Low-lying zeros of cuspidal Maass forms

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Introduction
From Random Matrix Theory to L-Function

Fundamental Problem: Spacing Between Events

General Formulation: Studying system, observe values at t_1, t_2, $t_3, \dots$.

Question: What rules govern the spacings between the t_i?

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Examples:

- Spacings b/w Energy Levels of Nuclei.
- Spacings b/w Eigenvalues of Matrices.
- Spacings b/w Primes.
- Spacings b/w Zeros of L-functions.

Random Matrix Ensembles

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1N} & a_{2N} & a_{3N} & \cdots & a_{NN} \end{pmatrix} = A^T, \quad a_{ij} = a_{ji}$$

Fix p, define

$$\text{Prob}(A) = \prod_{1 \le i \le j \le N} p(a_{ij}).$$

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Fix p, define

$$\text{Prob}(A) = \prod_{1 \le i \le j \le N} p(a_{ij}).$$

- Want to understand eigenvalues of A.
- Eigenvalues will be real since matrix is symmetric.

Eigenvalue Distribution

To each A, attach a probability measure using scaled eigenvalues:

$$\mu_{A,N}(x) \quad = \quad \frac{1}{N}\sum_{i=1}^{N}\delta\left(x - \frac{\lambda_i(A)}{2\sqrt{N}}\right)$$

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$$k^{\text{th}}\ moment = \frac{\sum_{i=1}^{N}\lambda_i(A)^k}{2^k N^{\frac{k}{2}+1}}.$$

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- We need the average k^{th} moment for typical A.
- This will give us information about the distribution of eigenvalues for typical A.

L-functions

Riemann zeta function:

$$\zeta(s) = \sum_n \frac{1}{n^s}$$

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L-functions generalizes the Riemann zeta-function:

$$L(s,f) \;=\; \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} \;=\; \prod_{p \text{ prime}} L_p\left(s,f\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

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Generalized Riemann Hypothesis (RH):

All non-trivial zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

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RMT vs *L*-functions

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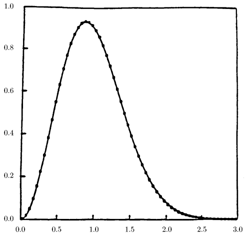
So can use RMT to make conjectures about L-functions zeros.
Example: Spacings b/w zeros same as b/w eigenvalues of Complex Hermitian matrices.



70 million spacings b/w adjacent zeros of $\zeta(s)$, starting at the $10^{20\text{th}}$ zero (from Odlyzko)

Sketch of proofs

Study L-functions like studied random matrix theory. Steps:

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1. Determine correct scale for events (like in RMT...)

2. Develop an explicit formula relating what we want to study to something we understand (like in RMT...)

3. Use an averaging formula to analyze the quantities above (like in RMT...)

n-Level Density

Measures of Spacings: n-Level Density

n-level density for one function

$$D_{n,f}(\phi) = \sum_{\substack{j_1,\dots,j_n \\ \text{distinct}}} \phi_1\left(L_f\gamma_f^{(j_1)}\right)\cdots\phi_n\left(L_f\gamma_f^{(j_n)}\right)$$

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- Fourier Transforms $\hat{\phi}$ has compact support: $(-\sigma, \sigma)$
- Zeros scaled by L_f
- Most of contribution is from low zeros.

Katz-Sarnak Conjecture

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For a 'nice' family of *L*-functions, the average *n*-level density depends only on a symmetry group attached to the family.

As $|\mathcal{F}| \to \infty$,

$$\frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} D_{n,f}(\phi) \;\; \to \;\; \int \cdots \int \phi(x) W_{n,\mathcal{G}(\mathcal{F})}(x) dx.$$

1-level density

The Fourier Transforms for the 1-level densities are

$$\begin{aligned}
\widehat{W_{1,\mathrm{SO(even)}}}(u) &= \delta_0(u) + \frac{1}{2}\eta(u) \\
\widehat{W_{1,\mathrm{SO}}}(u) &= \delta_0(u) + \frac{1}{2} \\
\widehat{W_{1,\mathrm{SO(odd)}}}(u) &= \delta_0(u) - \frac{1}{2}\eta(u) + 1 \\
\widehat{W_{1,Sp}}(u) &= \delta_0(u) - \frac{1}{2}\eta(u) \\
\widehat{W_{1,U}}(u) &= \delta_0(u)
\end{aligned}$$

where $\delta_0(u)$ is the Dirac Delta functional and

$$\eta(u) = \begin{cases} 1 & \text{if } |u| < 1 \\ \frac{1}{2} & \text{if } |u| = 1 \\ 0 & \text{if } |u| > 1 \end{cases}$$

Cuspidal Maass Forms

Maass Forms

Definition: Maass Forms

A Maass form on a group $\Gamma \subset PSL(2, \mathbb{R})$ is a function $f : \mathcal{H} \to \mathbb{R}$ which satisfies:

1. $f(\gamma z) = f(z)$ for all $\gamma \in \Gamma$,
2. f vanishes at the cusps of Γ, and
3. $\Delta f = \lambda f$ for some $\lambda > 0$, where

$$\Delta = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

is the Laplace-Beltrami operator on $\mathcal{H}$.

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is the Laplace-Beltrami operator on $\mathcal{H}$.

- Test Katz-Sarnak conjecture.
- Coefficients contain information about partitions.

L-function associated to Maass forms

Let u_j be a Maass form and write the Fourier expansion of u_j as

$$u_j(z) = \cosh(t_j) \sum_{n \neq 0} \sqrt{y} \lambda_j(n) K_{it_j}(2\pi |n| y) e^{2\pi i n x}.$$

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Then we define the L-function attached to u_j as:

$$\begin{aligned} L(s, u_j) &= \sum_{n \geq 1} \frac{\lambda_j(n)}{n^s} \\ &= \prod_p \left(1 - \frac{\alpha_j(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_j(p)}{p^s}\right)^{-1} \end{aligned}$$

where

$$\alpha_j(p) + \beta_j(p) = \lambda_j(p), \qquad \alpha_j(p)\beta_j(p) = 1.$$

Differences between L-functions

Dirichlet L-functions are in the GL_1 family:

$$L(s) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$$

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Maass/holomorphic forms L-functions are GL_2:

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n-level over a family

- Recall for Katz-Sarnak Conjecture, need to average n-level density over a family and take the limit of this parameter.

$$\frac{1}{|\mathcal{F}_N|}\sum_{f\in\mathcal{F}_N} D_{n,f}(\phi) = \frac{1}{|\mathcal{F}_N|}\sum_{f\in\mathcal{F}_N}\sum_{\substack{j_1,\dots,j_n \\ j_i\neq\pm j_k}}\prod_i \phi_i\left(L_f\gamma_E^{(j_i)}\right)$$

$$\to \int\cdots\int \phi(x) W_{n,\mathcal{G}(\mathcal{F})}(x)dx.$$

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- For Dirichlet/cuspidal newform L-functions, there are many with a given conductor.
- Problem: For Maass forms, expect at most one with a given conductor.

n-level over a family, continued

- Solution: Average over Laplace eigenvalues $\lambda_f = 1/4 + t_j^2$.

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- Use smooth weight function h,

$$h_T(t_j) = e^{-(t_j - T)^2/L^2},$$

which picks out eigenvalues in the window $[T-L, T+L]$.

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- Average 1-level density becomes

$$\frac{1}{\sum_j \frac{h_T(t_j)}{\|u_j\|^2}} \sum_j \frac{h_T(t_j)}{\|u_j\|^2} D_{n,u_j}(\phi)$$

$$= \frac{1}{\sum_j \frac{h_T(t_j)}{\|u_j\|^2}} \sum_j \frac{h_T(t_j)}{\|u_j\|^2} \sum_{\substack{j_1,\dots,j_n \\ j_i \neq \pm j_k}} \prod_i \phi_i\left(\frac{\gamma}{2\pi}\log R\right)$$

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1-level density for one function

$$D(u_j; \phi) \quad = \quad \sum_{\gamma} \phi\left(\frac{\gamma}{2\pi} \log R\right)$$

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1-level density for one function

$$D(u_j;\phi)$$
$$= \text{Terms involving } \Gamma + \frac{2}{\log R}\sum_p \frac{\log p}{p}\hat{\phi}\left(\frac{2\log p}{\log R}\right)$$
$$-\sum_p \frac{2\lambda_j(p)\log p}{p^{\frac{1}{2}}\log R}\hat{\phi}\left(\frac{\log p}{\log R}\right) - \sum_p \frac{2\lambda_j(p^2)\log p}{p\log R}\hat{\phi}\left(\frac{2\log p}{\log R}\right)$$

1. Complex Analysis (contour shifts) $\Rightarrow$ explicit formula to relate sums over zeros to sums over primes.

1-Level Density

1-level density for one function

$$\begin{aligned}
&D(u_j;\phi) \\
&= \hat{\phi}(0)\frac{\log(1+t_j^2)}{\log R} + \frac{2}{\log R}\sum_p \frac{\log p}{p}\hat{\phi}\left(\frac{2\log p}{\log R}\right) \\
&- \sum_p \frac{2\lambda_j(p)\log p}{p^{\frac{1}{2}}\log R}\hat{\phi}\left(\frac{\log p}{\log R}\right) - \sum_p \frac{2\lambda_j(p^2)\log p}{p\log R}\hat{\phi}\left(\frac{2\log p}{\log R}\right)
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2. Gamma function identities

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&D(u_j;\phi)\\
&= \hat{\phi}(0)\frac{\log(1+t_j^2)}{\log R} + \frac{\phi(0)}{2} + O\left(\frac{\log\log R}{\log R}\right)\\
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1. Complex Analysis (contour shifts) ⇒ explicit formula to relate sums over zeros to sums over primes.
2. Gamma function identities
3. Prime Number Theorem

Average 1-level density

The average 1-level density becomes:

$$\frac{1}{\sum_j \frac{h_T(t_j)}{\|u_j\|^2}} \sum_j \frac{h_t(t_j)}{\|u_j\|^2} D(u_j;\phi)$$

$$= \frac{\phi(0)}{2} + O\left(\frac{\log\log R}{\log R}\right) + \frac{1}{\sum_j \frac{h_t(t_j)}{\|u_j\|^2}} \sum_j \frac{h_t(t_j)}{\|u_j\|^2} \hat{\phi}(0) \frac{\log(1+t_j^2)}{\log R}$$

$$- \frac{1}{\sum_j \frac{h_T(t_j)}{\|u_j\|^2}} \sum_p \frac{2\log p}{p^{\frac{1}{2}}\log R} \hat{\phi}\left(\frac{\log p}{\log R}\right) \sum_j \frac{h_T(t_j)}{\|u_j\|^2} \lambda_j(p)$$

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Kuznetsov Trace Formula

To tackle terms with $\lambda_j(p)$ and $\lambda_j(p^2)$ we need the Kuznetsov Trace Formula:

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$$\sum_j \frac{h(t_j)}{\|u_j\|^2} \lambda_j(m) \overline{\lambda_j(n)}$$

$$= \text{ some function that depends just on h, m, and n}$$

Kuznetsov Trace Formula

$$\sum_j \frac{h(t_j)}{\|u_j\|^2}\lambda_j(m)\overline{\lambda_j(n)} + \frac{1}{4\pi}\int_{\mathbb{R}} \overline{\tau(m,r)}\tau(n,r)\frac{h(r)}{\cosh(\pi r)}dr =$$

$$\frac{\delta_{n,m}}{\pi^2}\int_{\mathbb{R}} r\tanh(r)h(r)dr + \frac{2i}{\pi}\sum_{c\geq 1}\frac{S(n,m;c)}{c}\int_{\mathbb{R}} J_{ir}\left(\frac{4\pi\sqrt{mn}}{c}\right)\frac{h(r)r}{\cosh(\pi r)}dr$$

where
$\tau(m,r) = \pi^{\frac{1}{2}+ir}\Gamma(1/2+ir)^{-1}\zeta(1+2ir)^{-1}n^{-\frac{1}{2}}\sum_{ab=|m|}\left(\frac{a}{b}\right)^{ir}$.

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$$S(n,m;c) = \sum_{0\leq x\leq c-1, gcd(x,c)=1} e^{2\pi i(nx+mx^*)/c}$$

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- The only $\lambda(m)\overline{\lambda(n)}$ term that contributes is when $m = n = 1$.
- The $m = 1, n = p$ and $m = 1, n = p^2$ terms do not contribute because of the $\delta_{m,n}$ function.

Result: 1-level density

Theorem (AILZ, 2011)

Then if $T \to \infty$, the average 1-level density is

$$\frac{1}{\sum_j \frac{h_T(t_j)}{\|u_j\|^2}} \sum_j \frac{h_T(t_j)}{\|u_j\|^2} D(u_j;\phi)$$
$$= \frac{\phi(0)}{2} + \hat{\phi}(0) + O\left(\frac{\log\log R}{\log R}\right) + O(T^{-\epsilon})$$

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- This matches with the orthogonal family density as predicted by Katz-Sarnak.
- Small support due to bounds on non-contributing terms in Kuznetsov formula.

2- Level Density

To differentiate between even and odd in orthogonal family, we calculated the 2-level density:

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$$= \frac{1}{\sum_j \frac{h_T(t_j)}{\|u_j\|^2}} \sum_j \frac{h_T(t_j)}{\|u_j\|^2} \prod_{i=1}^{2} \left[\frac{\phi_i(0)}{2} + \hat{\phi}_i(0)\frac{\log(1+t_j^2)}{\log R} + O\left(\frac{\log\log R}{\log R}\right) \right.$$

$$\left. - \sum_p \frac{2\lambda_j(p)\log p}{p^{\frac{1}{2}}\log R}\hat{\phi}_i\left(\frac{\log p}{\log R}\right) - \sum_p \frac{2\lambda_j(p^2)\log p}{p\log R}\hat{\phi}_i\left(\frac{2\log p}{\log R}\right) \right]$$

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To handle λ's, we use Kuznetsov again.

Result: 2-level density

Theorem (AILZ, 2011)

$$
\begin{aligned}
D^*_{2,\mathcal{F}} \quad = \quad & \prod_{i=1}^{2}\left[\frac{\phi_i(0)}{2} + \hat{\phi}_i(0)\right] + 2\int_{-\infty}^{\infty} |z|\hat{\phi}_1(z)\hat{\phi}_2(z)dz \\
& -\phi_1(0)\phi_1(0) - 2\widehat{\phi_1\phi_2}(0) + (\phi_1\phi_2)(0)\mathcal{N}(-1) \\
& +O\left(\frac{\log\log R}{\log R}\right)
\end{aligned}
$$

for $\sigma < 1/12$.

Note that $\mathcal{N}(-1)$ is a weighted percent that have odd sign in functional equation.

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Thank you!