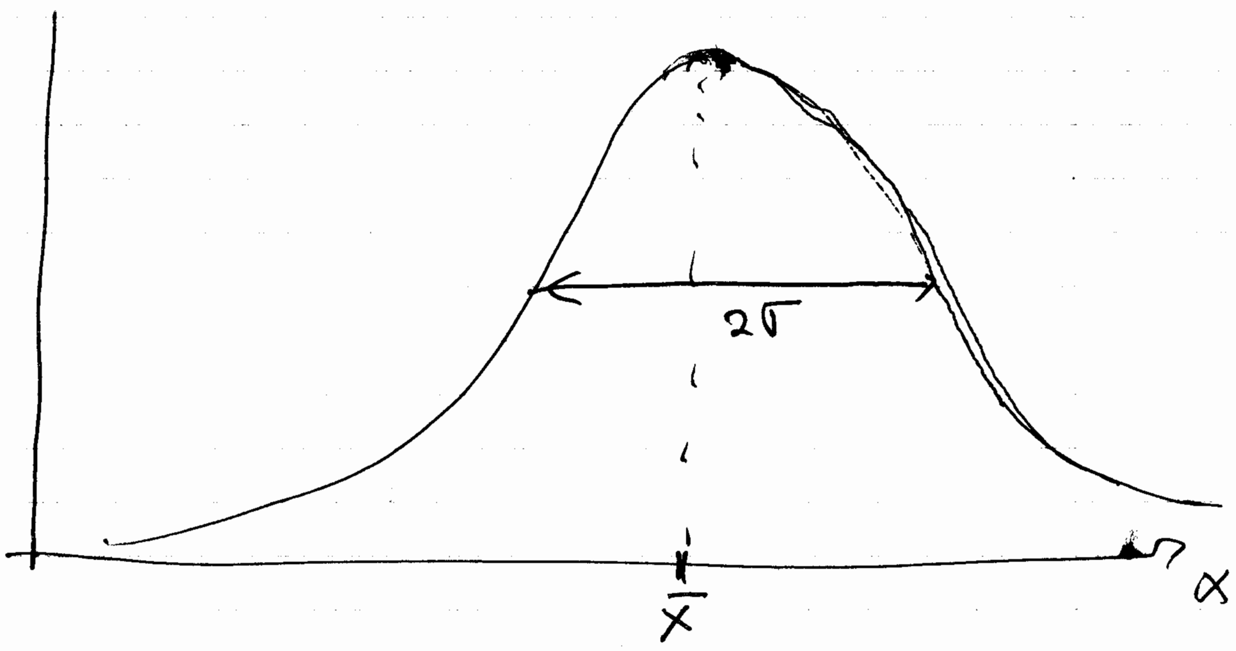
ERROR ANALYSIS

WHEN WE MEASURE X WITH RANDOM ERRORS

WE MEAN



Probability $$P(x) \sim e^{-\frac{(x-\bar{x})^2}{2\sigma_x^2}}$$

Approximate $$\bar{x} = \sum_{i=1}^{N} \frac{x_i}{N}$$

AVERAGE

REF: "AN INTRODUCTION TO Error Analysis" by J. R TAYLOR
UNIVERSITY SCIENCE: SAUSALITO, CA, 1997

$$\sigma_x = \sqrt{\sum_{i=1}^{N} \frac{(x_i - \bar{x})^2}{N-1}}$$

STANDARD DEVIATION

FOR MANY N- Measurements Exp

$$\bar{\sigma}_x = \frac{\sigma_x}{N},$$

STANDARD DEVIATION OF THE MEAN

$$\sigma_x \equiv \Delta x$$

ERROR

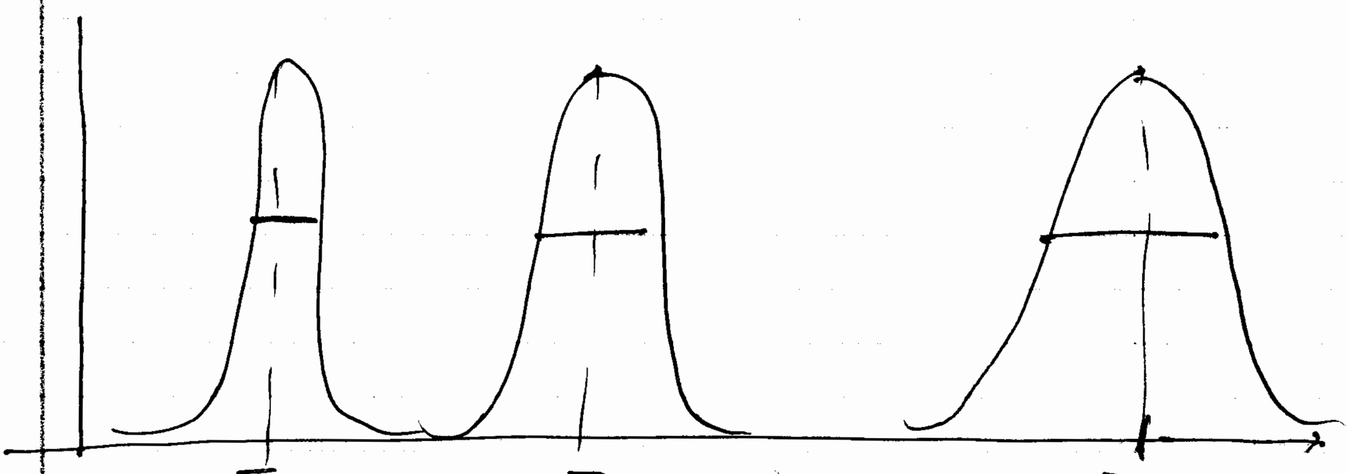
Now WE CONSIDER

$$A = x + y$$

$$\begin{aligned}
\bar{A} + \Delta A &= (\bar{x} \pm \Delta x) + (\bar{y} \pm \Delta y) \\
&= (\bar{x} + \bar{y}) \pm (\Delta x \pm \Delta y) \\
&= \bar{A} \pm \Delta A
\end{aligned}$$

$$\Delta A = |\Delta x| + |\Delta y|$$

WORST CASE (NO RANDOM ERRORS)



$$X \rightsquigarrow P(x) \sim e^{-\frac{(x-\bar{x})^2}{2\sigma_x^2}}$$

$$x = \bar{x} + \sigma_x$$

$$y = \bar{y} + \sigma_y$$

$$P(y) \sim e^{-\frac{(y-\bar{y})^2}{2\sigma_y^2}}$$

$$x + y \longrightarrow P(x)\,P(y)$$

$$\longrightarrow x + y = \bar{x} + \bar{y} + \sigma_{x+y}$$

$$\longrightarrow e^{-\frac{(x-\bar{x})^2}{2\sigma_x^2} - \frac{(y-\bar{y})^2}{2\sigma_y^2}}$$

$$\frac{(x-\bar{x})^2}{\sigma_x^2} + \frac{(y-\bar{y})^2}{\sigma_y^2} = \left(1+\frac{\sigma_y^2}{\sigma_x^2}\right)\frac{(x-\bar{x})^2}{\sigma_x^2+\sigma_y^2} + \left(1+\frac{\sigma_x^2}{\sigma_y^2}\right)\frac{(y-\bar{y})^2}{\sigma_x^2+\sigma_y^2}$$

$$= \frac{(x-\bar{x})^2+(y-\bar{y})^2}{\sigma_x^2+\sigma_y^2} + \frac{\frac{\sigma_y^2}{\sigma_x^2}(x-\bar{x})^2 + \frac{\sigma_x^2}{\sigma_y^2}(y-\bar{y})^2}{\sigma_x^2+\sigma_y^2}$$

$$= \frac{(x-\bar{x})^2 + 2(x-\bar{x})(y-\bar{y}) + (y-\bar{y})^2}{\sigma_x^2+\sigma_y^2}$$

$$+ \frac{\frac{\sigma_y^2}{\sigma_x^2}(x-\bar{x})^2 + \frac{\sigma_x^2}{\sigma_y^2}(y-\bar{y})^2 - 2(x-\bar{x})(y-\bar{y})}{\sigma_x^2+\sigma_y}$$

$$= \frac{\left[(x-\bar{x})+(y-\bar{y})\right]^2}{\sigma_x^2+\sigma_y^2} + \frac{\left[\frac{\sigma_y}{\sigma_x}(x-\bar{x}) - \frac{\sigma_x}{\sigma_y}(y-\bar{y})\right]^2}{\sigma_x^2+\sigma_y^2}$$

$$= \frac{\left[(x+y)-(\bar{x}+\bar{y})\right]^2}{\sigma_x^2+\sigma_y^2} + \frac{\left[\left(\frac{\sigma_y}{\sigma_x}x - \frac{\sigma_x}{\sigma_y}y\right) - \left(\frac{\sigma_y}{\sigma_x}\bar{x} - \frac{\sigma_x}{\sigma_y}\bar{y}\right)\right]^2}{\sigma_x^2+\sigma_y^2}$$

WE CAN REWRITE

$$P(x)\,P(y) \sim e^{-\frac{[(x+y)-(\bar{x}+\bar{y})]^2}{2(\sigma_x^2+\sigma_y^2)}}\; e^{-\frac{(z-\bar{z})^2}{2(\sigma_x^2+\sigma_y^2)}}$$

$$\sim P(x+y)\;P(z)$$

INTEGRATE OVER ALL POSSIBLE Z VALUES

$$P(x+y) \sim e^{-\frac{[(x+y)-(\bar{x}+\bar{y})]^2}{2(\sigma_x^2+\sigma_y^2)}}$$

$$\sigma_{x+y}^2 = \sigma_x^2 + \sigma_y^2$$

$$\boxed{\Delta_{x+y} = \sqrt{(\Delta X)^2 + (\Delta Y)^2}}$$

FOR RANDOM ERRORS.

EXAMPLE $r = x - y$

$x = 17.3 \pm 0.1$ ERROR $\sim 0.5\%$

$y = 17.1 \pm 0.1$ ERROR $\sim 0.5\%$

$$\Delta r = \sqrt{(.1)^2 + (.1)^2} = \sqrt{2(0.1)^2} = 1.41(.1)$$

$\Delta r = 0.14 \longrightarrow 0.1$ SIG FIGS

$\Delta r = 0.1$

$r = 0.2 \pm 0.1$ ERROR 50%

$$r = x \circledast y = xy$$

$$r \pm \Delta r = (x \pm \Delta x)(y \pm \Delta y)$$

$$= xy \pm x\Delta y \pm y\Delta x + \Delta x \Delta y$$

$$\frac{\Delta r}{r} = \pm \frac{x \Delta y}{xy} \pm \frac{y \Delta x}{xy} + \cancel{\Delta x \Delta y} \to 0$$

$$\frac{\Delta r}{r} = \frac{\Delta y}{y} + \frac{\Delta x}{x}$$ WORST CASE

$$\frac{\Delta r}{r} = \left|\frac{\Delta y}{y}\right| + \left|\frac{\Delta x}{x}\right|$$

FOR RANDOM ERRORS

$$\frac{\Delta r}{r} = \sqrt{\left(\frac{\Delta y}{y}\right)^2 + \left(\frac{\Delta x}{x}\right)^2}$$

$$z = f(x, y, w)$$

$$\bar{z} \pm \Delta z = f(\bar{x} \pm \Delta x, \bar{y} \pm \Delta y, \bar{w} \pm \Delta w)$$

TAYLOR EXPANSION

$$= f(\bar{x}, \bar{y}, \bar{w}) + \left.\frac{\partial f}{\partial x}\right|_{\bar{x}\bar{y}\bar{w}} \Delta x + \left.\frac{\partial f}{\partial y}\right|_{\bar{x}\bar{y}\bar{w}} \Delta y$$

$$+ \left.\frac{\partial f}{\partial w}\right|_{\bar{x}\bar{y}\bar{w}} \Delta w + O(\Delta^2)$$

WORSTCASE

$$\Delta z = \left|\left(\frac{\partial f}{\partial x}\right)_{\bar{x}\bar{y}\bar{w}}\right| \Delta x + \left|\left(\frac{\partial f}{\partial y}\right)_{\bar{x}\bar{y}\bar{w}}\right| \Delta y$$

$$+ \left|\left(\frac{\partial f}{\partial w}\right)_{\bar{x}\bar{y}\bar{w}}\right| \Delta w$$

FOR RANDOM ERRORS

$$\Delta z = \sqrt{\left[\left(\frac{\partial f}{\partial x}\right)_{\bar{x}\bar{y}\bar{w}} \Delta x\right]^2 + \left[\left(\frac{\partial f}{\partial y}\right)_{\bar{x}\bar{y}\bar{w}} \Delta y\right]^2 + \left[\left(\frac{\partial f}{\partial w}\right)_{\bar{x}\bar{y}\bar{w}} \Delta w\right]^2}$$

$$F(x, y) = z$$

$$\begin{array}{lll}
x_1 \ y_1 & \longrightarrow & z_1 \\
x_2 \ y_2 & \longrightarrow & z_2 \\
x_3 \ y_3 & \longrightarrow & z_3 \\
x_4 \ y_4 & \longrightarrow & z_4 \\
x_5 \ y_5 & \longrightarrow & z_5
\end{array}$$

$\bar{x} \pm \Delta x \qquad \bar{y} \pm \Delta y \qquad \bar{z} \pm \Delta z$

$$\Delta z = \sqrt{\left[\left(\frac{\partial f}{\partial x}\right)_{\bar{x}\bar{y}} \Delta x\right]^2 + \left[\left(\frac{\partial f}{\partial y}\right)_{\bar{x}\bar{y}} \Delta y\right]^2}$$

$$\bar{z} = f(\bar{x}, \bar{y})$$

$$k = A e^{-\frac{E}{RT}}$$

E9

$$\Delta k^2 = \left(\frac{\partial k}{\partial A}\Delta A\right)^2 + \left(\frac{\partial k}{\partial E}\Delta E\right)^2 + \left(\frac{\partial k}{\partial T}\Delta T\right)^2$$

$$(\Delta k)^2 = k^2\left(\frac{\Delta A}{A}\right)^2 + k^2\left(-\frac{1}{RT}\Delta E\right)^2 + k^2\left(\frac{E}{RT^2}\Delta T\right)^2$$

$$= k^2\left[\left(\frac{\Delta \bar{A}}{\bar{A}}\right)^2 + \left(\frac{\bar{E}}{R\bar{T}}\right)^2\left[\left(\frac{\Delta \bar{E}}{\bar{E}}\right)^2 + \cancel{\left(\frac{\Delta T}{T}\right)^2}^{\,0}\right]\right]$$

$$= k^2\left[\left(\frac{\Delta A}{A}\right)^2 + \left(\frac{\bar{E}}{R\bar{T}}\right)^2\left(\frac{\Delta E}{\bar{E}}\right)^2\right]$$

$$\frac{\Delta k}{k} = \sqrt{\left(\frac{\Delta A}{\bar{A}}\right)^2 + \left(\frac{\bar{E}}{R\bar{T}}\right)^2\left(\frac{\Delta E}{\bar{E}}\right)^2}$$

$$\ln k = \ln A - \frac{E}{R}\frac{1}{T}$$

$$y = b - m y$$

$\bar{b}$ $\boxed{\Delta b}$ $\bar{m}$ $\boxed{\Delta m}$

E10

$$\bar{A} = e^{\bar{b}}$$

$$\boxed{\Delta A = e^{\bar{b}} \Delta b}$$

$$\bar{E} = R\,\bar{m}$$

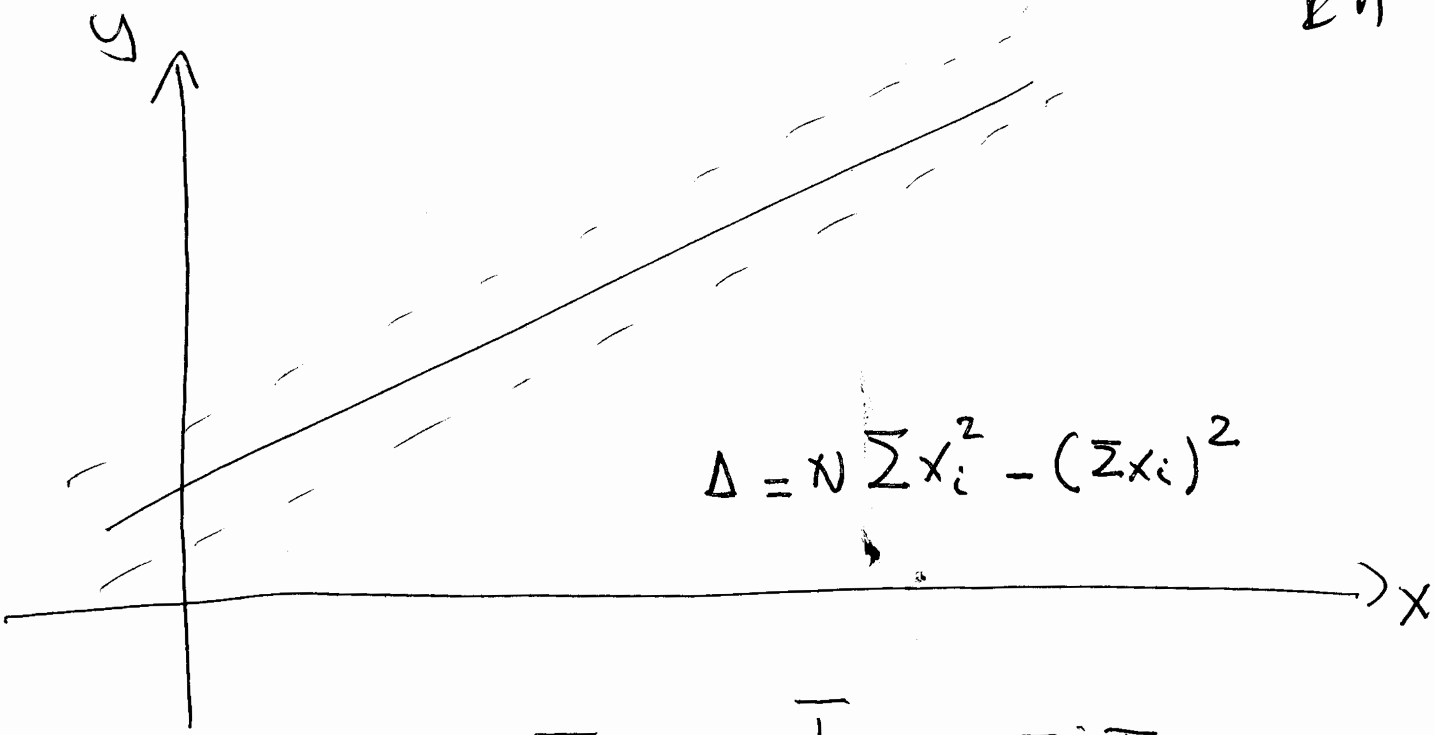
$$\boxed{\Delta E = R\,\Delta m}$$

$$\frac{\Delta A}{\bar{A}} = \Delta b \qquad \frac{\Delta E}{\bar{E}} = \frac{R\,\Delta m}{R\,\bar{m}} = \frac{\Delta m}{\bar{m}}$$

$$\frac{\Delta k}{k} = \sqrt{(\Delta b)^2 + \left(\frac{\bar{m}}{T}\right)^2 \left(\frac{\Delta m}{\bar{m}}\right)^2}$$

$$\boxed{\frac{\Delta k}{k} = \sqrt{(\Delta b)^2 + \frac{(\Delta m)^2}{T^2}}}$$

EY



$$\Delta = N \sum x_i^2 - \left(\sum x_i\right)^2$$

$$y = \overline{m}\, x + \overline{b} \qquad \text{FIT}$$

$$\overline{m} = \frac{N \sum x_i y_i - \sum x_i \sum y_i}{\Delta}$$

$$\overline{b} = \frac{\sum x_i^2 \sum y_i - \sum x_i \sum x_i y_i}{\Delta}$$

$$\sigma_y = \sqrt{\sum_i \frac{(y_i - b - m x_i)^2}{N-2}}$$

$$\sigma_b = \Delta b = \sigma_y \sqrt{\frac{\sum x_i^2}{\Delta}}$$

$$\Delta m = \sigma_m = \sigma_y \sqrt{\frac{N}{\Delta}}$$

$$data = \{\{1, \ \}, \{ \ , \ \}, \{ \ , \ \}, \{ \ , \ \}, 5, 3\}$$

MATHEMATICA

Regress[data, {1, x}]

$$y = mx + b$$

$$\Delta m, \ \Delta b$$

EXCEL

K