Problem Set 2

P13.3) Determine in each of the following cases if the function in the first column is an eigenfunction of the operator in the second column. If so, what is the eigenvalue?

a. x^3 d^3/dx^3

b. xy $x(\partial/\partial x)+y(\partial/\partial y)$

c. $\sin\theta\cos\phi$ $\partial^2/\partial\theta^2$

a) $\frac{d\,x^3}{d\,x^3}=6$ Not an eigenfunction

b) $x\frac{\partial\,xy}{\partial x}+y\frac{\partial\,xy}{\partial y}=2xy$ Eigenfunction with eigenvalue +2

c) $\frac{\partial^2}{\partial\theta^2}(\sin\theta\cos\phi)=-\sin\theta\cos\phi$ Eigenfunction with eigenvalue −1

P13.4) If two operators act on a wave function as indicated by $\hat{A}\hat{B}\,f(x)$, it is important to carry out the operations in succession with the first operation being that nearest to the function. Mathematically, $\hat{A}\hat{B}\,f(x)=\hat{A}(\hat{B}\,f(x))$ and $\hat{A}^2\,f(x)=\hat{A}(\hat{A}\,f(x))$. Evaluate the following successive operations $\hat{A}\hat{B}\,f(x)$. The operators $\hat{A}$ and $\hat{B}$ are listed in the first two columns $f(x)$ and is listed in the third column.

a. $\frac{d}{dx}$ x xe^{-ax^2}

b. x $\frac{d}{dx}$ xe^{-ax^2}

c. $y\frac{\partial}{\partial x}$ $x\frac{\partial}{\partial y}$ $e^{-a(x^2+y^2)}$

Note that your answers to parts (a) and (b) are not identical. As we will learn in Chapter 18, the fact that switching the order of the operators x and d/dx changes the outcome of the operation $\hat{A}\hat{B}\,f(x)$ is the basis for the Heisenberg uncertainty principle.

a) $\frac{d}{d\,x}\left[x\left(x\,e^{-ax^2}\right)\right]=2x\,e^{-ax^2}-2\,a\,x^3\,e^{-ax^2}$

b) $x\left[\frac{d}{d\,x}\left(x\,e^{-ax^2}\right)\right]=x\,e^{-ax^2}-2\,a\,x^3\,e^{-ax^2}$

c)

$$y\frac{\partial}{\partial x}\left[x\frac{\partial\left(e^{-a\left(x^2+y^2\right)}\right)}{\partial y}\right]=y\frac{\partial}{\partial x}\left[-2\,a\,x\,y\,e^{-a\left(x^2+y^2\right)}\right]$$

$$=-2\,a\,y^2e^{-a\left(x^2+y^2\right)}+4a^2x^2y^2e^{-a\left(x^2+y^2\right)}$$

P13.5) Let (1, 0) and (0, 1) represent the unit vectors along the x and y directions, respectively. The operator

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

effects a rotation in the x-y plane. Show that the length of an arbitrary vector

$$\begin{pmatrix} a \\ b \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

which is defined as $\sqrt{a^2+b^2}$, is unchanged by this rotation. See the Math Supplement (Appendix A) for a discussion of matrices.

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a\cos\theta - b\sin\theta \\ a\sin\theta + b\cos\theta \end{pmatrix}$$

The length of the vector is given by

$$\sqrt{\left(a\cos\theta - b\sin\theta\right)^2 + \left(a\sin\theta + b\cos\theta\right)^2} = (a^2\cos^2\theta + b^2\sin^2\theta - 2ab\sin\theta\cos\theta$$

$$+ a^2\sin^2\theta + b^2\cos^2\theta + 2ab\sin\theta\cos\theta)^{1/2}$$

$$= \sqrt{a^2\left(\cos^2\theta + \sin^2\theta\right) + b^2\left(\cos^2\theta + \sin^2\theta\right)} = \sqrt{a^2 + b^2}$$

This result shows that the length of the vector is not changed.

P13.7) Operators can also be expressed as matrices and wave functions as column vectors. The operator matrix

$$\begin{pmatrix} \alpha & \beta \\ \delta & \varepsilon \end{pmatrix}$$

acts on the wave function $\begin{pmatrix} a \\ b \end{pmatrix}$ according to the rule

$$\begin{pmatrix} \alpha & \beta \\ \delta & \varepsilon \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \alpha a + \beta b \\ \delta a + \varepsilon b \end{pmatrix}$$

In words, the 2 × 2 matrix operator acting on the two-element column wave function generates another two-element column wave function. If the wave function generated by the operation is the original wave function multiplied by a constant, the wave function is an eigenfunction of the operator. What is the effect of the operator

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

on the column vectors (1, 0), (0, 1), (1, 1), and (–1, 1)? Are these wave functions eigenfunctions of the operator? See the Math Supplement (Appendix A) for a discussion of matrices.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} = (-1)\begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Only $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ are eigenfunctions with the eigenvalues 1 and –1, respectively.

P13.8) Show that

$$\frac{a+ib}{c+id} = \frac{ac+bd+i(bc-ad)}{c^2+d^2}$$

$$\frac{a+ib}{c+id} = \left(\frac{a+ib}{c+id}\right)\left(\frac{c-id}{c-id}\right) = \frac{ac+bd+ibc-iad}{c^2+d^2} = \frac{ac+bd+i\left(bc-ad\right)}{c^2+d^2}$$

P13.10) Show that the set of functions $\phi_n(\theta) = e^{in\theta}$, $0 \le \theta \le 2\pi$, is orthogonal if n and m are integers. To do so, you need to show that the integral $\int_0^{2\pi} \phi_m^*(\theta)\phi_n(\theta)d\theta = 0$ for $m \neq n$ if n and m are integers.

$$\int_0^{2\pi} \phi_m^*(\theta)\phi_n(\theta)d\theta = \int_0^{2\pi} e^{-im\theta}\, e^{in\theta}\, d\theta = \int_0^{2\pi} e^{i(n-m)\theta} d\theta = \left[\frac{1}{i\left(n-m\right)} e^{i(n-m)\theta}\right]_0^{2\pi}$$

$$= \frac{1}{i\left(n-m\right)}\left[e^{i(n-m)2\pi} - e^0\right] = \frac{1}{i\left(n-m\right)}\left[\cos\left[\left(n-m\right)2\pi\right] + i\sin\left[\left(n-m\right)2\pi\right] - 1\right]$$

Because n and m are integers, $\left(n-m\right)$ is an integer and the arguments of the sine and cosine functions are integral multiples of 2π.

$$\int_0^{2\pi} \phi_m^*(\theta)\phi_n(\theta)d\theta = \frac{1}{i\left(n-m\right)}\left[1+0-1\right] = 0$$

P13.12) Which of the following wave functions are eigenfunctions of the operator d^2/dx^2? If they are eigenfunctions, what is the eigenvalue?

a. $a e^{-3x} + b e^{-3ix}$

b. $\sin^2 x$

c. e^{-ix}

d. $\cos a x$

e. e^{-ix^2}

a) $\dfrac{d^2\left(a e^{-3x} + b e^{-3ix}\right)}{dx^2} = 9 a e^{-3x} - 9 b e^{-3ix}$ Not an eigenfunction

b) $\dfrac{d^2 \sin^2 x}{dx^2} = -2\sin^2 x + 2\cos^2 x$ Not an eigenfunction

c) $\dfrac{d^2 e^{-ix}}{dx^2} = -e^{-ix}$ Eigenfunction with eigenvalue –1

d) $\dfrac{d^2 \cos a x}{dx^2} = -a^2 \cos a x$ Eigenfunction with eigenvalue $-a^2$

e) $\dfrac{d^2 e^{-ix^2}}{dx^2} = -2i e^{-ix^2} - 4x^2 e^{-ix^2}$ Not an eigenfunction

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P13.17) Determine in each of the following cases if the function in the first column is an eigenfunction of the operator in the second column. If so, what is the eigenvalue?

a.	$e^{-i(3x+2y)}$	$\dfrac{\partial^2}{\partial x^2}$
b.	$\sqrt{x^2+y^2}$	$(1/x)(x^2+y^2)\dfrac{\partial}{\partial x}$
c.	$\sin\theta\cos\theta$	$\sin\theta\dfrac{d}{d\theta}\left(\sin\theta\dfrac{d}{d\theta}\right)+6\sin^2\theta$

a)

$\dfrac{\partial^2 e^{-i(3x+2y)}}{\partial x^2} = -9e^{-i(3x+2y)}$ Eigenfunction with eigenvalue –9.

b)

$$\frac{1}{x}\left(x^2+y^2\right)\frac{\partial\sqrt{x^2+y^2}}{\partial x}=\sqrt{x^2+y^2}$$ Eigenfunction with eigenvalue +1.

c)

$$\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{d\sin\theta\cos\theta}{d\theta}\right)+6\sin^3\theta\cos\theta=\sin\theta\frac{d}{d\theta}\left(\sin\theta\left[1-2\sin^2\theta\right]\right)+6\sin^3\theta\cos\theta$$

$$=\sin\theta\left(\cos\theta-6\sin^2\theta\cos\theta\right)+6\sin^3\theta\cos\theta=\sin\theta\cos\theta$$

Eigenfunction with eigenvalue +1.

P13.18) Assume that a system has a very large number of energy levels given by the formula $\varepsilon = \varepsilon_0 l^2$ with $\varepsilon_0 = 2.34\times10^{-22}\,\mathrm{J}$, where l takes on the integral values 1, 2, 3, …. Assume further that the degeneracy of a level is given by $g_l = 2l$. Calculate the ratios n_3/n_1 and n_{15}/n_1 for T = 250. K and T = 900. K, respectively.

$$\frac{n_3}{n_1}=\frac{g_3}{g_1}\exp\left[\frac{-(\varepsilon_3-\varepsilon_1)}{kT}\right]=\frac{2\times3}{2\times1}\exp\left[\frac{-\left(3^2\varepsilon_0-\varepsilon_0\right)}{kT}\right]$$

$$\frac{n_3}{n_1}(250.\ \mathrm{K})=\frac{6}{2}\exp\left[\frac{-2.34\times10^{-22}\ \mathrm{J}\times(9-1)}{1.381\times10^{-23}\,\mathrm{J\ K^{-1}}\times250.\ \mathrm{K}}\right]=1.74$$

$$\frac{n_3}{n_1}(900.\ \mathrm{K})=\frac{6}{2}\exp\left[\frac{-2.34\times10^{-22}\ \mathrm{J}\times(9-1)}{1.381\times10^{-23}\,\mathrm{J\ K^{-1}}\times900.\ \mathrm{K}}\right]=2.58$$

$$\frac{n_{15}}{n_1}(250.\ \mathrm{K})=\frac{30}{2}\exp\left[\frac{-2.34\times10^{-22}\ \mathrm{J}\times(225-1)}{1.381\times10^{-23}\,\mathrm{J\ K^{-1}}\times250.\ \mathrm{K}}\right]=3.82\times10^{-6}$$

$$\frac{n_{15}}{n_1}(900.\ \mathrm{K})=\frac{30}{2}\exp\left[\frac{-2.34\times10^{-22}\ \mathrm{J}\times(225-1)}{1.381\times10^{-23}\,\mathrm{J\ K^{-1}}\times900.\ \mathrm{K}}\right]=0.221$$

P13.27) Find the result of operating with $(1/r^2)(d/dr)(r^2\, d/dr) + 2/r$ on the function Ae^{-br}. What must the values of A and b be to make this function an eigenfunction of the operator?

$$\frac{1}{r^2}\frac{d}{dr}r^2\frac{dAe^{-br}}{dr} + \frac{2Ae^{-br}}{r} = \frac{1}{r^2}\frac{d}{dr}\left(-bAr^2e^{-br}\right) + \frac{2Ae^{-br}}{r}$$

$$= \frac{1}{r^2}\left(-2brAe^{-br} + b^2r^2Ae^{-br}\right) + \frac{2Ae^{-br}}{r}$$

$$= \frac{2Ae^{-br}}{r}(2-2b) + b^2Ae^{-br}$$

To be an eigenfunction of the operator, the terms in $\frac{Ae^{-br}}{r}$ must vanish. This requires that $b = 1$. There are no restrictions on the value of A.

P13.32) Use a Fourier series expansion to express the function $f(x) = x, \quad -b \le x \le b$, in the form

$$f(x) = d_0 + \sum_{n=1}^{m} c_n \sin\left(\frac{n\pi x}{b}\right) + d_n \cos\left(\frac{n\pi x}{b}\right)$$

Obtain d_0 and the first five coefficients c_n and d_n.

The coeficents can be obtained as outlined in Section 2.7.

$$d_0 = \frac{1}{2b}\int_{-b}^{b} f(x)\,dx = \frac{1}{2b}\int_{-b}^{b} x\,dx = 0$$

$$d_n = \frac{1}{b}\int_{-b}^{b} f(x)\cos\left(\frac{n\pi x}{b}\right)dx = \frac{1}{b}\int_{-b}^{b} x\cos\left(\frac{n\pi x}{b}\right)dx$$

Using the standard integral $\int x\cos ax = \frac{\cos ax}{a^2} + \frac{x\sin ax}{a}$

$$d_n = \frac{1}{b}\left[\left(\frac{b}{n\pi}\right)^2 \cos\left(\frac{n\pi b}{b}\right) + \left(\frac{b}{n\pi}\right) b\sin\left(\frac{n\pi b}{b}\right) - \left(\frac{b}{n\pi}\right)^2 \cos\left(\frac{n\pi(-b)}{b}\right) - \left(\frac{b}{n\pi}\right)(-b)\sin\left(\frac{n\pi(-b)}{b}\right)\right] = 0$$

All the $d_n = 0$ because $f(x)$ is an odd function of x.

$$c_n = \frac{1}{b}\int_{-b}^{b} f(x)\sin\left(\frac{n\pi x}{b}\right)dx = \frac{1}{b}\int_{-b}^{b} x\sin\left(\frac{n\pi x}{b}\right)dx$$

Using the standard integral $\int x\sin ax = -\frac{x\cos ax}{a} + \frac{\sin ax}{a^2}$

$$c_n = \frac{1}{b}\left[\left(\frac{b}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{b}\right) - \left(\frac{b}{n\pi}\right) x\cos\left(\frac{n\pi x}{b}\right)\right]_{-b}^{b}$$

$$c_n = \frac{1}{b}\left[\left(\frac{b}{n\pi}\right)^2 \sin\left(\frac{n\pi b}{b}\right) - \left(\frac{b}{n\pi}\right) b\cos\left(\frac{n\pi b}{b}\right) - \left(\frac{b}{n\pi}\right)^2 \sin\left(\frac{n\pi(-b)}{b}\right) + \left(\frac{b}{n\pi}\right)(-b)\cos\left(\frac{n\pi(-b)}{b}\right)\right]$$

$$c_n = \frac{2}{b}\left[\left(\frac{b}{n\pi}\right)^2 \sin(n\pi) - \frac{b^2}{n\pi}\cos(n\pi)\right]$$

$$c_n = \left(\frac{2b}{n\pi}\right)\cos n\pi = \frac{2b}{n\pi}(-1)^{n+1}$$

We conclude that $d_0 = d_1 = d_2 = d_3 = d_4 = d_5 = 0$.

$c_1 = \frac{2b}{\pi}$, $c_2 = -\frac{b}{\pi}$, $c_3 = \frac{2b}{3\pi}$, $c_4 = -\frac{b}{2\pi}$, and $c_5 = \frac{2b}{5\pi}$