PROBLEM SET 1

P12.1) When a molecule absorbs a photon, both the energy and momentum are conserved. If a H_2 molecule at 750. K absorbs an ultraviolet photon of wavelength 225 nm, what is the change in its velocity Δv? Given that its average speed is $v_{rms} = \sqrt{3kT/m}$, what is $\Delta v/v_{rms}$?

Because momentum is conserved,

$$p_{photon} = \frac{h}{\lambda} = \frac{6.626\times10^{-34}\text{ J s}}{225\times10^{-9}\text{ m}} = 2.94\times10^{-27}\text{ kg m s}^{-1}$$

All this momentum is transferred to the H_2 molecule

$$\Delta p_{H_2} = 2.94\times10^{-27}\text{ kg m s}^{-1} = m\Delta v$$

$$\Delta v_{H_2} = \frac{2.94\times10^{-27}\text{ kg m s}^{-1}}{2.016\text{ amu}\times1.661\times10^{-27}\text{ kg/amu}} = 0.879\text{ m s}^{-1}$$

$$\frac{\Delta v}{v} = \frac{0.879\text{ m s}^{-1}}{\sqrt{\frac{3RT}{M}}} = \frac{0.879\text{ m s}^{-1}}{\sqrt{\frac{3\times8.314\text{ J mol}^{-1}\text{ K}^{-1}\times298\text{ K}}{2.016\times10^{-3}\text{ kg mol}^{-1}}}} = 2.89\times10^{-4}$$

P12.2) A more accurate expression for $\overline{E}_{osc}$ would be obtained by including additional terms in the Taylor-Mclaurin series. The Taylor-Mclaurin series expansion of *f(x)* in the vicinity of x_0 is given by (see Math Supplement)

$$f(x) = f(x_0) + \left(\frac{d\,f\,(x)}{d\,x}\right)_{x=x_0}(x-x_0) + \frac{1}{2!}\left(\frac{d^2 f\,(x)}{d\,x^2}\right)_{x=x_0}(x-x_0)^2 + \frac{1}{3!}\left(\frac{d^3 f\,(x)}{d\,x^3}\right)_{x=x_0}(x-x_0)^3 + \ldots$$

Use this formalism to better approximate $\overline{E}_{osc}$ by expanding $e^{\frac{h\nu}{kT}}$ in powers of $h\nu/kT$ out to $(h\nu/kT)^3$ in the vicinity of $h\nu/kT = 0$. Calculate the relative error, $\frac{\overline{E}_{osc} - kT}{\overline{E}_{osc}}$, if you had not included the additional terms for $\nu = 1.00\times10^{12}\text{ s}^{-1}$ at temperatures of 800., 500., and 250. K. Explain the trend you see.

The Taylor series expansion of $e^{\frac{h\nu}{kT}}$ is $1+\frac{h\nu}{kT}+\frac{1}{2}\left(\frac{h\nu}{kT}\right)^2+\frac{1}{6}\left(\frac{h\nu}{kT}\right)^3+\ldots$. Therefore including terms up to $\left(h\nu/kT\right)^3$,

$$\overline{E}_{osc}=\frac{h\nu}{\frac{h\nu}{kT}+\frac{1}{2}\left(\frac{h\nu}{kT}\right)^2+\frac{1}{6}\left(\frac{h\nu}{kT}\right)^3}$$

$$\frac{\overline{E}_{osc}-kT}{\overline{E}_{osc}}=\left[\frac{h\nu}{\frac{h\nu}{kT}+\frac{1}{2}\left(\frac{h\nu}{kT}\right)^2+\frac{1}{6}\left(\frac{h\nu}{kT}\right)^3}-kT\right]\Bigg/\frac{h\nu}{\frac{h\nu}{kT}+\frac{1}{2}\left(\frac{h\nu}{kT}\right)^2+\frac{1}{6}\left(\frac{h\nu}{kT}\right)^3}$$

$$\frac{\overline{E}_{osc}-kT}{\overline{E}_{osc}}=$$

$$\left[6.626\times10^{-34}\text{ J s}\times1.00\times10^{12}\text{ s}^{-1}\Bigg/\left\{\begin{array}{l}\frac{6.626\times10^{-34}\text{ J s}\times1.00\times10^{12}\text{ s}^{-1}}{1.381\times10^{-23}\text{ J K}^{-1}\times800\text{K}}+\frac{1}{2}\left(\frac{6.626\times10^{-34}\text{ J s}\times1.00\times10^{12}\text{ s}^{-1}}{1.381\times10^{-23}\text{ J K}^{-1}\times800\text{K}}\right)^2\\+\frac{1}{6}\left(\frac{6.626\times10^{-34}\text{ J s}\times1.00\times10^{12}\text{ s}^{-1}}{1.381\times10^{-23}\text{ J K}^{-1}\times800\text{K}}\right)^3\end{array}\right\}-1.381\times10^{-23}\text{ J K}^{-1}\times800\text{K}\right]\times$$

$$\left[6.626\times10^{-34}\text{ J s}\times1.00\times10^{12}\text{ s}^{-1}\Bigg/\left\{\begin{array}{l}\frac{6.626\times10^{-34}\text{ J s}\times1.00\times10^{12}\text{ s}^{-1}}{1.381\times10^{-23}\text{ J K}^{-1}\times800\text{K}}+\frac{1}{2}\left(\frac{6.626\times10^{-34}\text{ J s}\times1.00\times10^{12}\text{ s}^{-1}}{1.381\times10^{-23}\text{ J K}^{-1}\times800\text{K}}\right)^2\\+\frac{1}{6}\left(\frac{6.626\times10^{-34}\text{ J s}\times1.00\times10^{12}\text{ s}^{-1}}{1.381\times10^{-23}\text{ J K}^{-1}\times800\text{K}}\right)^3\end{array}\right\}\right]$$

$\frac{\overline{E}_{osc}-kT}{\overline{E}_{osc}}=-0.0306$ for 800. K. The corresponding values for 500. K and 250. K are –0.0495 and –0.102. We expect the high temperature limit of $\overline{E}_{osc}$ to be kT. Therefore the error is small at 800. K but becomes larger as T decreases.

P12.7) Show that the energy density radiated by a blackbody

$$\frac{E_{total}(T)}{V}=\int_0^\infty \rho(\nu,T)d\nu=\int_0^\infty \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/kT}-1}d\nu$$

depends on the temperature as T^4. (*Hint:* Make the substitution of variables $x=h\nu/kT$.) The definite integral $\int_0^\infty [x^3/(e^x-1)]dx=\pi^4/15$. Using your result, calculate the energy density radiated by a blackbody at 1350. K and 5250. K.

$\frac{E_{total}}{V}=\int_0^\infty \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/kT}-1}d\nu$. Let $x=h\nu/kT$; $dx=\frac{h}{kT}d\nu$

$$\int_0^\infty \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/kT}-1}d\nu = \frac{8\pi k^4T^4}{h^3c^3}\int_0^\infty \frac{x^3}{e^x-1}dx = \frac{8\pi^5 k^4T^4}{15h^3c^3}$$

At 1350. K, $\frac{E_{total}}{V} = \frac{8\pi^5 k^4T^4}{15h^3c^3} = \frac{8\pi^5 (1.381\times10^{-23}\ \text{J K}^{-1})^4\times(1350.\ \text{K})^4}{15\times(6.626\times10^{-34}\ \text{J s})^3(2.998\times10^8\ \text{m s}^{-1})^3} = 2.51\times10^{-3}\ \text{J m}^{-3}$ At 5250. K,

$$\frac{E_{total}}{V} = \frac{8\pi^5\times(1.381\times10^{-23}\ \text{J K}^{-1})^4\times(5250.\ \text{K})^4}{15\times(6.626\times10^{-34}\ \text{J s})^3\times(2.998\times10^8\ \text{m s}^{-1})^3} = 0.575\ \text{J m}^{-3}$$

P12.9) A newly developed substance that emits 315 W of photons with a wavelength of 275 nm is mounted in a small rocket initially at rest in outer space such that all of the radiation is released in the same direction. Because momentum is conserved, the rocket will be accelerated in the opposite direction. If the total mass of the rocket is 6.75 kg, how fast will it be traveling at the end of 365 days in the absence of frictional forces?

The number of photons is given by

$$n' = \frac{E_{total}}{E_{photon}} = \frac{\text{Watts}\times\frac{1\ \text{Js}^{-1}}{\text{W}}}{\frac{hc}{\lambda}} = \frac{315\,\text{W}\times\frac{1\ \text{Js}^{-1}}{\text{W}}}{\frac{6.626\times10^{-34}\ \text{J s}\times2.998\times10^8\ \text{m s}^{-1}}{275\times10^{-9}\ \text{m}}} = 4.36\times10^{20}\ \text{s}^{-1}$$

The momentum of one photon is

$$p = \frac{h}{\lambda} = \frac{6.626\times10^{-34}\ \text{J s}}{275\times10^{-9}\ \text{m}} = 2.41\times10^{-27}\text{kg m s}^{-1}$$

The force is given by the rate of change of momentum.

$$F = \frac{d(n'p)}{dt} = 2.41\times10^{-27}\ \text{kg m s}^{-1}\times4.36\times10^{20}\ \text{s}^{-1} = 1.05\times10^{-6}\ \text{kg m s}^{-2}$$

The final speed is given by

$$\text{v} = \text{v}_0 + at = \frac{F}{m}t = \frac{1.05\times10^{-6}\ \text{kg m s}^{-2}}{6.75\text{kg}}\times365\,\text{days}\times\frac{86400\ \text{s}}{\text{day}} = 4.91\ \text{m s}^{-1}$$

P12.10) In our discussion of blackbody radiation, the average energy of an oscillator $\overline{E}_{osc} = h\nu/(e^{h\nu/kT} - 1)$ was approximated as $\overline{E}_{osc} = h\nu/[(1 + h\nu/kT) - 1] = kT$ for $h\nu/kT << 1$. Calculate the relative error $= (E - E_{approx})/E$ in making this approximation for $\nu = 4.00\times 10^{12}\ \text{s}^{-1}$ at temperatures of 6000., 2000., and 500. K. Can you predict what the sign of the relative error will be without a detailed calculation?

$$\text{Relative Error} = \frac{E - E_{approx}}{E} = \frac{\left(\dfrac{h\nu}{\exp\left(\dfrac{h\nu}{kT}\right) - 1}\right) - kT}{\left(\dfrac{h\nu}{\exp\left(\dfrac{h\nu}{kT}\right) - 1}\right)}$$

$$\frac{\left(\dfrac{h\nu}{\exp\left(\dfrac{h\nu}{kT}\right) - 1}\right) - kT}{\left(\dfrac{h\nu}{\exp\left(\dfrac{h\nu}{kT}\right) - 1}\right)}$$

$$= \frac{\left(\dfrac{6.626\times 10^{-34}\ \text{J s}\times 4.00\times 10^{12}\ \text{s}^{-1}}{\exp\left(\dfrac{6.626\times 10^{-34}\ \text{J s}\times 4.00\times 10^{12}\ \text{s}^{-1}}{1.381\times 10^{-23}\ \text{J K}^{-1}\times 6000\,\text{K}}\right) - 1}\right) - 1.381\times 10^{-23}\ \text{J K}^{-1}\times 6000\,\text{K}}{\left(\dfrac{6.626\times 10^{-34}\ \text{J s}\times 4.00\times 10^{12}\ \text{s}^{-1}}{\exp\left(\dfrac{6.626\times 10^{-34}\ \text{J s}\times 4.00\times 10^{12}\ \text{s}^{-1}}{1.381\times 10^{-23}\ \text{J K}^{-1}\times 6000\,\text{K}}\right) - 1}\right)}$$

$$\text{Relative Error} = \frac{E - E_{approx}}{E} = -0.0162 \text{ for } T = 6000.\ \text{K}.$$

The results for 2000. K, and 500. K are –0.0496 and –0.219.

Because $E_{approx} = kT > E$, the relative error $= \dfrac{E - E_{approx}}{E}$ is always a negative number.

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P12.14) The distribution in wavelengths of the light emitted from a radiating blackbody is a sensitive function of the temperature. This dependence is used to measure the temperature of hot objects, without making physical contact with those objects, in a technique called *optical pyrometry*. In the limit $(hc/\lambda kT) >> 1$, the maximum in a plot of $\rho(\lambda, T)$ versus λ is given by $\lambda_{max} = hc/5kT$. At what wavelength does the maximum in $\rho(\lambda, T)$ occur for T = 850., 1300., and 5500. K?

According to Example Problem 14.1, $\lambda_{max} = \dfrac{hc}{5kT}$.

$\lambda_{max} = \dfrac{6.626\times10^{-34}\text{ J s}\times2.998\times10^{8}\text{ m s}^{-1}}{5\times1.381\times10^{-23}\text{ J K}^{-1}\times850.\text{K}} = 3.38\times10^{-6}$ m at 850. K. λ_{max} for 1300. K and 5500. K is 2.21×10^{-6} m and 5.23×10^{-7} m, respectively.

P12.15) A beam of electrons with a speed of 4.75×10^{4} m/s is incident on a slit of width 235 nm. The distance to the detector plane is chosen such that the distance between the central maximum of the diffraction pattern and the first diffraction minimum is 0.375 cm. How far is the detector plane from the slit?

The diffraction minima satisfy the condition $\sin\theta = \dfrac{n\lambda}{a}, n = \pm1, \pm2, ...$ and the first minimum is at $\sin\theta = \pm\dfrac{\lambda}{a}$. We choose the plus sign (the minus sign gives the distance from the slit in the opposite direction) giving

$$\sin\theta = \frac{\lambda}{a} = \frac{h}{mva} = \frac{6.626\times10^{-34}\text{ J s}}{9.109\times10^{-31}\text{ kg}\times4.75\times10^{4}\text{ m s}^{-1}\times235\times10^{-9}\text{ m}} = 0.06517$$
$$\theta = 5.74\text{ degrees}$$

The distance *d* from the screen and the position of the first minimum *s* are related by

$$d = \frac{s}{\tan\theta} = \frac{0.375\text{ cm}}{0.0652} = 5.74\text{ cm}$$

P12.18) X-rays can be generated by accelerating electrons in a vacuum and letting them impact on atoms in a metal surface. If the 1525-eV kinetic energy of the electrons is completely converted to the photon energy, what is the wavelength of the X-rays produced? If the electron current is 2.25×10^{-5} A, how many photons are produced per second?

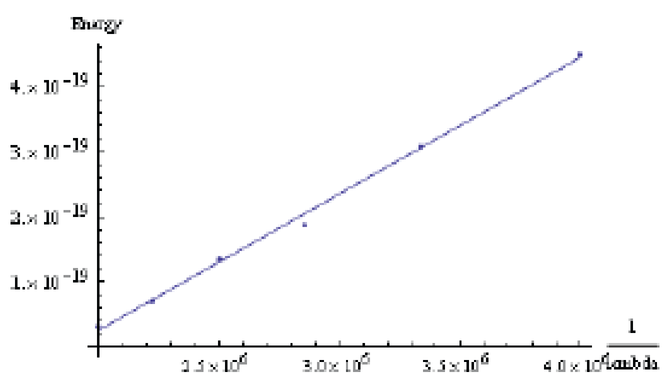
$$\lambda = \frac{hc}{E} = \frac{6.626\times10^{-34}\ \text{J s}\times2.998\times10^{8}\ \text{m s}^{-1}}{1525\ \text{eV}\times\dfrac{1.602\times10^{-19}\ \text{J}}{\text{eV}}} = 0.813\ \text{nm}$$

$$n = \frac{\text{current}}{\text{charge per electron}} = \frac{2.25\times10^{-5}\ \text{C s}^{-1}}{1.602\times10^{-19}\ \text{C}} = 1.40\times10^{14}\ \text{s}^{-1}.$$

P12.19) The following data were observed in an experiment on the photoelectric effect from potassium:

10^{19} Kinetic Energy (J)	4.49	3.09	1.89	1.34	0.700	0.311
Wavelength (nm)	250.	300.	350.	400.	450.	500.

Graphically evaluate these data to obtain values for the work function and Planck's constant.



The best-fit line is given by $E(J) = -3.95831\times10^{-19} + 2.1032\times10^{-25}\,\frac{1}{\lambda}$.

Because the slope is hc, $h = \frac{2.11171\times10^{-25}\ \text{J m}}{2.998\times10^{8}\ \text{m s}^{-1}} = 7.02\times10^{-34}$ J s. The work function is given by the intercept of the line with the x axis at $y = 0$. $\phi = \frac{hc}{\lambda_0} - E$ where $\lambda_0 = \frac{2.1171\times10^{-25}\ \text{J m}}{3.97362\times10^{-19}\ \text{J}} = 5.31\times10^{-7}$ m

This gives $\phi = 3.74\times10^{-19}$ J or 2.33eV.

P12.22) Assume that water absorbs light of wavelength 3.50×10^{-6} m with 100% efficiency. How many photons are required to heat 2.50 g of water by 1.00 K? The heat capacity of water is 75.3 J mol^{-1} K^{-1}.

$$E = Nh\nu = N\frac{hc}{\lambda} = nC_{p,m}\Delta T$$

$$N = \frac{m}{M}\frac{C_{p,m}\Delta T\lambda}{hc} = \frac{2.50\text{ g}}{18.02\text{ g mol}^{-1}}\frac{75.3\text{ JK}^{-1}\text{ mol}^{-1}\times 1.00\text{K}\times 3.50\times 10^{-6}\text{ m}}{6.626\times 10^{-34}\text{ Js}\times 2.998\times 10^{8}\text{ m s}^{-1}}$$

$$= 1.84\times 10^{20}$$
