From Nuclear Physics to Number Theory

How the Manhattan Project helped us understand primes

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Fundamental Problem: Spacing Between Events

General Formulation: Studying system, observe values at $t_1, t_2, t_3, \ldots$.

Question: what rules govern the spacings between the t_i?

Examples:

- Spacings b/w Primes.
- Spacings b/w Energy Levels of Nuclei.
- Spacings b/w Eigenvalues of Matrices.
- Spacings b/w Zeros of Functions.

Goals of the Talk

- Determine correct scale to study spacings.
- See similar behavior in different systems.
- Discuss tools / techniques needed to prove the results.

PART I

NORMALIZED SPACINGS

Normalized Spacing

Example: Fractional Parts

For $\alpha \notin \mathbb{Q}$, set $x_n = n\alpha \bmod 1$.

Order $x_1, \dots, x_N$: $0 \ \leq \ y_1 \ \leq \ \cdots \ \leq \ y_N \leq \ 1$.

Expect spacings between adjacent y's of size $\frac{1}{N}$.

Should study $\frac{y_{n+1} - y_n}{1/N}$.

Normalized Spacing

Example: Primes

$$\pi(x) = \#\{p : p \text{ prime}, p \leq x\} \approx \frac{x}{\log x}$$

$$(\text{Ave Spacing b/w Primes at most } x) = \frac{x}{\pi(x)} \approx \log x.$$

If $p_n, p_{n+1} \approx x$, study $\frac{p_{n+1}-p_n}{\log x}$.

One reason why twin primes are so hard: $\frac{2}{\log x} \to 0$.

PART II

PROBABILITY AND LINEAR ALGEBRA REVIEW

Probability Review

Let $p(x)$ be a probability density:

$$\begin{aligned} p(x) &\geq 0 \\ \int_{-\infty}^{\infty} p(x)dx &= 1. \end{aligned}$$

Thus

$$\text{Prob}\left(x \in [a,b]\right) = \int_a^b p(x)dx.$$

Moments:

$$k^{\text{th}}\text{-moment} = \int_{-\infty}^{\infty} x^k p(x)dx.$$

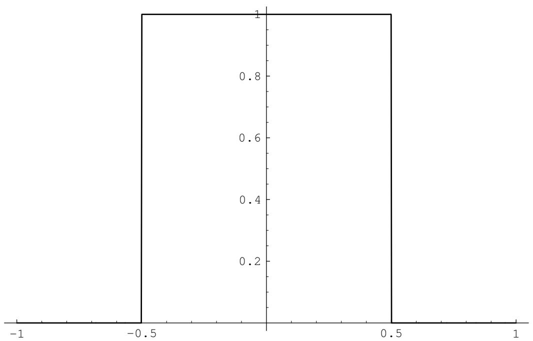
Good function uniquely determined by its Taylor Series, a good probability density is uniquely determined by its moments.

Probability Review (cont)

Important quantities:

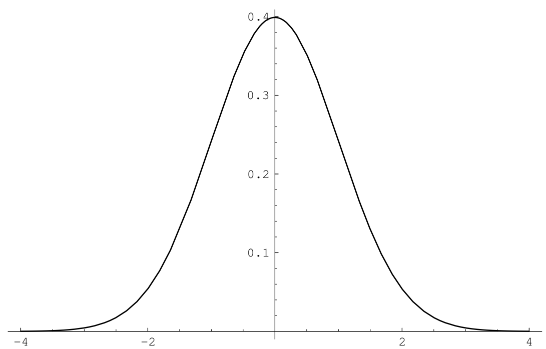
1. Mean $\mu = \int x p(x) dx$.

2. Variance $\sigma^2 = \int (x-\mu)^2 p(x) dx$.



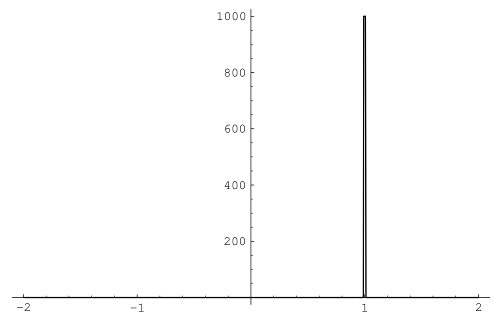
Uniform:$\mu = 0, \sigma^2 = \frac{1}{12}$

$$p(x) = \begin{cases} 1 & \text{if } |x| \le \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$



Gaussian:$\mu = 0, \sigma^2 = 1$

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$



Delta Spike:$\mu = 1, \sigma^2 = 0$

$$\delta(x-1)$$

Linear Algebra Review

$$\begin{pmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & \ddots & \vdots \\ a_{N1} & \cdots & a_{NN} \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_N \end{pmatrix} = \begin{pmatrix} w_1 \\ \vdots \\ w_N \end{pmatrix}$$

In general, in $A\overrightarrow{v} = \overrightarrow{w}$, $\overrightarrow{w}$ will have different **magnitude** and **direction** than $\overrightarrow{v}$.

$\overrightarrow{v} \neq \overrightarrow{0}$ is an **eigenvector** with **eigenvalue** λ if

$$A\overrightarrow{v} = \lambda\overrightarrow{v}.$$

Note

$$A^2\overrightarrow{v} = A(A\overrightarrow{v}) = A(\lambda\overrightarrow{v}) = \lambda^2\overrightarrow{v}.$$

Linear Algebra Review (cont)

Say $\overrightarrow{v_i}$ eigenvectors with eigenvalues λ_i.

Assume

$$\overrightarrow{v} = c_1\overrightarrow{v_1} + \cdots + c_k\overrightarrow{v_k}.$$

Then

$$\begin{aligned}
A^m\overrightarrow{v} &= A^m(c_1\overrightarrow{v_1} + \cdots + c_k\overrightarrow{v_k}) \\
&= A^m(c_1\overrightarrow{v_1}) + \cdots + A^m(c_k\overrightarrow{v_k}) \\
&= c_1A^m\overrightarrow{v_1} + \cdots + c_kA^m\overrightarrow{v_k} \\
&= c_1\lambda_1^m\overrightarrow{v_1} + \cdots + c_k\lambda_k^m\overrightarrow{v_k}.
\end{aligned}$$

PART III

RANDOM MATRIX THEORY

Origins of Random Matrix Theory

Classical Mechanics: 3 Body Problem Intractable.

Heavy nuclei like Uranium ($200+$ protons / neutrons) even worse!

Get some info by shooting high-energy neutrons into nucleus, see what comes out.

Fundamental Equation:

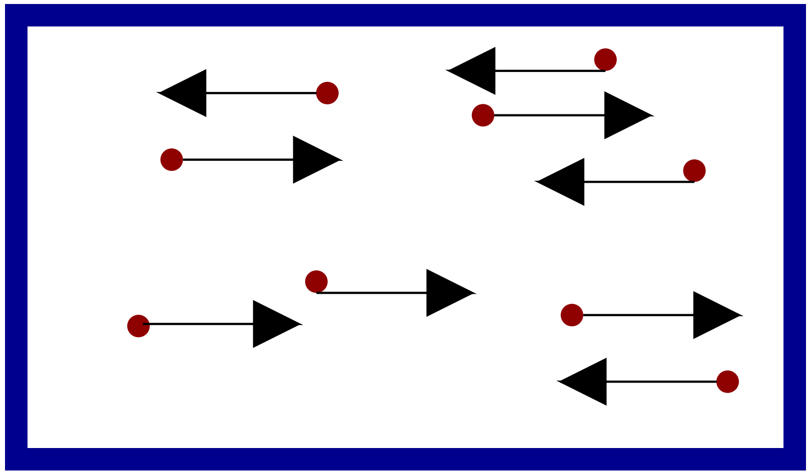
$$H\psi_n \ = \ E_n\psi_n$$

H : matrix, entries depend on system

E_n : are the energy levels

ψ_n : are the energy eigenfunctions

Origins (cont)



Statistical Mechanics: for each configuration, calculate quantity (say pressure).

Average over all configurations – most configurations close to system average.

Nuclear physics: choose matrix at random, calculate eigenvalues, average over matrices.

Look at: Real Symmetric ($A = A^T$), Complex Hermitian ($\overline{A}^T = A$).

Random Matrix Ensembles

Real Symmetric Matrices:

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1N} & a_{2N} & a_{3N} & \cdots & a_{NN} \end{pmatrix} = A^T, \quad a_{ij} = a_{ji}$$

Fix p, define

$$\text{Prob}(A) = \prod_{1 \leq i \leq j \leq N} p(a_{ij}).$$

This means

$$\text{Prob}\left(A : a_{ij} \in [\alpha_{ij}, \beta_{ij}]\right) = \prod_{1 \leq i \leq j \leq N} \int_{x_{ij}=\alpha_{ij}}^{\beta_{ij}} p(x_{ij}) dx_{ij}.$$

Want to understand eigenvalues of A.

Eigenvalue Distribution

$\delta(x - x_0)$ is a unit point mass at x_0.

To each A, attach a probability measure:

$$\mu_{A,N}(x) \; = \; \frac{1}{N} \sum_{i=1}^{N} \delta \left(x - \frac{\lambda_i(A)}{2\sqrt{N}} \right)$$

Equivalently

$$\int_a^b \mu_{A,N}(x) dx \; = \; \frac{\# \left\{ \lambda_i : \frac{\lambda_i(A)}{2\sqrt{N}} \; \in \; [a, b] \right\}}{N}$$

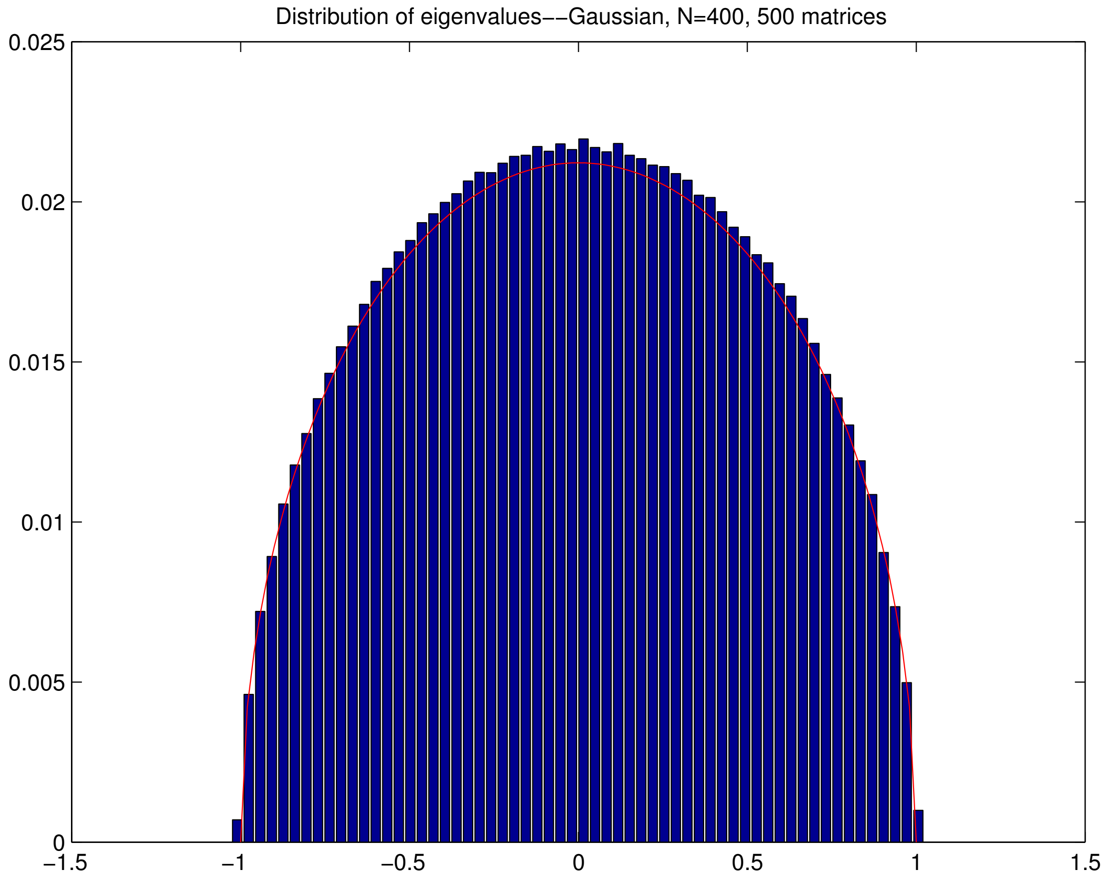
Semi-Circle Law

$N \times N$ real symmetric matrices, entries i.i.d.r.v. from a fixed $p(x)$.

Semi-Circle Law: Assume p has mean 0, variance 1, other moments finite. Then for almost all A, as $N \to \infty$

$$\mu_{A,N}(x) \longrightarrow \begin{cases} \frac{2}{\pi}\sqrt{1-x^2} & \text{if } |x| \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

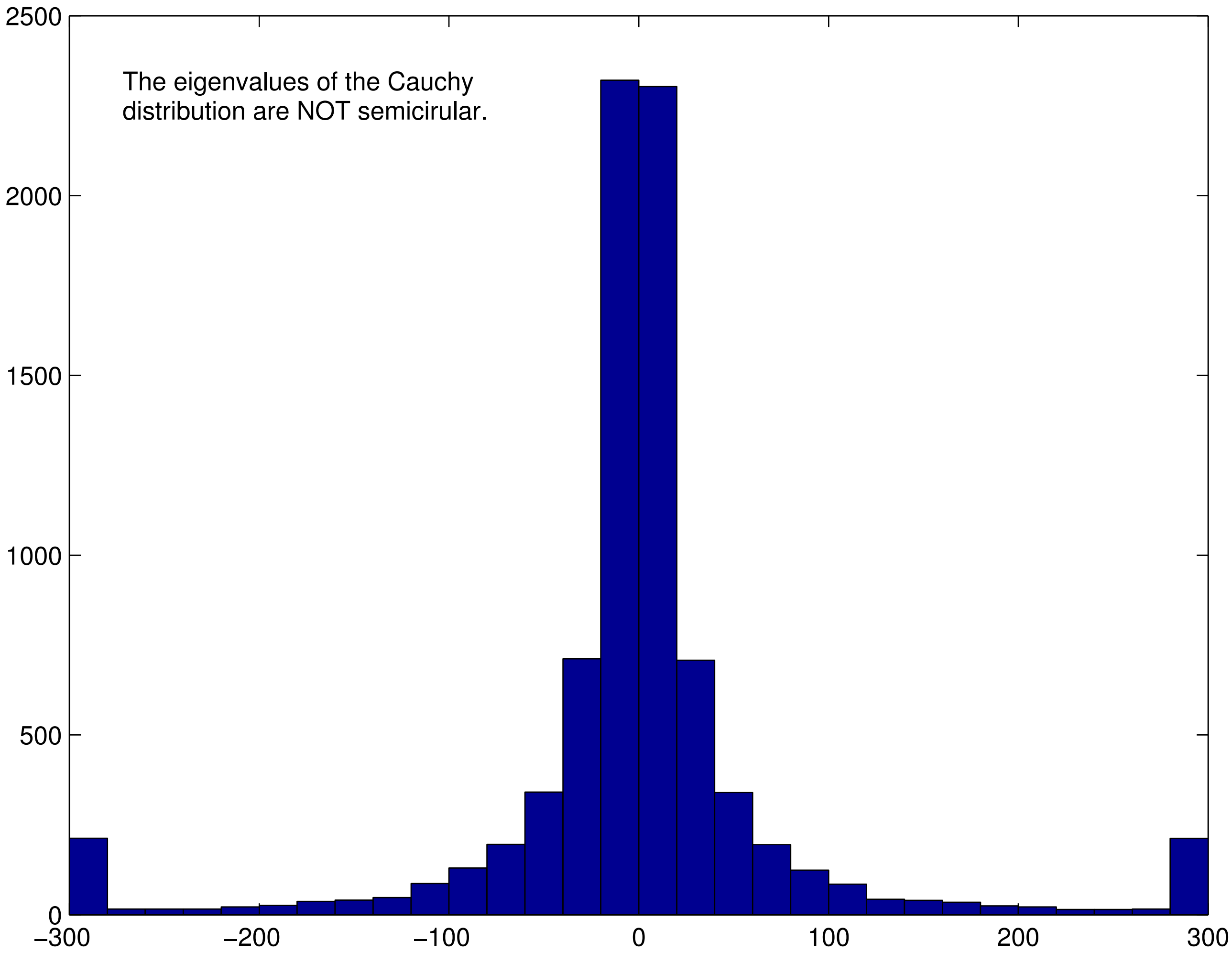
Random Matrix Theory: Semi-Circle Law



500 **Matrices: Gaussian** 400×400

$$p(x) \; = \; \frac{1}{\sqrt{2\pi}} \, e^{-x^2/2}$$

Random Matrix Theory: Semi-Circle Law



Cauchy Distr: $p(x) = \frac{1}{\pi(1+x^2)}$

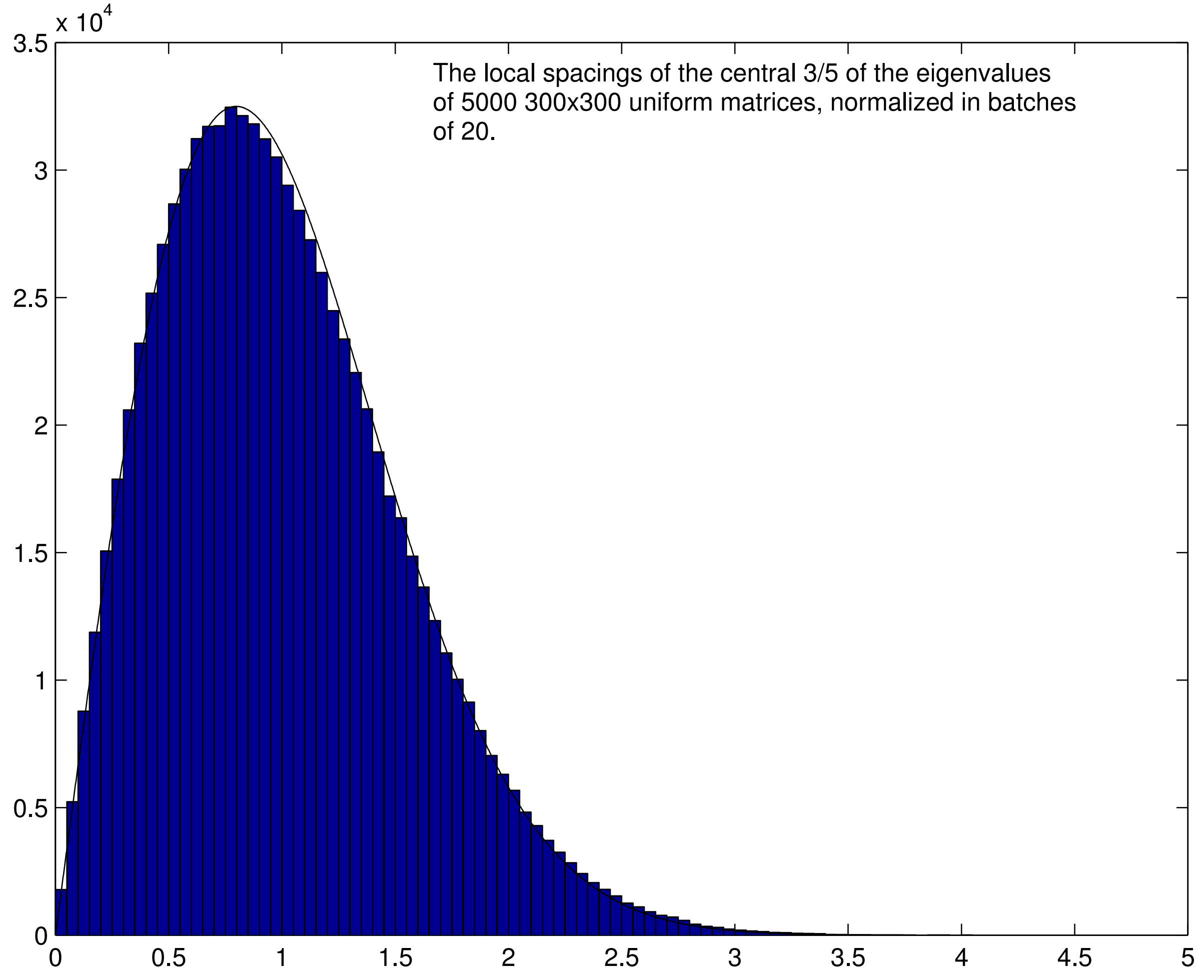
GOE Conjecture

GOE Conjecture: As $N \to \infty$, the probability density of the spacing b/w consecutive normalized eigenvalues approaches a limit independent of p.

Only known if p is a Gaussian.

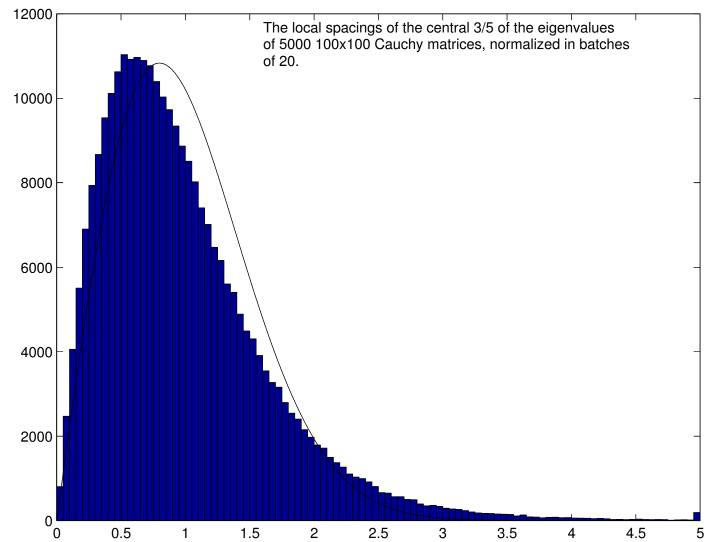
$$\text{GOE}(x) \ \approx \ Axe^{-Bx^2}.$$

Uniform Distribution: $p(x) = \frac{1}{2}$ for $|x| \leq 1$

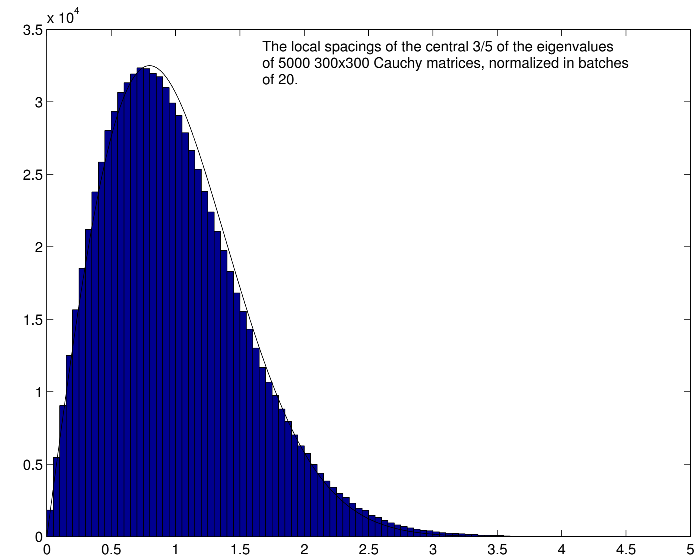


5000: 300×300 **uniform on** $[-1, 1]$

Cauchy Distribution: $p(x) = \frac{1}{\pi(1+x^2)}$

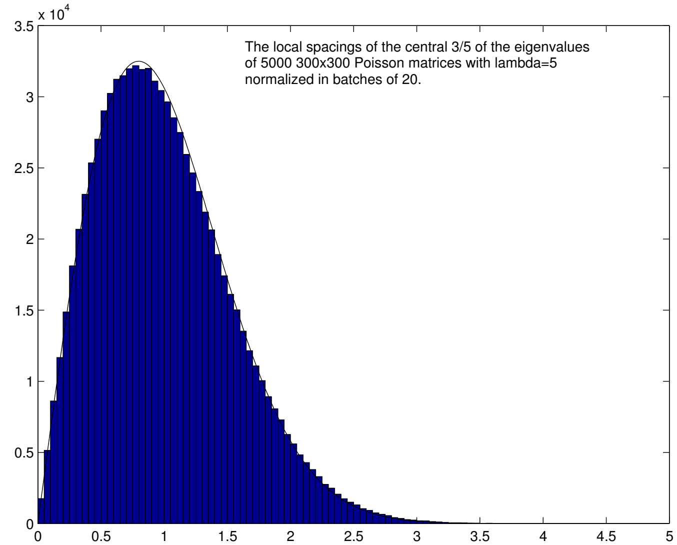


5000: 100×100 Cauchy

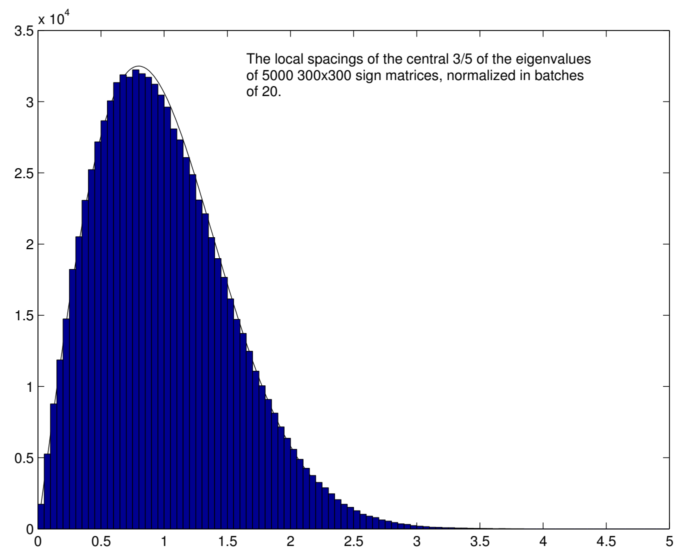


5000: 300×300 Cauchy

Poisson Distribution: $p(n) = \frac{\lambda^n}{n!}\, e^{-\lambda}$



5000: 300×300 Poisson, $\lambda = 5$



5000: 300×300 Poisson, $\lambda = 20$

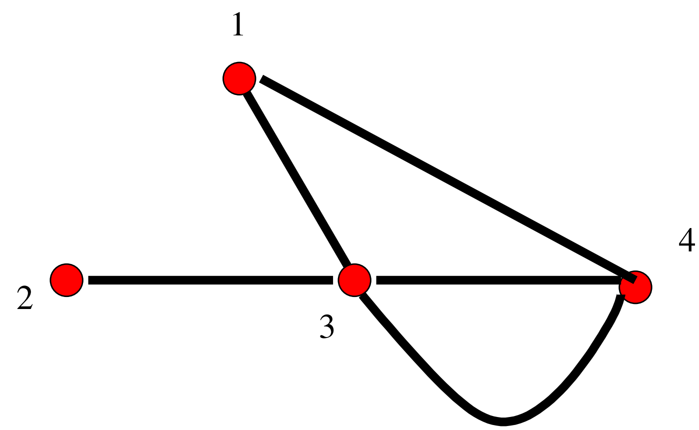
Fat Thin Families

Need a family ***FAT*** enough to do averaging.

Need a family ***THIN*** enough so that everything isn't averaged out.

Real Symmetric Matrices have $\frac{N(N+1)}{2}$ independent entries.

Random Graphs



Degree of a vertex = number of edges leaving the vertex.

Adjacency matrix:a_{ij} =number edges b/w Vertex i & Vertex j.

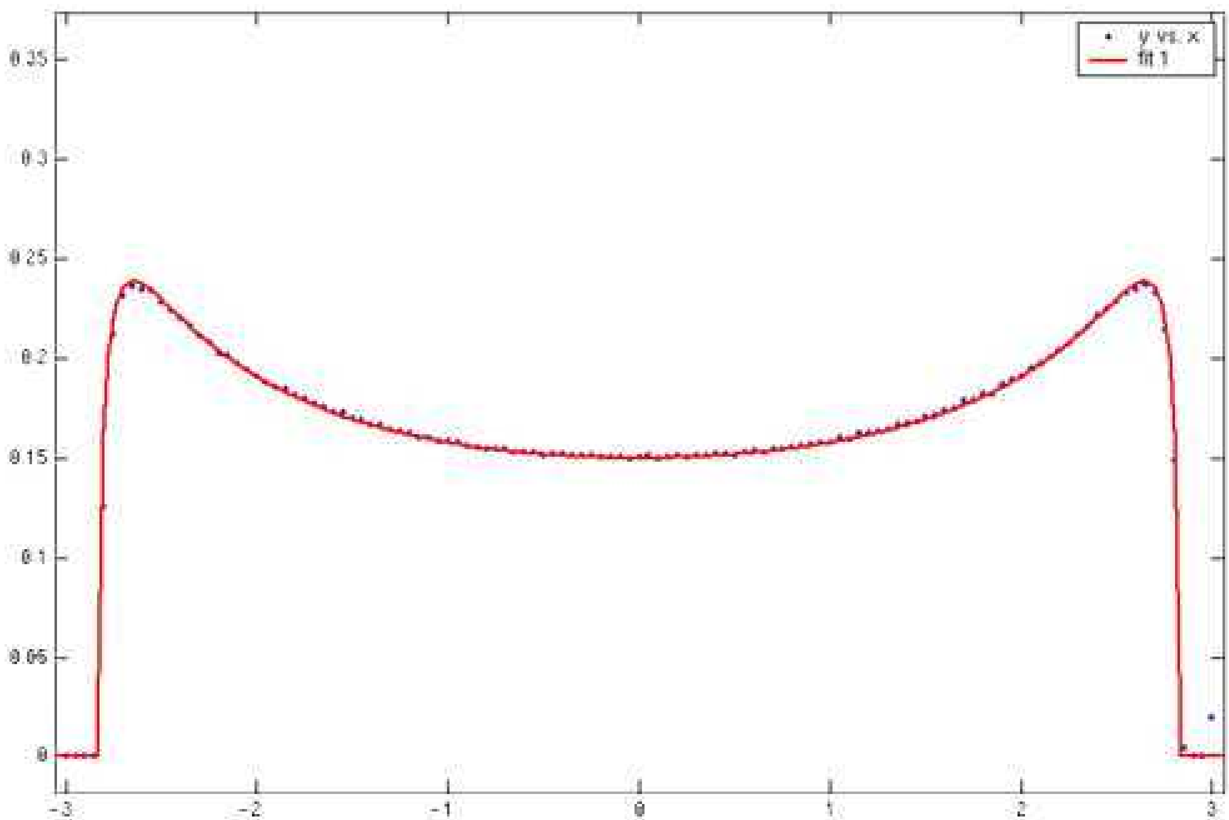
$$A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \end{pmatrix}$$

These are Real Symmetric Matrices.

McKay's Law (Kesten Measure)

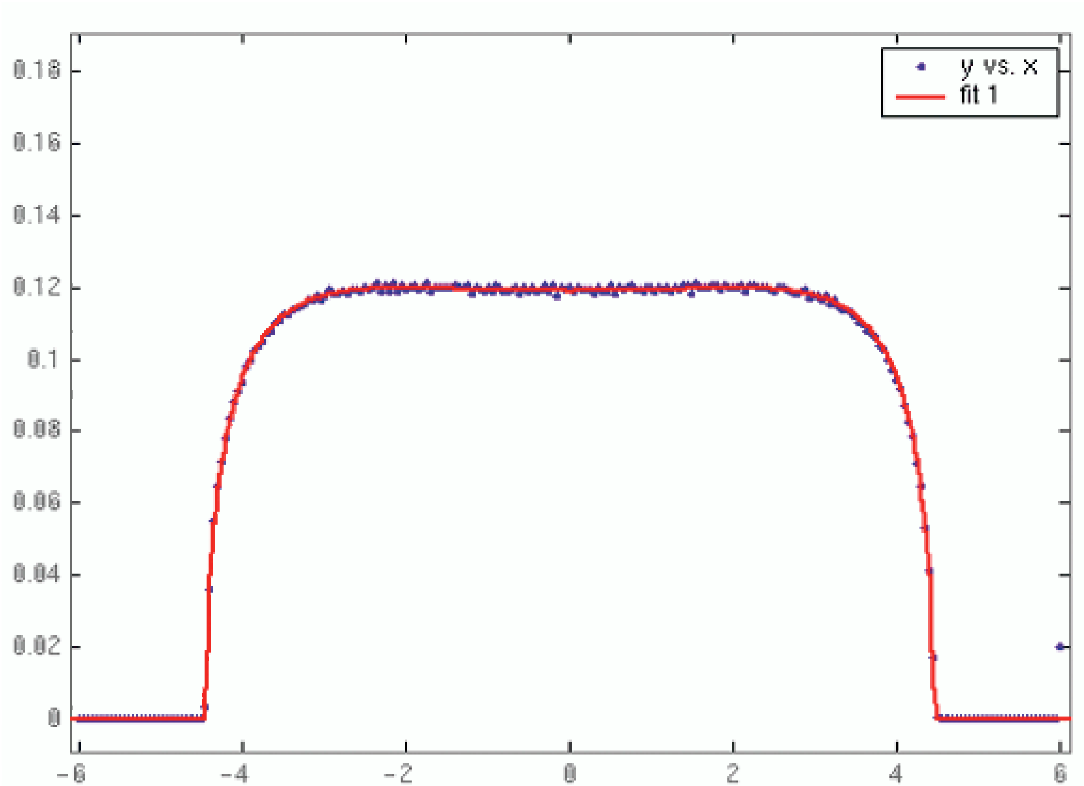
Density of Eigenvalues for d-regular graphs

$$f(x) \;=\; \begin{cases} \frac{d}{2\pi(d^2-x^2)}\sqrt{4(d-1)-x^2} & |x| \le 2\sqrt{d-1} \\ 0 & \text{otherwise.} \end{cases}$$



$$d \;=\; 3.$$

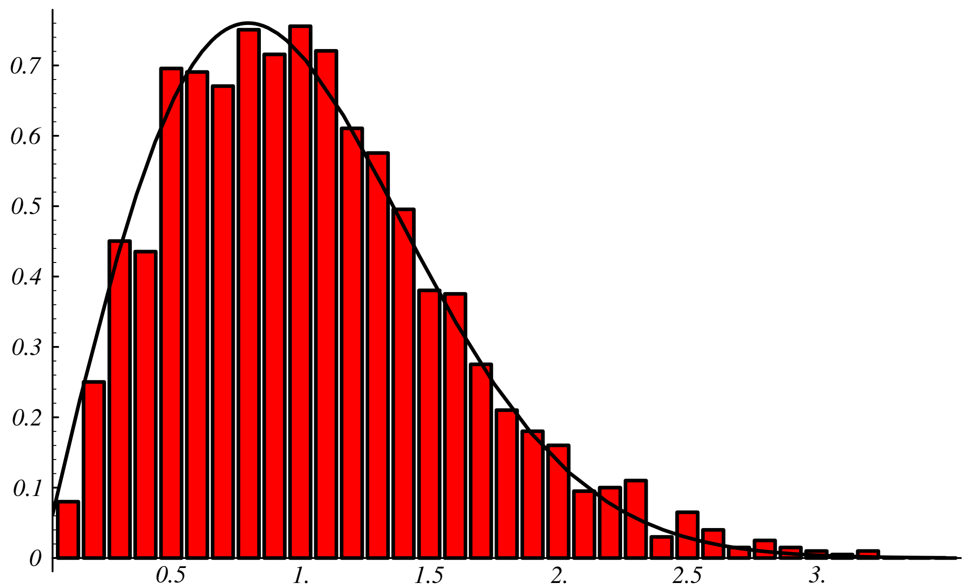
McKay's Law (Kesten Measure)



$$d = 6.$$

Fat Thin: fat enough to average, thin enough to get something different than Semi-circle.

3-Regular, 2000 Vertices and GOE



PART IV

NUMBER THEORY

Infinitude of Primes

Theorem (Euclid): There are infinitely many primes.

Proof: Assume not: $p_1, \ldots, p_N$.

Look at $p_1 \cdots p_N + 1$.

Gives weak bound for number of primes at most x.

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Geometric Series (and Extending Functions): If $|u| < 1$,

$$\sum_{k=0}^{\infty} u^k = 1 + u + u^2 + u^3 + \cdots = \frac{1}{1-u}$$

Unique Factorization: $n = p_1^{r_1} \cdots p_m^{r_m}$.

$$\prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \left[1 + \frac{1}{2^s} + \left(\frac{1}{2^s}\right)^2 + \cdots\right]\left[1 + \frac{1}{3^s} + \left(\frac{1}{3^s}\right)^2 + \cdots\right] \cdots$$
$$= \sum_n \frac{1}{n^s}.$$

Riemann Zeta Function (cont):

$$\zeta(s) = \sum_n \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1$$

$$\pi(x) = \#\{p : p \text{ is prime}, p \leq x\}$$

Properties of $\zeta(s)$ and Primes:

- $\lim_{s \to 1^+} \zeta(s) = \infty$, $\pi(x) \to \infty$.

- $\zeta(2) = \frac{\pi^2}{6}$, $\pi(x) \to \infty$.

Riemann Zeta Function (cont):

$$\zeta(s) \;=\; \sum_{n=1}^{\infty} \frac{1}{n^s} \;=\; \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1.$$

Functional Equation:

$$\xi(s) \;=\; \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) \;=\; \xi(1-s).$$

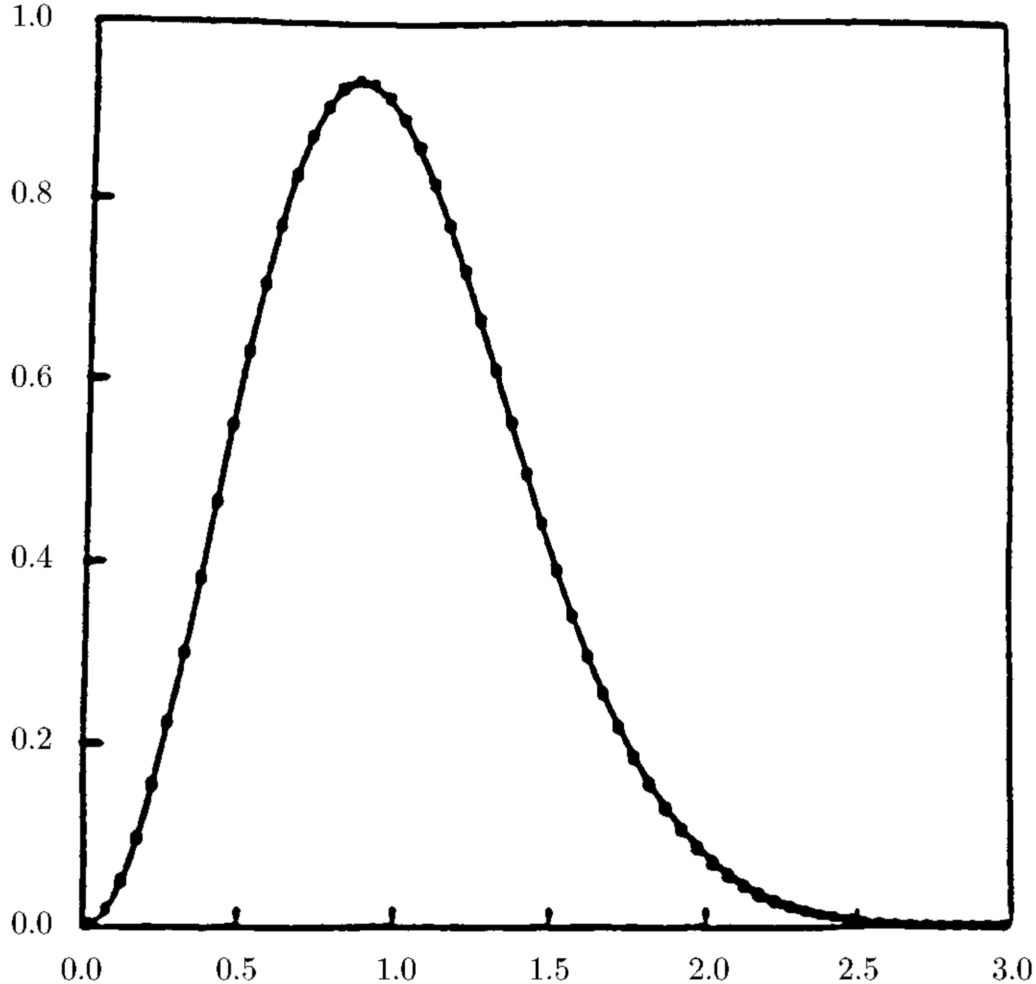
Riemann Hypothesis:

All zeros have $\text{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

Observation:

Spacings b/w zeros appear same as b/w eigenvalues of Complex Hermitian matrices $(\overline{A}^T = A)$.

Zeros of $\zeta(s)$ vs GUE



70 million spacings b/w adjacent zeros of $\zeta(s)$, starting at the $10^{20\text{th}}$ zero (from Odlyzko)

PART V

SKETCH OF PROOFS

SKETCH OF PROOF: Eigenvalue Trace Lemma

$$\text{Trace}(A) \;=\; a_{11} + a_{22} + \cdots + a_{NN}.$$

THEOREM: $$\text{Trace}(A^k) \;=\; \sum_{i=1}^{N} \lambda_i(A)^k.$$

$k = 1$ *Case:* Say $\overrightarrow{v} \neq \overrightarrow{0}$ is an eigenvector with eigenvalue λ:

$$A\overrightarrow{v} \;=\; \lambda\overrightarrow{v} \;\;\Leftrightarrow\;\; (\lambda I - A)\overrightarrow{v} \;=\; \overrightarrow{0}.$$

Means $\lambda I - A$ isn't invertible, so

$$p(\lambda) \;=\; \det(\lambda I - A) \;= 0.$$

Write $p(\lambda) = (\lambda - \lambda_1)\cdots(\lambda - \lambda_N)$, compare coefficients with $\det(\lambda I - A)$.

SKETCH OF PROOF: Probability Review

Probability density p of mean $\mu = 0$, variance $\sigma^2 = 1$:

$$\begin{aligned} \mu &= \int_{-\infty}^{\infty} x p(x) dx \\ \sigma^2 &= \int_{-\infty}^{\infty} (x - \mu)^2 p(x) dx. \end{aligned}$$

As $\sigma^2 = 1$, expect $x^2 \approx 1$.

SKETCH OF PROOF: Correct Scale

$$\text{Trace}(A^2) \;=\; \sum_{i=1}^{N} \lambda_i(A)^2.$$

By the Central Limit Theorem:

$$\text{Trace}(A^2) \;=\; \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}a_{ji} \;=\; \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^2 \;\sim\; N^2$$

$$\sum_{i=1}^{N} \lambda_i(A)^2 \;\sim\; N^2$$

Gives $N\text{Ave}(\lambda_i(A)^2) \;\sim\; N^2$ or $\text{Ave}(\lambda_i(A)) \sim \sqrt{N}$.

SKETCH OF PROOF: Eigenvalue Distribution

$\delta(x - x_0)$ is a unit point mass at x_0.

To each A, attach a probability measure:

$$\mu_{A,N}(x) \;=\; \frac{1}{N}\sum_{i=1}^{N} \delta\left(x - \frac{\lambda_i(A)}{2\sqrt{N}}\right)$$

Obtain:

$$k^{th}\text{-moment} \;=\; \int_{-\infty}^{\infty} x^k \mu_{A,N}(x)dx \;=\; \frac{1}{N}\sum_{i=1}^{N} \frac{\lambda_i(A)^k}{(2\sqrt{N})^k} \;=\; \frac{\text{Trace}(A^k)}{2^k N^{\frac{k}{2}+1}}$$

SKETCH OF PROOF: Semi-Circle Law

$N \times N$ real symmetric matrices, entries i.i.d.r.v. from a fixed $p(x)$.

Semi-Circle Law: Assume p has mean 0, variance 1, other moments finite. Then for almost all A, as $N \to \infty$

$$\mu_{A,N}(x) \longrightarrow \begin{cases} \frac{2}{\pi}\sqrt{1-x^2} & \text{if } |x| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

Trace formula converts sums over eigenvalues to sums over entries of A.

Expected value of k^{th}-moment of $\mu_{A,N}(x)$ is

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{\text{Trace}(A^k)}{2^k N^{\frac{k}{2}+1}} \prod_{i \leq j} p(a_{ij}) da_{ij}$$

SKETCH OF PROOF: 2^{nd}-Moment

$$\text{Trace}(A^2) \;=\; \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}a_{ji} \;=\; \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^2.$$

Substituting into expansion gives

$$\frac{1}{2^2N^2}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ji}^2 \cdot p(a_{11})da_{11}\cdots p(a_{NN})da_{NN}$$

Integration factors as

$$\int_{a_{ij}=-\infty}^{\infty} a_{ij}^2 p(a_{ij})da_{ij} \quad\cdot \prod_{\substack{(k,l)\neq(ij)\\ k<l}} \int_{a_{kl}=-\infty}^{\infty} p(a_{kl})da_{kl} \;=\; 1.$$

Have N^2 summands, answer is $\frac{1}{4}$.

SKETCH OF PROOF: Zero Knowledge (Heuristic)

$P(x)$ polynomial, zeros $r_1, \ldots, r_n$. Then

$$\begin{aligned} P(x) &= (x-r_1)(x-r_2)\cdots(x-r_n) \\ &= x^n + a_{n-1}(r_1,\ldots,r_n)x^{n-1} + \cdots + a_0(r_1,\ldots,r_n) \end{aligned}$$

where

$$\begin{aligned} a_{n-1}(r_1,\ldots,r_n) &= -(r_1+\cdots+r_n) \\ &\vdots \\ a_0(r_1,\ldots,r_n) &= r_1 r_2 \cdots r_n. \end{aligned}$$

Knowledge of zeros gives info on coefficients.

PART VI

CONCLUSIONS

Correspondences

Similarities b/w Nuclei and Primes:

Energy Levels $\longleftrightarrow$ Zeros of $\zeta(s)$

Neutron Energy $\longleftrightarrow$ Summation Lemmas

Different Elements: U, Pu, ... $\longleftrightarrow$ Different L-Functions

Summary

- Similar behavior in different systems.
- Find correct scale.
- Average over similar elements.
- Need a Trace Lemma.
- Thin subsets can exhibit very different behavior.

Open Problems

Real Symmetric Band Matrices

$$\begin{pmatrix} a_{1,1} & a_{1,2} & 0 & \cdots & 0 & 0 & 0 \\ a_{1,2} & a_{2,2} & a_{2,3} & \cdots & 0 & 0 & 0 \\ 0 & a_{2,3} & a_{3,3} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{N-2,N-2} & a_{N-2,N} & 0 \\ 0 & 0 & 0 & \cdots & a_{N-2,N} & a_{N-1,N-1} & a_{N-1,N} \\ 0 & 0 & 0 & \cdots & 0 & a_{N-1,N} & a_{N,N} \end{pmatrix}$$

Real Symmetric Toeplitz Matrices

$$\begin{pmatrix} b_0 & b_1 & b_2 & \cdots & b_{N-1} \\ b_1 & b_0 & b_1 & \cdots & b_{N-2} \\ b_2 & b_1 & b_0 & \cdots & b_{N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{N-1} & b_{N-2} & b_{N-3} & \cdots & b_0 \end{pmatrix}$$

Rates of Convergence

APPENDIX: Contour Integration and L-Fns

Zero Knowledge: (Contour Integration)

$$\begin{aligned} -\frac{\zeta'(s)}{\zeta(s)} &= -\frac{\mathrm{d}}{\mathrm{d}s}\log\zeta(s) \\ &= \frac{\mathrm{d}}{\mathrm{d}s}\sum_p \log\left(1-p^{-s}\right) \\ &= \sum_p \frac{\log p \ \cdot \ p^{-s}}{1-p^{-s}} \\ &= \sum_p \frac{\log p}{p^s} \ + \ \mathrm{Good}(s). \end{aligned}$$

Contour Integration:

$$\int -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}ds \quad \text{vs} \quad \sum_p \log p \int \left(\frac{x}{p}\right)^s \frac{ds}{s}.$$

Knowledge of zeros gives info on coefficients.

Families of L-Functions

More generally, we may consider an L-function

$$L(s,f) \;=\; \sum_{n=1}^{\infty} \frac{a_n(f)}{n^s} \;=\; \prod_p L_p(p^{-s},f)^{-1}, \quad \mathrm{Re}(s) > s_0.$$

Examples:

- Dirichlet Characters: $a_n(f) \;=\; \chi_f(n)$.
- Elliptic Curves: $y^2 \;=\; x^3 + A_f x + B_f$, $a_p(f)$ is related to number of solns mod p.

General Riemann Hypothesis: All L-functions (after normalization) have their zeros on the critical line.