We began by proving our conjecture from last time.

**Proposition 1.** Given two linear maps $f, g : \mathbb{R}^2 \to \mathbb{R}^2$. Then

$$\det(f \circ g) = (\det f)(\det g).$$

**Proof.** Since $f$ and $g$ are linear, we can write them as matrices, say,

$$f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} \ell & m \\ n & p \end{pmatrix}$$

It follows from our earlier work that $f \circ g$ is also a linear map, and has the matrix

$$f \circ g = \begin{pmatrix} a\ell + bn & am + bp \\ c\ell + dn & cm + dp \end{pmatrix}$$

Thus we have

$$\det f = ad - bc \quad \text{det } g = \ell p - mn$$

$$\text{det } f \circ g = (a\ell + bn)(cm + dp) - (am + bp)(c\ell + dn)$$

A straightforward computation verifies the identity $\det(f \circ g) = (\det f)(\det g)$. □

Recall from last lecture that we asserted (without proof) that the quantity $\det f$ determines how $f$ scales area. Thus far we’ve rigorously proved only one special case of this: that the image under $f$ of $S$ (the unit square with lower left corner at the origin) has area $\det f$. We now use the proposition above to generalize this a bit:

**Proposition 2.** Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map. Then

$$\text{area } f(\mathcal{R}) = (\det f) \cdot \text{area } \mathcal{R}$$

where $\mathcal{R}$ denotes the $r \times s$ rectangle with lower left corner at the origin.

The proof is most easily understood via the illustration below:

**Proof of Proposition.** We first observe that $\mathcal{R}$ is the image of the unit square $S$ under a pretty simple linear map:

$$\mathcal{R} = \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} S.$$  

It follows that

$$f(\mathcal{R}) = f \circ \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} S.$$
Using Proposition 13.1 and Proposition 1, we see that

\[
\text{area } f(\mathcal{R}) = \det \begin{pmatrix} f \circ \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} \end{pmatrix} = (\det f) \cdot \det \begin{pmatrix} r & 0 \\ 0 & s \end{pmatrix} \\
= (\det f) \cdot rs = (\det f) \cdot \text{area } \mathcal{R} \]

\[\square\]

Next, we turned to a new topic:

**VECTORS**

Consider the linear map \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) defined by

\[
f(x, y) := (x - y, 2x - y).
\]

From your homework, you know that \( f \) maps line segments to line segments. For example:

Or another example:

The curious thing about the above is that if we remove the coordinate axes, both of them look like this:
Having removed the axes (and therefore any hope of labeling the endpoints), it’s a bit trickier to describe $\mathcal{L}$. Certainly, we can no longer write $\mathcal{L}$ as a set of points, since we have no way to label them! How else can we describe it? Well, we can view $\mathcal{L}$ as a path from one endpoint to the other, and describe that path. Better yet, all we have to do is to describe the relationship between the two endpoints; once we do that, it’s an easy task to connect them with a straight line segment. For example, the relationship between the endpoints of $\mathcal{L}$ can be described as follows:

\textit{Move 2 units to the right and 1 unit up.}

Note that this instruction \textit{does not depend on where we start}: pick any point $A$, follow the instructions above to arrive at a second point $B$, and now connect $A$ to $B$, and you have drawn $\mathcal{L}$!

Because it’s tiresome to write everything in words, we introduce a couple of symbols. Let $\vec{e}_1$ denote ‘move one unit to the right’ and $\vec{e}_2$ denote ‘move one unit up’. (The arrows are there to remind you that these variables aren’t numbers or points, but instead are a direction of motion.) Using this notation, we can write $\mathcal{L}$ more succinctly as

$$\mathcal{L} = 2\vec{e}_1 + \vec{e}_2.$$ 

Similarly,

$$f(\mathcal{L}) = \vec{e}_1 + 3\vec{e}_2.$$ 

Or to put it all in a single equation:

$$f(2\vec{e}_1 + \vec{e}_2) = \vec{e}_1 + 3\vec{e}_2.$$

(†)

Now, we haven’t actually proved that this is true yet – all we did was to verify this in the two cases of starting point $(0,0)$ and starting point $(-1,-1)$. By contrast, (†) asserts a relationship which holds \textit{no matter which starting point you choose}. We will pick this up next time.