

Low Lying Zeros of Number Field L -functions

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Random Matrices and Number Theory

Random Matrix Ensembles

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1N} & a_{2N} & a_{3N} & \cdots & a_{NN} \end{pmatrix} = A^T, \quad a_{ij} = a_{ji}$$

Fix p , define

$$\text{Prob}(A) = \prod_{1 \leq i < j \leq N} p(a_{ij}).$$

This means

$$\text{Prob}(A : a_{ij} \in [\alpha_{ij}, \beta_{ij}]) = \prod_{1 \leq i < j \leq N} \int_{x_{ij}=\alpha_{ij}}^{\beta_{ij}} p(x_{ij}) dx_{ij}.$$

Want to understand eigenvalues of A .

Eigenvalue Distribution

$\delta(\mathbf{x} - \mathbf{x}_0)$ is a unit point mass at \mathbf{x}_0 :

$$\int f(\mathbf{x})\delta(\mathbf{x} - \mathbf{x}_0)d\mathbf{x} = f(\mathbf{x}_0).$$

To each A , attach a probability measure:

$$\begin{aligned}\mu_{A,N}(\mathbf{x}) &= \frac{1}{N} \sum_{i=1}^N \delta\left(\mathbf{x} - \frac{\lambda_i(A)}{2\sqrt{N}}\right) \\ \int_a^b \mu_{A,N}(\mathbf{x})d\mathbf{x} &= \frac{\#\left\{\lambda_i : \frac{\lambda_i(A)}{2\sqrt{N}} \in [a, b]\right\}}{N}\end{aligned}$$

Random Matrix Theory: Eigenvalue Trace Formula

Want to understand the eigenvalues of A , but it is the matrix elements that are chosen randomly and independently.

Eigenvalue Trace Lemma

Let A be an $N \times N$ matrix with eigenvalues $\lambda_i(A)$. Then

$$\sum_{n=1}^N \lambda_i(A)^k = \text{Trace}(A^k),$$

where

$$\text{Trace}(A^k) = \sum_{i_1=1}^N \cdots \sum_{i_k=1}^N a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1}.$$

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \operatorname{Re}(s) > 1.$$

Functional Equation:

$$\xi(s) = \Gamma\left(\frac{s}{2}\right) \pi^{-\frac{s}{2}} \zeta(s) = \xi(1-s).$$

Riemann Hypothesis (RH):

All non-trivial zeros have $\operatorname{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

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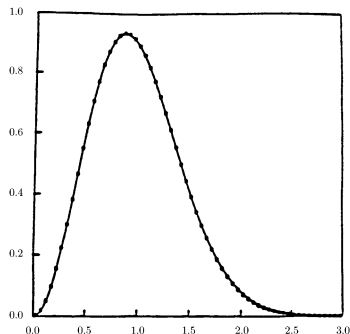
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Observation: Spacings b/w zeros appear same as b/w eigenvalues of Complex Hermitian matrices $\overline{A}^T = A$.

Zeros of $\zeta(s)$ vs GUE



70 million spacings b/w adjacent zeros of $\zeta(s)$, starting at the 10^{20} th zero (from Odlyzko)

General L-functions

$$L(s, f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} = \prod_{p \text{ prime}} L_p(s, f)^{-1}, \quad \operatorname{Re}(s) > 1.$$

Functional Equation:

$$\Lambda(s, f) = \Lambda_{\infty}(s, f)L(s, f) = \Lambda(1 - s, f).$$

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Measures of Spacings: 1-Level Density

$\phi(x)$ even Schwartz function whose Fourier Transform is compactly supported.

1-level density

$$D_f(\phi) = \sum_j \phi(L_f \gamma_{j,f})$$

- 1 Individual zeros contribute in limit.
- 2 Most of contribution is from low zeros.

ϕ decays too rapidly for this sum to be evaluated asymptotically.

Families of L -functions

Instead of looking at a single L -function $L(s, f)$, need to average over a family $\{L(s, f) : f \in \mathcal{F}\}$ of ‘similar’ L -functions.

1-level density for families

$$\begin{aligned} D_{\mathcal{F}}(\phi) &= \frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} D_f(\phi) \\ &= \frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} \sum_j \phi(L_f \gamma_{j;f}) \end{aligned}$$

Katz-Sarnak Conjecture

For a ‘nice’ family of L -functions, the 1-level density depends only on a symmetry group attached to the family.

Number Fields

Definitions

A *number field* is a subfield K of \mathbb{C} which forms a finite-dimensional vector space over \mathbb{Q} .

Examples

1 $\mathbb{Q}(\sqrt{-5}) = \{a + b\sqrt{-5} : a, b \in \mathbb{Q}\}$

2 $\mathbb{R}, \mathbb{C}, \bar{\mathbb{Q}}$ are not number fields

Definitions

Every number field K/\mathbb{Q} contains a *ring of algebraic integers* \mathcal{O}_K with properties similar to those of \mathbb{Z} .

Examples

① $K = \mathbb{Q}(\sqrt{-5})$. Have $\mathcal{O}_K = \mathbb{Z}[\sqrt{-5}]$, and

$$6 = 2 \cdot 3 = (1 - \sqrt{-5})(1 + \sqrt{-5})$$

② $K = \mathbb{Q}(i)$. Have $\mathcal{O}_K = \mathbb{Z}[i]$, and

$$2 = (1 - i)(1 + i)$$

③ $K = \mathbb{Q}(\sqrt{-3})$. Have $\mathcal{O}_K = \mathbb{Z} \left[\frac{1 + \sqrt{-3}}{2} \right]$.

Prime ideals

Because of non-unique factorization, doesn't help to study prime elements of \mathcal{O}_K .

Theorem (Unique factorization of ideals)

Let \mathfrak{a} be an ideal of \mathcal{O}_K . Then \mathfrak{a} factors uniquely as a product of prime ideals:

$$\mathfrak{a} = \mathfrak{p}_1^{e_1} \mathfrak{p}_2^{e_2} \cdots \mathfrak{p}_r^{e_r}.$$

Thus, we study prime ideals instead of prime elements.

Prime ideals

Fact: Every non-zero prime ideal \mathfrak{p} of \mathcal{O}_K is maximal.
Thus $\mathcal{O}_K/\mathfrak{p}$ is a field.

Lemma

The field $\mathcal{O}_K/\mathfrak{p}$ is finite.

We define $N\mathfrak{p} := |\mathcal{O}_K/\mathfrak{p}|$. If \mathfrak{a} is an arbitrary ideal of \mathcal{O}_K ,
define

$$N\mathfrak{a} = N\mathfrak{p}_1^{e_1} \cdots N\mathfrak{p}_r^{e_r}.$$

Dedekind zeta function

Given a number field K , we define for $\text{Re}(s) > 1$

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{N\mathfrak{a}^s}.$$

Have an Euler product

$$\zeta_K(s) = \prod_{\mathfrak{p}} \left(1 - \frac{1}{N\mathfrak{p}^s} \right)^{-1}.$$

How to attach L -functions to K ?

Number Field L -functions

Ideal class characters

An *ideal class character* is a function χ which assigns complex numbers to ideals of \mathcal{O}_K in a multiplicative fashion, i.e. $\chi(\mathfrak{a}\mathfrak{b}) = \chi(\mathfrak{a})\chi(\mathfrak{b})$, and which is trivial on principal ideals.

Given such a character, define the L -function

$$L(s, \chi) = \sum_{\mathfrak{a}} \frac{\chi(\mathfrak{a})}{N\mathfrak{a}^s} = \prod_{\mathfrak{p}} \left(1 - \frac{\chi(\mathfrak{p})}{N\mathfrak{p}^s} \right)^{-1}.$$

How to study the zeros of this function?

Explicit formula

Need an *explicit formula*, which relates sums over zeros to sums over primes.

Weil's explicit formula

$$\begin{aligned} \sum_{L(1/2+i\gamma_x, \chi)=0} \widehat{\phi}(\gamma_x) &= 4\delta_x \int_0^\infty \phi(x) \cosh(x/2) dx \\ &\quad + \phi(0) \left(\log \Delta - N\gamma - N \log 8\pi - \frac{r_1 \pi}{2} \right) \\ &\quad - \sum_{\mathfrak{p}} \log N\mathfrak{p} \sum_{m=1}^{\infty} \frac{\phi(m \log N\mathfrak{p})}{N\mathfrak{p}^{m/2}} (\chi(\mathfrak{p})^m + \chi(\mathfrak{p})^{-m}) \\ &\quad + r_1 \int_0^\infty \frac{\phi(0) - \phi(x)}{2 \cosh(x/2)} dx + N \int_0^\infty \frac{\phi(0) - \phi(x)}{2 \sinh(x/2)} dx \end{aligned}$$

Explicit formula

Gives a nice expression for the 1-level density:

$$\begin{aligned}
 D_x(\phi) &= \sum_{L(1/2+i\gamma_x)=0} \phi\left(\frac{\log \Delta}{2\pi}\gamma_x\right) \\
 &= \frac{1}{\log \Delta} \left[4\delta_x \int_0^\infty \widehat{\phi}\left(x\frac{2\pi}{\log \Delta}\right) \cosh(x/2) dx \right. \\
 &\quad + \widehat{\phi}(0)(\log \Delta - N\gamma - N \log 8\pi - \frac{r_1\pi}{2}) \\
 &\quad - \sum_{\mathfrak{p}} \log N_{\mathfrak{p}} \sum_{m=1}^{\infty} \frac{\widehat{\phi}\left(m\frac{2\pi}{\log \Delta} \log N_{\mathfrak{p}}\right)}{N_{\mathfrak{p}}^{m/2}} (\chi(\mathfrak{p})^m + \chi(\mathfrak{p})^{-m}) \\
 &\quad \left. + r_1 \int_0^\infty \frac{\widehat{\phi}(0) - \widehat{\phi}(x)}{2\cosh(x/2)} dx + N \int_0^\infty \frac{\widehat{\phi}(0) - \widehat{\phi}(x)}{2\sinh(x/2)} dx \right].
 \end{aligned}$$

Class number

The 1-level density for the family is then

$$D_{\mathcal{F}}(\phi) = \frac{1}{h} \sum_x D_x(\phi)$$

where h is the *class number* of K .

Need to understand what happens to h as K changes.

Main Difficulty: If there are many prime ideals with small norm, then the sum

$$\sum_{\mathfrak{p}} \log N_{\mathfrak{p}} \sum_{m=1}^{\infty} \frac{\widehat{\phi} \left(m 2\pi \frac{\log N_{\mathfrak{p}}}{\log \Delta} \right)}{N_{\mathfrak{p}}^{m/2}} (\chi(\mathfrak{p})^m + \chi(\mathfrak{p})^{-m})$$

might blow up.

Use algebraic number theory to control norms of primes.

Results

Fouvry and Iwaniec

Fouvry and Iwaniec studied the family $\{\mathbb{Q}(\sqrt{-D})\}$ for suitable D .

$$\rho = \frac{m^2 + Dn^2}{4}$$
$$\Rightarrow \rho \geq \frac{D}{4}$$

Theorem (Fouvry and Iwaniec, 2003)

Let ϕ be an even Schwartz function whose Fourier transform is supported in $(-1, 1)$. Then for the family $\{\mathbb{Q}(\sqrt{-D})\}$, the 1-level density is

$$D_{\mathcal{F}}(\phi) \sim \hat{\phi}(0) - \frac{1}{2}\phi(0) = \int \phi(x) W_{\text{Sp}}(x) dx$$

Main result

Theorem

Let K_0 be a normal, totally real field of class number one. Then for the family $\{K\}$ of all CM-fields such that $K^+ = K_0$, the 1-level density is again given by the symplectic distribution:

$$D_{\mathcal{F}}(\phi) \sim \hat{\phi}(0) - \frac{1}{2}\phi(0) = \int \phi(x) W_{\text{Sp}}(x) dx$$

THANK YOU!