensembles.
- Reconcile theory and data.

Elliptic Curves and L -functions

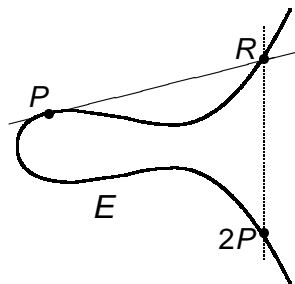
Mordell-Weil Group

Elliptic curve $y^2 = x^3 + ax + b$ with rational solutions

$P = (x_1, y_1)$ and $Q = (x_2, y_2)$ and connecting line $y = mx + b$.



Addition of distinct points P and Q



Adding a point P to itself

$$E(\mathbb{Q}) \approx E(\mathbb{Q})_{\text{tors}} \oplus \mathbb{Z}^r$$

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \operatorname{Re}(s) > 1.$$

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \operatorname{Re}(s) > 1.$$

Unique Factorization: $n = p_1^{r_1} \cdots p_m^{r_m}$.

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \operatorname{Re}(s) > 1.$$

Unique Factorization: $n = p_1^{r_1} \cdots p_m^{r_m}$.

$$\begin{aligned} \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} &= \left[1 + \frac{1}{2^s} + \left(\frac{1}{2^s}\right)^2 + \cdots\right] \left[1 + \frac{1}{3^s} + \left(\frac{1}{3^s}\right)^2 + \cdots\right] \cdots \\ &= \sum_n \frac{1}{n^s}. \end{aligned}$$

Riemann Zeta Function (cont)

$$\zeta(s) = \sum_n \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1$$
$$\pi(x) = \#\{p : p \text{ is prime}, p \leq x\}$$

Properties of $\zeta(s)$ and Primes:

Riemann Zeta Function (cont)

$$\zeta(s) = \sum_n \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \operatorname{Re}(s) > 1$$
$$\pi(x) = \#\{p : p \text{ is prime}, p \leq x\}$$

Properties of $\zeta(s)$ and Primes:

- $\lim_{s \rightarrow 1^+} \zeta(s) = \infty, \pi(x) \rightarrow \infty.$

Riemann Zeta Function (cont)

$$\zeta(s) = \sum_n \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1$$

$$\pi(x) = \#\{p : p \text{ is prime}, p \leq x\}$$

Properties of $\zeta(s)$ and Primes:

- $\lim_{s \rightarrow 1^+} \zeta(s) = \infty, \pi(x) \rightarrow \infty.$
- $\zeta(2) = \frac{\pi^2}{6}, \pi(x) \rightarrow \infty.$

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \operatorname{Re}(s) > 1.$$

Functional Equation:

$$\xi(s) = \Gamma\left(\frac{s}{2}\right) \pi^{-\frac{s}{2}} \zeta(s) = \xi(1-s).$$

Riemann Hypothesis (RH):

All non-trivial zeros have $\operatorname{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

General L -functions

$$L(s, f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} = \prod_{p \text{ prime}} L_p(s, f)^{-1}, \quad \operatorname{Re}(s) > 1.$$

Functional Equation:

$$\Lambda(s, f) = \Lambda_{\infty}(s, f)L(s, f) = \Lambda(1 - s, f).$$

Generalized Riemann Hypothesis (GRH):

All non-trivial zeros have $\operatorname{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

Elliptic curve L -function

$E : y^2 = x^3 + ax + b$, associate L -function

$$L(s, E) = \sum_{n=1}^{\infty} \frac{a_E(n)}{n^s} = \prod_{p \text{ prime}} L_E(p^{-s}),$$

where

$$a_E(p) = p - \#\{(x, y) \in (\mathbb{Z}/p\mathbb{Z})^2 : y^2 \equiv x^3 + ax + b \pmod{p}\}.$$

Elliptic curve L -function

$E : y^2 = x^3 + ax + b$, associate L -function

$$L(s, E) = \sum_{n=1}^{\infty} \frac{a_E(n)}{n^s} = \prod_{p \text{ prime}} L_E(p^{-s}),$$

where

$$a_E(p) = p - \#\{(x, y) \in (\mathbb{Z}/p\mathbb{Z})^2 : y^2 \equiv x^3 + ax + b \pmod{p}\}.$$

Birch and Swinnerton-Dyer Conjecture

Rank of group of rational solutions equals order of vanishing of $L(s, E)$ at $s = 1/2$.

Classical Random Matrix Theory

Fundamental Problem: Spacing Between Events

General Formulation: Studying system, observe values at t_1, t_2, t_3, \dots

Question: What rules govern the spacings between the t_j ?

Examples:

- Spacings b/w Energy Levels of Nuclei.
- Spacings b/w Eigenvalues of Matrices.
- Spacings b/w Primes.
- Spacings b/w $n^k \alpha \bmod 1$.
- Spacings b/w Zeros of L -functions.

Sketch of proofs

In studying many statistics, often three key steps:

- 1 Determine correct scale for events.
- 2 Develop an explicit formula relating what we want to study to something we understand.
- 3 Use an averaging formula to analyze the quantities above.

It is not always trivial to figure out what is the correct statistic to study!

Origins of Random Matrix Theory

Classical Mechanics: 3 Body Problem Intractable.

Origins of Random Matrix Theory

Classical Mechanics: 3 Body Problem Intractable.

Heavy nuclei (Uranium: 200+ protons / neutrons) worse!

Origins of Random Matrix Theory

Classical Mechanics: 3 Body Problem Intractable.

Heavy nuclei (Uranium: 200+ protons / neutrons) worse!

Get some info by shooting high-energy neutrons into nucleus, see what comes out.

Origins of Random Matrix Theory

Classical Mechanics: 3 Body Problem Intractable.

Heavy nuclei (Uranium: 200+ protons / neutrons) worse!

Get some info by shooting high-energy neutrons into nucleus, see what comes out.

Fundamental Equation:

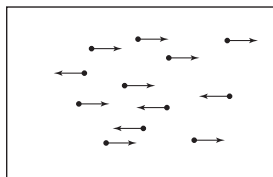
$$H\psi_n = E_n\psi_n$$

H : matrix, entries depend on system

E_n : energy levels

ψ_n : energy eigenfunctions

Origins (continued)



- Statistical Mechanics: for each configuration, calculate quantity (say pressure).
- Average over all configurations – most configurations close to system average.
- Nuclear physics: choose matrix at random, calculate eigenvalues, average over matrices (real Symmetric $A = A^T$, complex Hermitian $\bar{A}^T = A$).

Random Matrix Ensembles

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1N} & a_{2N} & a_{3N} & \cdots & a_{NN} \end{pmatrix} = A^T, \quad a_{ij} = a_{ji}$$

Fix p , define

$$\text{Prob}(A) = \prod_{1 \leq i < j \leq N} p(a_{ij}).$$

This means

$$\text{Prob}(A : a_{ij} \in [\alpha_{ij}, \beta_{ij}]) = \prod_{1 \leq i < j \leq N} \int_{x_{ij}=\alpha_{ij}}^{\beta_{ij}} p(x_{ij}) dx_{ij}.$$

Eigenvalue Distribution

$\delta(x - x_0)$ is a unit point mass at x_0 :

$$\int_{-\infty}^{\infty} f(x)\delta(x - x_0)dx = f(x_0).$$

Eigenvalue Distribution

$\delta(\mathbf{x} - \mathbf{x}_0)$ is a unit point mass at \mathbf{x}_0 :

$$\int_{-\infty}^{\infty} f(\mathbf{x})\delta(\mathbf{x} - \mathbf{x}_0)d\mathbf{x} = f(\mathbf{x}_0).$$

To each A , attach a probability measure:

$$\begin{aligned} \mu_{A,N}(\mathbf{x}) &= \frac{1}{N} \sum_{i=1}^N \delta\left(\mathbf{x} - \frac{\lambda_i(A)}{2\sqrt{N}}\right) \\ \int_a^b \mu_{A,N}(\mathbf{x})d\mathbf{x} &= \frac{\#\left\{\lambda_i : \frac{\lambda_i(A)}{2\sqrt{N}} \in [a, b]\right\}}{N} \\ \text{k}^{\text{th}} \text{ moment} &= \frac{\sum_{i=1}^N \lambda_i(A)^k}{2^k N^{\frac{k}{2}+1}}. \end{aligned}$$

Eigenvalue Trace Lemma

Want to understand the eigenvalues of A , but it is the matrix elements that are chosen randomly and independently.

Eigenvalue Trace Lemma

Want to understand the eigenvalues of A , but it is the matrix elements that are chosen randomly and independently.

Eigenvalue Trace Lemma

Let A be an $N \times N$ matrix with eigenvalues $\lambda_i(A)$. Then

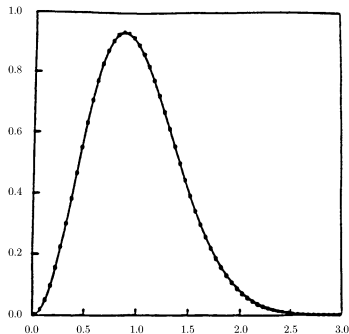
$$\text{Trace}(A^k) = \sum_{n=1}^N \lambda_i(A)^k,$$

where

$$\text{Trace}(A^k) = \sum_{i_1=1}^N \cdots \sum_{i_k=1}^N a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_N i_1}.$$

Results, Questions
and Conjectures

Zeros of $\zeta(s)$ vs GUE



70 million spacings b/w adjacent zeros of $\zeta(s)$, starting at the $10^{20\text{th}}$ zero (from Odlyzko) versus RMT prediction.

1-Level Density

L -function $L(s, f)$: by RH non-trivial zeros $\frac{1}{2} + i\gamma_{f,j}$.

C_f : analytic conductor.

$\varphi(x)$: compactly supported even Schwartz function.

$$D_{1,f}(\varphi) = \sum_j \varphi\left(\frac{\log C_f}{2\pi} \gamma_{f,j}\right)$$

1-Level Density

L -function $L(s, f)$: by RH non-trivial zeros $\frac{1}{2} + i\gamma_{f,j}$.

C_f : analytic conductor.

$\varphi(x)$: compactly supported even Schwartz function.

$$D_{1,f}(\varphi) = \sum_j \varphi\left(\frac{\log C_f}{2\pi} \gamma_{f,j}\right)$$

- individual zeros contribute in limit
- most of contribution is from low zeros

1-Level Density

L -function $L(s, f)$: by RH non-trivial zeros $\frac{1}{2} + i\gamma_{f,j}$.

C_f : analytic conductor.

$\varphi(\mathbf{x})$: compactly supported even Schwartz function.

$$D_{1,f}(\varphi) = \sum_j \varphi\left(\frac{\log C_f}{2\pi} \gamma_{f,j}\right)$$

- individual zeros contribute in limit
- most of contribution is from low zeros

Katz-Sarnak Conjecture:

$$D_{1,\mathcal{F}}(\varphi) = \lim_{N \rightarrow \infty} \frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} D_{1,f}(\varphi) = \int \varphi(\mathbf{x}) \rho_{G(\mathcal{F})}(\mathbf{x}) d\mathbf{x}.$$

Comparing the RMT Models

Theorem: M- '04

For small support, one-param family of rank r over $\mathbb{Q}(T)$:

$$\lim_{N \rightarrow \infty} \frac{1}{|\mathcal{F}_N|} \sum_{E_t \in \mathcal{F}_N} \sum_j \varphi \left(\frac{\log C_{E_t}}{2\pi} \gamma_{E_t, j} \right) = \int \varphi(\mathbf{x}) \rho_{\mathcal{G}}(\mathbf{x}) d\mathbf{x} + r\varphi(0)$$

where

$$\mathcal{G} = \begin{cases} \text{SO} & \text{if half odd} \\ \text{SO}(\text{even}) & \text{if all even} \\ \text{SO}(\text{odd}) & \text{if all odd} \end{cases}$$

Confirm Katz-Sarnak, B-SD predictions for small support.

Supports Independent and not Interaction model in the limit.

Sketch of Proof

- **Explicit Formula:** Relates sums over zeros to sums over primes.
- **Averaging Formulas:** Orthogonality of characters, Petersson formula.
- **Control of conductors:** Monotone.

Explicit Formula (Contour Integration)

$$-\frac{\zeta'(s)}{\zeta(s)} = -\frac{d}{ds} \log \zeta(s) = -\frac{d}{ds} \log \prod_p (1 - p^{-s})^{-1}$$

Explicit Formula (Contour Integration)

$$\begin{aligned}
 -\frac{\zeta'(s)}{\zeta(s)} &= -\frac{d}{ds} \log \zeta(s) = -\frac{d}{ds} \log \prod_p (1 - p^{-s})^{-1} \\
 &= \frac{d}{ds} \sum_p \log (1 - p^{-s}) \\
 &= \sum_p \frac{\log p \cdot p^{-s}}{1 - p^{-s}} = \sum_p \frac{\log p}{p^s} + \text{Good}(s).
 \end{aligned}$$

Explicit Formula (Contour Integration)

$$\begin{aligned}
 -\frac{\zeta'(s)}{\zeta(s)} &= -\frac{d}{ds} \log \zeta(s) = -\frac{d}{ds} \log \prod_p (1 - p^{-s})^{-1} \\
 &= \frac{d}{ds} \sum_p \log (1 - p^{-s}) \\
 &= \sum_p \frac{\log p \cdot p^{-s}}{1 - p^{-s}} = \sum_p \frac{\log p}{p^s} + \text{Good}(s).
 \end{aligned}$$

Contour Integration:

$$\int -\frac{\zeta'(s)}{\zeta(s)} \phi(s) ds \quad \text{vs} \quad \sum_p \log p \int \phi(s) p^{-s} ds.$$

Explicit Formula (Contour Integration)

$$\begin{aligned}
 -\frac{\zeta'(s)}{\zeta(s)} &= -\frac{d}{ds} \log \zeta(s) = -\frac{d}{ds} \log \prod_p (1 - p^{-s})^{-1} \\
 &= \frac{d}{ds} \sum_p \log (1 - p^{-s}) \\
 &= \sum_p \frac{\log p \cdot p^{-s}}{1 - p^{-s}} = \sum_p \frac{\log p}{p^s} + \text{Good}(s).
 \end{aligned}$$

Contour Integration (see Fourier Transform arising):

$$\int -\frac{\zeta'(s)}{\zeta(s)} \phi(s) ds \quad \text{vs} \quad \sum_p \frac{\log p}{p^s} \int \phi(s) e^{-it \log p} ds.$$

Knowledge of zeros gives info on coefficients.

Explicit Formula: Examples

Cuspidal Newforms: Let \mathcal{F} be a family of cuspidal newforms (say weight k , prime level N and possibly split by sign)

$L(s, f) = \sum_n \lambda_f(n)/n^s$. Then

$$\frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} \sum_{\gamma_f} \phi \left(\frac{\log R}{2\pi} \gamma_f \right) = \widehat{\phi}(0) + \frac{1}{2} \phi(0) - \frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} P(f; \phi) + O \left(\frac{\log \log R}{\log R} \right)$$

$$P(f; \phi) = \sum_{p \nmid N} \lambda_f(p) \widehat{\phi} \left(\frac{\log p}{\log R} \right) \frac{2 \log p}{\sqrt{p} \log R}.$$

RMT: Theoretical Results ($N \rightarrow \infty$)

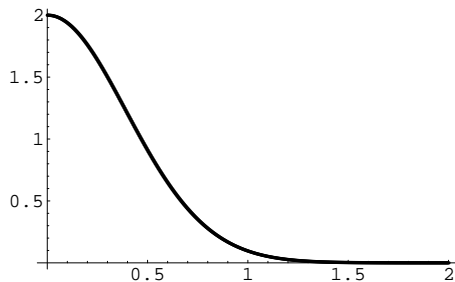


Figure 1a: 1st norm. eval. above 1: SO(even)

RMT: Theoretical Results ($N \rightarrow \infty$)

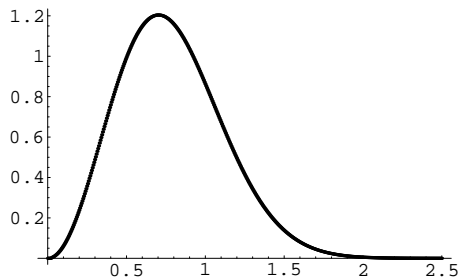


Figure 1b: 1st norm. eval. above 1: SO(odd)

Rank 0 Curves: 1st Normalized Zero above Central Point

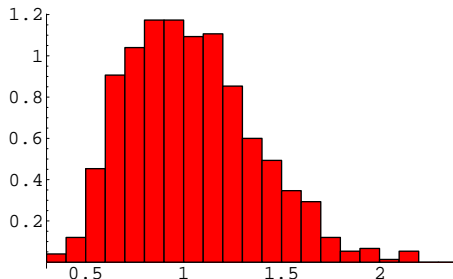


Figure 2a: 750 rank 0 curves from
 $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$.
 $\log(\text{cond}) \in [3.2, 12.6]$, median = 1.00 mean = 1.04, $\sigma_\mu = .32$

Rank 0 Curves: 1st Normalized Zero above Central Point

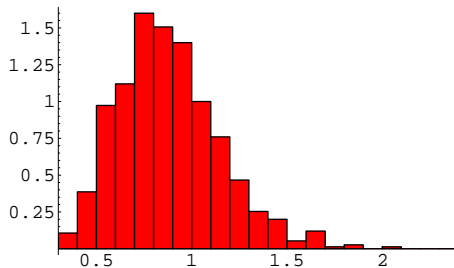
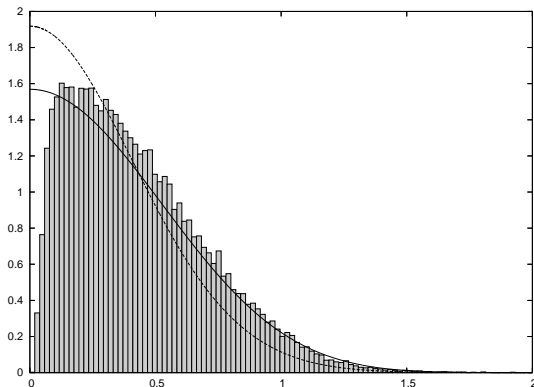


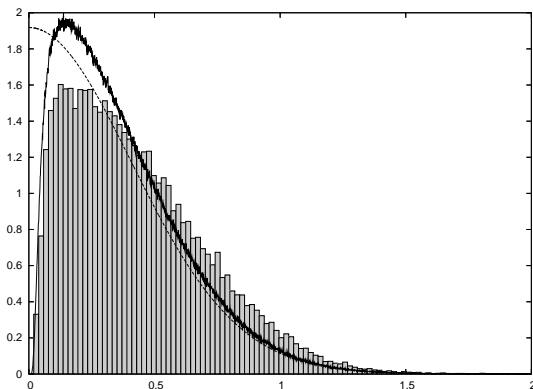
Figure 2b: 750 rank 0 curves from
 $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$.
 $\log(\text{cond}) \in [12.6, 14.9]$, median = .85, mean = .88, $\sigma_\mu = .27$

Modeling lowest zero of $L_{E_{11}}(s, \chi_d)$ with $0 < d < 400,000$



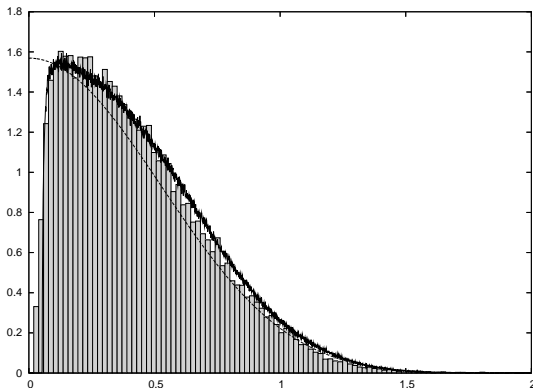
Lowest zero for $L_{E_{11}}(s, \chi_d)$ (bar chart), lowest eigenvalue of $SO(2N)$ with N_{eff} (solid), standard N_0 (dashed).

Modeling lowest zero of $L_{E_{11}}(s, \chi_d)$ with $0 < d < 400,000$



Lowest zero for $L_{E_{11}}(s, \chi_d)$ (bar chart), lowest eigenvalue of $SO(2N)$ with $N_0 = 12$ (solid) with discretisation and with standard $N_0 = 12.26$ (dashed) without discretisation.

Modeling lowest zero of $L_{E_{11}}(s, \chi_d)$ with $0 < d < 400,000$



Lowest zero for $L_{E_{11}}(s, \chi_d)$ (bar chart), lowest eigenvalue of $SO(2N)$ effective N of $N_{\text{eff}} = 2$ (solid) with discretisation and with effective N of $N_{\text{eff}} = 2.32$ (dashed) without discretisation.

Conclusions

Conclusions

- L -functions encode arithmetic.
- Understand behavior as conductors tend to infinity.
- New random matrix model (incorporates arithmetic and discretization).
- **Similarities between L -Functions and Nuclei:**

Zeros \longleftrightarrow Energy Levels

Schwartz test function \longrightarrow Neutron

Support of test function \longleftrightarrow Neutron Energy.

References

Caveat: this bibliography is only meant to be a first reference.



B. Bektemirov, B. Mazur and M. Watkins, *Average Ranks of Elliptic Curves*, preprint.



B. Birch and H. Swinnerton-Dyer, *Notes on elliptic curves. I*, J. reine angew. Math. **212**, 1963, 7 – 25.



B. Birch and H. Swinnerton-Dyer, *Notes on elliptic curves. II*, J. reine angew. Math. **218**, 1965, 79 – 108.



C. Breuil, B. Conrad, F. Diamond and R. Taylor, *On the modularity of elliptic curves over \mathbf{Q} : wild 3-adic exercises*, J. Amer. Math. Soc. **14**, no. 4, 2001, 843 – 939.



A. Brumer, *The average rank of elliptic curves I*, Invent. Math. **109**, 1992, 445 – 472.



A. Brumer and R. Heath-Brown, *The average rank of elliptic curves III*, preprint.



A. Brumer and R. Heath-Brown, *The average rank of elliptic curves V*, preprint.



A. Brumer and O. McGuinness, *The behaviour of the Mordell-Weil group of elliptic curves*, Bull. AMS **23**, 1991, 375 – 382.



J. Coates and A. Wiles, *On the conjecture of Birch and Swinnerton-Dyer*, Invent. Math. **39**, 1977, 43 – 67.



Cremona, *Algorithms for Modular Elliptic Curves*, Cambridge University Press, 1992.



F. Diamond, *On deformation rings and Hecke rings*, Ann. Math. **144**, 1996, 137 – 166.



S. Fermigier, *Zéros des fonctions L de courbes elliptiques*, Exper. Math. **1**, 1992, 167 – 173.



S. Fermigier, *Étude expérimentale du rang de familles de courbes elliptiques sur \mathbb{Q}* , Exper. Math. **5**, 1996, 119 – 130.



E. Fouvrey and J. Pomykala, *Rang des courbes elliptiques et sommes d'exponentelles*, Monat. Math. **116**, 1993, 111 – 125.



F. Gouvêa and B. Mazur, *The square-free sieve and the rank of elliptic curves*, J. Amer. Math. Soc. **4**, 1991, 45 – 65.



D. Goldfeld, *Conjectures on elliptic curves over quadratic fields*, Number Theory (Proc. Conf. in Carbondale, 1979), Lecture Notes in Math. **751**, Springer-Verlag, 1979, 108 – 118.



H. Iwaniec, W. Luo and P. Sarnak, *Low lying zeros of families of L -functions*, Inst. Hautes Études Sci. Publ. Math. **91**, 2000, 55 – 131.



A. Knapp, *Elliptic Curves*, Princeton University Press, Princeton, 1992.



N. Katz and P. Sarnak, *Random Matrices, Frobenius Eigenvalues and Monodromy*, AMS Colloquium Publications **45**, AMS, Providence, 1999.



N. Katz and P. Sarnak, *Zeros of zeta functions and symmetries*, Bull. AMS **36**, 1999, 1 – 26.



V. Kolyvagin, *On the Mordell-Weil group and the Shafarevich-Tate group of modular elliptic curves*, Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990), Math. Soc. Japan, Tokyo, 1991, 429 – 436.



L. Mai, *The analytic rank of a family of elliptic curves*, Canadian Journal of Mathematics **45**, 1993, 847 – 862.



J. Mestre, *Formules explicites et minorations de conducteurs de variétés algébriques*, Compositio Mathematica **58**, 1986, 209 – 232.



J. Mestre, *Courbes elliptiques de rang ≥ 11 sur $\mathbf{Q}(t)$* , C. R. Acad. Sci. Paris, ser. 1, **313**, 1991, 139 – 142.



J. Mestre, *Courbes elliptiques de rang ≥ 12 sur $\mathbf{Q}(t)$* , C. R. Acad. Sci. Paris, ser. 1, **313**, 1991, 171 – 174.



P. Michel, *Rang moyen de familles de courbes elliptiques et lois de Sato-Tate*, *Monat. Math.* **120**, 1995, 127 – 136.



S. J. Miller, *1- and 2-Level Densities for Families of Elliptic Curves: Evidence for the Underlying Group Symmetries*, P.H.D. Thesis, Princeton University, 2002,
<http://www.math.princeton.edu/~sjmiller/thesis/thesis.pdf>.



S. J. Miller, *1- and 2-level densities for families of elliptic curves: evidence for the underlying group symmetries*, *Compositio Mathematica* **140** (2004), 952-992.



S. J. Miller, *Variation in the number of points on elliptic curves and applications to excess rank*, *C. R. Math. Rep. Acad. Sci. Canada* **27** (2005), no. 4, 111–120.



S. J. Miller, *Investigations of zeros near the central point of elliptic curve L -functions* (with an appendix by E. Dueñez), *Experimental Mathematics* **15** (2006), no. 3, 257–279.



K. Nagao, *Construction of high-rank elliptic curves*, *Kobe J. Math.* **11**, 1994, 211 – 219.



K. Nagao, *$\mathbb{Q}(t)$ -rank of elliptic curves and certain limit coming from the local points*, *Manuscr. Math.* **92**, 1997, 13 – 32.



Rizzo, *Average root numbers for a non-constant family of elliptic curves*, preprint.



D. Rohrlich, *Variation of the root number in families of elliptic curves*, Compos. Math. **87**, 1993, 119 – 151.



M. Rosen and J. Silverman, *On the rank of an elliptic surface*, Invent. Math. **133**, 1998, 43 – 67.



M. Rubinstein, *Evidence for a spectral interpretation of the zeros of L -functions*, P.H.D. Thesis, Princeton University, 1998, <http://www.ma.utexas.edu/users/miker/thesis/thesis.html>.



Z. Rudnick and P. Sarnak, *Zeros of principal L -functions and random matrix theory*, Duke Journal of Math. **81**, 1996, 269 – 322.



T. Shioda, *Construction of elliptic curves with high-rank via the invariants of the Weyl groups*, J. Math. Soc. Japan **43**, 1991, 673 – 719.



J. Silverman, *The Arithmetic of Elliptic Curves*, Graduate Texts in Mathematics **106**, Springer-Verlag, Berlin - New York, 1986.



J. Silverman, *Advanced Topics in the Arithmetic of Elliptic Curves*, Graduate Texts in Mathematics **151**, Springer-Verlag, Berlin - New York, 1994.



J. Silverman, *The average rank of an algebraic family of elliptic curves*, J. reine angew. Math. **504**, 1998, 227 – 236.



N. Snaith, *Derivatives of random matrix characteristic polynomials with applications to elliptic curves*, preprint.



C. Stewart and J. Top, *On ranks of twists of elliptic curves and power-free values of binary forms*, Journal of the American Mathematical Society **40**, number 4, 1995.



R. Taylor and A. Wiles, *Ring-theoretic properties of certain Hecke algebras*, Ann. Math. **141**, 1995, 553 – 572.



A. Wiles, *Modular elliptic curves and Fermat's last theorem*, Ann. Math **141**, 1995, 443 – 551.



M. Young, *Lower order terms of the 1-level density of families of elliptic curves*, IMRN **10** (2005), 587–633.



M. Young, *Low-Lying Zeros of Families of Elliptic Curves*, JAMS, to appear.



D. Zagier and G. Kramarz, *Numerical investigations related to the L -series of certain elliptic curves*, J. Indian Math. Soc. **52** (1987), 51–69.