Last time we considered (a special case of) the rotation map

\[ R_\theta : \mathbb{R}^2 \to \mathbb{R}^2 \]

which rotates the plane counterclockwise about the origin by angle \( \theta \). Today we explored two questions:

1. Is \( R_\theta \) linear?
2. Can we find a nice formula for \( R_\theta(x, y) \)?

Some of you suggested a geometric approach. The main idea, captured in the illustrations below, is that \( R_\theta \) is a **rigid motion**: it doesn’t affect shapes, and more precisely, moves shapes to congruent shapes. This gives two ways to describe rotated image of points like \( x + y \) and \( \alpha x \):

**Proof of Additivity**

On one hand, \( P = R_\theta(x + y) \). On the other, \( P = R_\theta(x) + R_\theta(y) \).

**Proof of Scaling**

On one hand, \( Q = R_\theta(\alpha x) \). On the other, \( Q = \alpha R_\theta(x) \).

These pictures give an appealing way to think about the problem, not only because they’re pretty, but because they give a sense of why \( R_\theta \) must be additive and scale. However, they’re unsatisfying in a different way: it’s difficult to tell how rigorous they are. For example, do these pictures represent the general situation? The illustration of additivity does not: what if \( x \) and \( y \) are collinear with the origin? And how do we know that there aren’t some other configurations of \( x \) and \( y \) which behave differently than the above pictures suggest?

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A different approach combines geometry with algebra. Given a point \((x, y) \in \mathbb{R}^2\), write it in polar coordinates as \((r, \alpha)\), say. We can convert back and forth between rectangular and polar coordinates using the following dictionary:

\[
x = r \cos \alpha \quad y = r \sin \alpha
\]

\[
r = \sqrt{x^2 + y^2} \quad \alpha = \arctan(y/x)
\]

All these formulas can be read off from the following picture:

The point \((x, y)\) labelled in polar coordinates as \((r, \alpha)\).

The advantage of working in polar coordinates is that rotation becomes easy: we have

\[
R_\theta(x, y) = (r, \theta + \alpha)
\]

where the right hand side is in polar. Translating this back to rectangular coordinates, we find

\[
R_\theta(x, y) = \left( r \cos(\theta + \alpha), r \sin(\theta + \alpha) \right).
\]

Although technically this tells us where \(R_\theta(x, y)\) is located, it’s not a very satisfying answer because it’s not in terms of \(x\) and \(y\). This is easy to rectify using trig addition rules:

\[
R_\theta(x, y) = \left( r \cos(\alpha + \theta), r \sin(\alpha + \theta) \right)
\]

\[
= \left( r(\cos \alpha \cos \theta - \sin \alpha \sin \theta), r(\sin \alpha \cos \theta + \cos \alpha \sin \theta) \right)
\]

\[
= \left( x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta \right)
\]

This formula will allow us to easily prove the linearity of \(R_\theta\). More importantly, it will give us a hint about the structure of linear maps from \(\mathbb{R}^2 \to \mathbb{R}^2\). We’ll pick this up next lecture.