Problem Set 5

P17.3) Evaluate the commutator $[d/dr, 1/r]$ by applying the operators to an arbitrary function $f(r)$.

$$\left[\frac{d}{dr},\frac{1}{r}\right]f(r)=\frac{d}{dr}\left(\frac{f(r)}{r}\right)-\frac{1}{r}\frac{df(r)}{dr}$$

$$=-\frac{1}{r^2}f(r)+\frac{1}{r}\frac{df(r)}{dr}-\frac{1}{r}\frac{df(r)}{dr}=-\frac{1}{r^2}f(r)$$

Therefore, $\left[\frac{d}{dr},\frac{1}{r}\right]=-\frac{1}{r^2}$

P17.7) Evaluate $\left[\hat{A},\hat{B}\right]$ if $\hat{A}=x^2-d^2/dx^2$ and $\hat{B}=x-d/dx$.

$$\hat{A}f(x)=x^2f(x)-d^2f(x)/dx^2$$

$$\hat{B}f(x)=xf(x)-df(x)/dx$$

$$\hat{A}\hat{B}f(x)=x^2\left[xf(x)-df(x)/dx\right]-d^2\left[xf(x)-df(x)/dx\right]/dx^2$$

$$=x^2\left[xf(x)-df(x)/dx\right]-2df(x)/dx-x\,d^2f(x)/dx^2\;+d^3f(x)/dx^3$$

$$\hat{B}\hat{A}f(x)=x\left[x^2f(x)-d^2f(x)/dx^2\;\right]-d\left[x^2f(x)-d^2f(x)/dx^2\;\right]/dx$$

$$=-2xf(x)-x^2f(x)+x^3f(x)-x\,d^2f(x)/dx^2\;+d^3f(x)/dx^3$$

$$\hat{A}\hat{B}f(x)-\hat{B}\hat{A}f(x)=2xf(x)+x^3f(x)-x^2\,df(x)/dx-2\,df(x)/dx+x^2\,df(x)/dx$$

$$-x^3f(x)+x\,d^2f(x)/dx^2\;-x\,d^2f(x)/dx^2$$

$$=2xf(x)-2\,df(x)/dx$$

$$\left[\hat{A},\hat{B}\right]=2x-2\,d/dx$$

P17.11) Evaluate the commutator $[d^2/dx^2, x]$ by applying the operators to an arbitrary function $f(x)$.

$$\left[\frac{d^2}{dx^2},x\right]f(x)=\frac{d^2}{dx^2}(x\,f(x))-x\left(\frac{d^2}{dx^2}f(x)\right)$$

$$=\left\{2\frac{df(x)}{d\,x}+x\frac{d^2f(x)}{d\,x^2}-x\frac{d^2f(x)}{d\,x^2}\right\}=\left(2\frac{d}{d\,x}\right)f(x)$$

$$\left[\frac{d^2}{dx^2},x\right]=2\frac{d}{d\,x}$$

P17.12) Revisit the TV picture tube of Example Problem 17.3. Keeping all other parameters the same, what electron energy would result in a position uncertainty of 7.7×10^{-9} m?

$$\Delta p=\frac{\hbar}{2\Delta x}=\frac{1.055\times10^{-34}\text{ J s}}{2\times7.7\times10^{-9}\text{ m}}=6.9\times10^{-27}\text{ kg m s}^{-1}$$

$$p=\frac{\Delta p}{0.01}=6.9\times10^{-25}\text{ kg m s}^{-1}$$

$$E=\frac{p^2}{2m}=\frac{\left(6.9\times10^{-25}\text{ kg m s}^{-1}\right)^2}{2\times9.11\times10^{-31}\text{ kg}}=2.6\times10^{-19}\text{ J}=1.6\text{ eV}$$

P17.13) Evaluate the commutator $\left[(d^2/dx^2)-x,(d/dx)+x^2\right]$ by applying the operators to an arbitrary function *f*(*x*).

$$\left[\hat{A},\hat{B}\right]=\left[d^2/dx^2-x,(d/dx)+x^2\right]$$
$$\hat{A}f(x)=d^2f(x)/dx^2-xf(x)$$
$$\hat{B}f(x)=df(x)/dx+x^2f(x)$$
$$\hat{B}\hat{A}f(x)=d\left[d^2f(x)/dx^2-xf(x)\right]/dx+x^2\left[d^2f(x)/dx^2-xf(x)\right]$$
$$\hat{A}\hat{B}f(x)=d^2\left[df(x)/dx+x^2f(x)\right]/dx^2-x\left[df(x)/dx+x^2f(x)\right]$$
$$\hat{A}\hat{B}f(x)-\hat{B}\hat{A}f(x)=d^3f(x)/dx^3+d\left[2xf(x)+x^2\,df(x)/dx\right]/dx-x\,df(x)/dx-x^3f(x)$$
$$-d^3f(x)/dx^3+f(x)+x\,df(x)/dx-x^2d^2f(x)/dx^2+x^3f(x)$$
$$=d^3f(x)/dx^3+2f(x)+2x\,df(x)/dx+2x\,df(x)/dx+x^2d^2f(x)/dx^2-x\,df(x)/dx-x^3f(x)$$
$$-d^3f(x)/dx^3+f(x)+x\,df(x)/dx-x^2d^2f(x)/dx^2+x^3f(x)$$
$$=4x\,df(x)/dx+3f(x)$$
$$\left[\hat{A},\hat{B}\right]=3+4x\,d/dx$$

P17.15) Apply the Heisenberg uncertainty principle to estimate the zero point energy for the particle in the box.

a. First justify the assumption that $\Delta x\le a$ and that, as a result, $\Delta p\ge\hbar/2a$. Justify the statement that, if $\Delta p\ge 0$, we cannot know that $E=p^2/2m$ is identically zero.

b. Make this application more quantitative. Assume that $\Delta x=0.50a$ and $\Delta p=0.50\,p$ where p is the momentum in the lowest energy state. Calculate the total energy of this state based on these assumptions and compare your result with the ground-state energy for the particle in the box.

c. Compare your estimates for Δp and Δx with the more rigorously derived uncertainties σ_p and σ_x of Equation (17.13).

a) Because the particle is in the box, $\Delta x \le a$. From the uncertainty principle,

$\Delta p_x \Delta x \ge \frac{\hbar}{2}$ so that $\Delta p_x \ge \frac{\hbar}{2a}$.

b)

$\Delta p \, \Delta x \ge \frac{\hbar}{2}$, therefore, $\frac{a}{2}\frac{p}{2} \ge \frac{\hbar}{2}$ or $p \ge \frac{2\hbar}{a} = \frac{h}{\pi a}$

$$E = \frac{p^2}{2m} \ge \frac{h^2}{2\pi^2 m a^2}$$

This energy is smaller than the true ground state energy by the factor $\frac{2\pi^2}{8} = 2.48$.

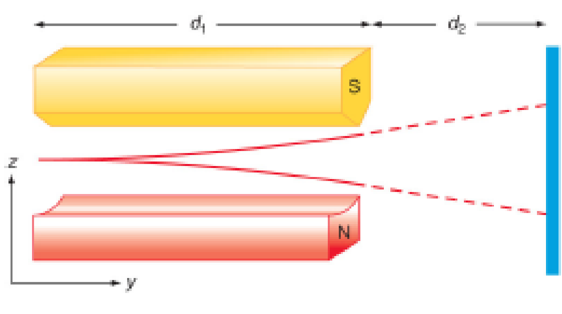
c)

From Equation (17.12), $\frac{\sigma_p}{p} = \frac{\pi\hbar/a}{\sqrt{2mE}} = \frac{\pi\hbar/a}{\sqrt{2m \times \frac{h^2}{8ma^2}}} = \frac{h/2a}{h/2a} = 1.$ $\quad \sigma_x = \sqrt{a^2\left(\frac{1}{3} - \frac{1}{2\pi^2}\right) - \frac{a^2}{4}} = 0.181a.$

Our estimate of the uncertainty in p was low and our estimate of the uncertainty in x was high.

P17.18) In this problem, you will carry out the calculations that describe the Stern-Gerlach experiment shown in Figure 17.2. Classically, a magnetic dipole $\boldsymbol{\mu}$ has the potential energy $E = -\boldsymbol{\mu} \cdot \mathbf{B}$. If the field has a gradient in the z direction, the magnetic moment will experience a force, leading it to be deflected in the z direction. Because classically $\boldsymbol{\mu}$ can take on any value in the range $-|\mu| \le \mu_z \le |\mu|$, a continuous range of positive and negative z deflections of a beam along the y direction will be observed. From a quantum mechanical perspective, the forces are the same as in the classical picture, but μ_z can only take on a discrete set of values. Therefore, the incident beam will be split into a discrete set of beams that have different deflections in the z direction.

a. The geometry of the experiment is shown here. In the region of the magnet indicated by d_1, the Ag atom experiences a constant force. It continues its motion in the force-free region indicated by d_2.



If the force inside the magnet is F_z, show that $|z| = 1/2\,(F_z/m_{Ag})\,t_1^2 + t_2 v_z(t_1)$. The times t_1 and t_2 correspond to the regions d_1 and d_2.

b. Show that assuming a small deflection,

$$|z| = F_z\left(\frac{d_1 d_2 + \frac{1}{2}d_1^2}{m_{Ag}\mathrm{v}_y^2}\right)$$

c. The magnetic moment of the electron is given by $|\mu| = g_S \mu_B / 2$. In this equation, μ_B is the Bohr magneton and has the value 9.274×10^{-24} J/T. The gyromagnetic ratio of the electron g_S has the value 2.00231. If $\partial B_z / \partial z = 1000.$ T m^{-1}, and d_1 and d_2 are 0.200 and 0.250 m, respectively, and $\mathrm{v}_y = 500.$ m s^{-1}, what values of z will be observed?

$$F_z = \frac{\partial B_z}{\partial z} \times \frac{g_S \mu_B}{2} = 1000.\,\text{tesla cm}^{-1} \times \frac{2.00231 \times 9.274 \times 10^{-24}\ \text{J tesla}^{-1}}{2} = 9.29 \times 10^{-21}\ \text{N}$$

$$|z| = F_z\left(\frac{d_1 d_2 + \frac{1}{2}d_1^2}{m_{Ag}\mathrm{v}_y^2}\right)$$

$$|z| = 9.29 \times 10^{-21}\text{N} \times \left(\frac{0.200\,\text{m} \times 0.250\,\text{m} + \frac{1}{2}(0.200\,\text{m})^2}{107.9\,\text{amu} \times 1.661 \times 10^{-27}\ \text{kg amu}^{-1} \times (500.\text{m s}^{-1})^2}\right) = 1.45 \times 10^{-2}\ \text{m}$$

$$z = \pm 1.45 \times 10^{-2}\ \text{m}$$

P17.19) Evaluate the commutator $[(d/dx) - x, (d/dx) + x]$ by applying the operators to an arbitrary function $f(x)$.

$$\left[\frac{d}{dx} - x, \frac{d}{dx} + x\right] f(x) = \left(\frac{d}{dx} - x\right)\left(\frac{d}{dx} + x\right) f(x) - \left(\frac{d}{dx} + x\right)\left(\frac{d}{dx} - x\right) f(x)$$

$$= \left(\frac{d}{dx} - x\right)\left(\frac{df(x)}{dx} + xf(x)\right) - \left(\frac{d}{dx} + x\right)\left(\frac{df(x)}{dx} - xf(x)\right)$$

$$= \frac{d^2 f(x)}{dx^2} + f(x) + x\frac{df(x)}{dx} - x\frac{df(x)}{dx} - x^2 f(x) - \frac{d^2 f(x)}{dx^2} + f(x) + x\frac{df(x)}{dx}$$

$$-x\frac{df(x)}{dx} + x^2 f(x)$$

$= 2f(x)$ Therefore,

$$\left[\frac{d}{dx} - x, \frac{d}{dx} + x\right] = 2$$

P17.21) What is wrong with the following argument? We know that the functions $\psi_n(x) = \sqrt{2/a}\sin(n\pi x/a)$ are eigenfunctions of the total energy operator for the particle in the infinitely deep box. We also know that in the box, $E = \frac{p_x^2}{2m} + V(x) = \frac{p_x^2}{2m}$ Therefore, the operator for E_{total} is proportional to the operator for p_x^2. Because the operators for p_x^2 and p_x commute as you demonstrated in Problem P17.9, the functions $\psi_n(x) = \sqrt{2/a}\sin(n\pi x/a)$ are eigenfunctions of both the total energy and momentum operators.

As shown in Example Problem 17.1, the momentum and total energy operators only commute if $\frac{dV(x)}{dx} = 0$ over the whole range of x for the system of interest. However, $\frac{dV(x)}{dx} \neq 0$ at both ends of the box. Therefore, the total energy and momentum operators do not commute for this potential. Therefore the total energy eigenfunctions are not also eigenfunctions of the momentum operator.

P17.22) For linear operators A, B, and C, show that $[\hat{A},\hat{B}\hat{C}] = [\hat{A},\hat{B}]\hat{C} + \hat{B}[\hat{A},\hat{C}]$.

$$\left[\hat{A},\hat{B}\right]\hat{C}\,f(x) + \hat{B}\left[\hat{A},\hat{C}\right]f(x) = \hat{A}\,\hat{B}\,\hat{C}\,f(x) - \hat{B}\,\hat{A}\,\hat{C}\,f(x) + \hat{B}\,\hat{A}\,\hat{C}\,f(x) - \hat{B}\,\hat{C}\,\hat{A}\,f(x)$$
$$= \hat{A}\,\hat{B}\,\hat{C}\,f(x) - \hat{B}\,\hat{C}\,\hat{A}\,f(x) = \left[\hat{A},\hat{B}\,\hat{C}\right]f(x)$$