1. A Comment on Notation

Thus far, we’ve been writing points in the form \((a, b)\) or \(\begin{pmatrix} a \\ b \end{pmatrix}\) and vectors in the form \(a\vec{e}_1 + b\vec{e}_2\). I’ve been insisting on this to emphasize the distinction between points and vectors. However, as we’ve seen, linear maps can’t distinguish between the two. Since this is a course on linear algebra (and therefore concerned almost exclusively with linear maps) we will use point notation to denote both points and vectors. In other words, we will write \(\begin{pmatrix} a \\ b \end{pmatrix}\) instead of \(a\vec{e}_1 + b\vec{e}_2\).

2. An Explicit Formula for Fibonacci Numbers

Recall that we are trying to find a formula for the \(n\)th Fibonacci number \(f_n\) using matrices. Our strategy (described last time) is to find a diagonal matrix which is similar to \(\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\), say,

\[
\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = P^{-1} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} P. \quad (1)
\]

We thus rephrase our goal as follows:

**Question 1.** Do there exist \(\lambda_1, \lambda_2 \in \mathbb{R}\) and an invertible matrix \(P\) such that (1) holds?

If the answer to this question is affirmative, then it would follows that

\[
\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} P = P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}
\]

whence

\[
\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} P(\vec{e}_1) = P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} (\vec{e}_1) = P(\lambda_1 \vec{e}_1) = \lambda_1 P(\vec{e}_1). \quad (2)
\]

Further, note that \(P(\vec{e}_1) \neq 0\), since we are hoping to find an invertible map \(P\). (Can you justify this sentence?) Thus, to have any hope of answering Question 1 in the affirmative, we must be able to find a number \(\lambda\) and a vector \(\vec{v}\) such that

\[
\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \vec{v} = \lambda \vec{v}. \quad (3)
\]

In other words, we wish to find some nonzero vector \(\vec{v}\) such that when we apply \(\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\) to it we get the same vector back, just stretched out by a factor of \(\lambda\). Note that no matter what \(\lambda\) and \(\vec{v}\) are, there’s one map which has the desired effect:

\[
\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \vec{v} = \lambda \vec{v}.
\]

Thus, we wish to find \(\lambda\) and \(\vec{v}\) such that

\[
\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \vec{v} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \vec{v}
\]
or equivalently
\[
\begin{pmatrix} 1 - \lambda & 1 \\ 1 & -\lambda \end{pmatrix} \vec{v} = \mathbf{0}.
\]
Recall from above that \( \vec{v} \neq \mathbf{0} \). What sort of linear map sends a nonzero vector to the zero vector? Only a singular one! (Can you explain why?) Thus, we deduce that we must have
\[
\det \left( \begin{pmatrix} 1 - \lambda & 1 \\ 1 & -\lambda \end{pmatrix} \right) = 0,
\]
or in other words,
\[
\lambda^2 - \lambda - 1 = 0.
\]
Recall that we’re searching for a number \( \lambda \) and a nonzero vector \( \vec{v} \) which satisfy the relation (3). What we’ve just proved is that if these exist, then
\[
\lambda = \frac{1 \pm \sqrt{5}}{2}
\]
Set
\[
\lambda_1 := \frac{1 + \sqrt{5}}{2} \quad \text{and} \quad \lambda_2 := \frac{1 - \sqrt{5}}{2}
\]
Returning to (3), it remains to find some nonzero vector \( \vec{v}_1 \) such that
\[
\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \vec{v}_1 = \lambda_1 \vec{v}_1.
\]
How do we construct such a vector? The most natural approach is to write \( \vec{v}_1 = \begin{pmatrix} x \\ y \end{pmatrix} \), plug it in above, and solve for \( x \) and \( y \). When we did this in class, we discovered that \( x = \lambda_1 y \) and that \( y \) satisfies \( (\lambda_2^2 - \lambda_1 - 1)y = 0 \). Note that the latter relationship holds for every \( y \)! (Why is this?) Although this looks like a failure at first glance, it’s actually a success – this tells us that we can choose \( y \) to be anything, and then set \( x = \lambda_1 y \). For example, we can take
\[
\vec{v}_1 := \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}
\]
With this choice, it is easy to verify that
\[
\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \vec{v}_1 = \lambda_1 \vec{v}_1.
\]
Going back to (2) shows that we’d like to find an invertible map \( P \) such that
\[
P(\vec{e}_1) = \vec{v}_1 = \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}
\]
The exact same arguments show that we’d like
\[
P(\vec{e}_2) = \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}
\]
Combining the two previous statements tells us how to choose \( P \):
\[
P = \begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}
\]
Now that we’ve determined \( \lambda_1, \lambda_2, \) and \( P \), it’s straightforward to verify (1).

Having done all of this, it’s not so hard to find an explicit formula for the \( n \)th Fibonacci number. Manipulating (1), we see that
\[
\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P^{-1}
\]
(make sure you can explain why!). Raising both sides to the \( n \)\(^{th} \) power gives
\[
\begin{pmatrix}
  f_{n+1} \\
  f_n
\end{pmatrix}
= \begin{pmatrix}
  1 & 1 \\
  1 & 0
\end{pmatrix}^n
= P \begin{pmatrix}
  \lambda_1^n & 0 \\
  0 & \lambda_2^n
\end{pmatrix} P^{-1}
= \begin{pmatrix}
  \lambda_1 & \lambda_2 \\
  1 & 1
\end{pmatrix} \begin{pmatrix}
  \lambda_1^n & 0 \\
  0 & \lambda_2^n
\end{pmatrix} \begin{pmatrix}
  \lambda_1 & \lambda_2 \\
  1 & 1
\end{pmatrix}^{-1}
= \begin{pmatrix}
  * & \lambda_1^n - \lambda_2^n \\
  \lambda_1 - \lambda_2 & *
\end{pmatrix}
\]

It follows that
\[
f_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}
\]
where \( \lambda_1, \lambda_2 \) are the two solutions to the equation \( \lambda^2 - \lambda - 1 = 0 \). We’ve discovered our formula!

3. Spectral Theory

The key to figuring out the formula for \( f_n \) above was finding numbers \( \lambda_1 \) and \( \lambda_2 \), along with an invertible matrix \( P \), such that
\[
\begin{pmatrix}
  1 & 1 \\
  1 & 0
\end{pmatrix}
= P \begin{pmatrix}
  \lambda_1 & 0 \\
  0 & \lambda_2
\end{pmatrix} P^{-1}
\]

In addition to being useful for calculating powers of \( \begin{pmatrix}
  1 & 1 \\
  1 & 0
\end{pmatrix} \), this gives us a nice geometric interpretation of the action of \( \begin{pmatrix}
  1 & 1 \\
  1 & 0
\end{pmatrix} \) on the plane. Recall from above that
\[
\begin{pmatrix}
  1 & 1 \\
  1 & 0
\end{pmatrix} \vec{v}_j = \lambda_j \vec{v}_j
\]
where \( P(\vec{e}_1) = \vec{v}_1 \) and \( P(\vec{e}_2) = \vec{v}_2 \). In other words, \( P \) is the change-of-basis map from \( \vec{e}_1, \vec{e}_2 \) to \( \vec{v}_1, \vec{v}_2 \), and if we replace the usual coordinate system (generated by \( \vec{e}_1 \) and \( \vec{e}_2 \)) by the one generated by \( \vec{v}_1 \) and \( \vec{v}_2 \), then \( \begin{pmatrix}
  1 & 1 \\
  1 & 0
\end{pmatrix} \) has a simple geometric description: it stretches the plane out by a factor of \( \lambda_1 \) in the \( \vec{v}_1 \) direction and by \( \lambda_2 \) in the \( \vec{v}_2 \) direction.

Let’s generalize this. Suppose \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) is a linear map. Suppose we can find numbers \( \lambda_1, \lambda_2 \) and an invertible matrix \( P \) such that
\[
f = P \begin{pmatrix}
  \lambda_1 & 0 \\
  0 & \lambda_2
\end{pmatrix} P^{-1}.
\]
This is called the **spectral decomposition of** \( f \), and it gives us a nice way to interpret \( f \). Think of \( P \) as the change-of-basis matrix from \( \vec{e}_1, \vec{e}_2 \) to some vectors \( \vec{v}_1, \vec{v}_2 \). Then it is straightforward to verify that
\[
f(\vec{v}_1) = \lambda_1 \vec{v}_1 \quad \text{and} \quad f(\vec{v}_2) = \lambda_2 \vec{v}_2
\]

Now label each point of the plane in terms of how to get there using \( \vec{v}_1 \) and \( \vec{v}_2 \). For example:
Once we adopt this perspective, it’s very easy to describe what \( f \) does to any point: it stretches the first coordinate by \( \lambda_1 \) and the second by \( \lambda_2 \). For example, where does \( f \) send the point \((3,-2)\) indicated above? (Note: this isn’t the usual \((3,-2)\); it’s \(3\vec{v}_1 - 2\vec{v}_2\).) Easy: \( f(3,-2) = (3\lambda_1, -2\lambda_2) \). The quantities \( \lambda_j \) and \( \vec{v}_j \) play a pivotal role in understanding the spectral decomposition, so they get a special name.

**Definition.** Given a linear map \( f : \mathbb{R}^2 \to \mathbb{R}^2 \). We say a number \( \lambda \) is an *eigenvalue* of \( f \) if and only if there exists a nonzero vector \( \vec{v} \) such that \( f(\vec{v}) = \lambda \vec{v} \).

In this case, we say \( \vec{v} \) is an *eigenvector* corresponding to the eigenvalue \( \lambda \).

**Example 1.** We discovered above that the map \(
\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}
\) has eigenvalues \( \lambda_1 = \frac{1 + \sqrt{5}}{2} \) and \( \lambda_2 = \frac{1 - \sqrt{5}}{2} \) and that their corresponding eigenvectors are \( \vec{v}_1 = \left( \frac{\lambda_1}{1} \right) \) and \( \vec{v}_2 = \left( \frac{\lambda_2}{1} \right) \).

One immediate remark is that eigenvectors aren’t uniquely determined.

**Proposition 1.** Suppose \( f \) has eigenvalue \( \lambda \) with corresponding eigenvector \( \vec{v} \). Then \( \alpha \vec{v} \) is an eigenvector corresponding to \( \lambda \) for every \( \alpha \neq 0 \).

### 3.1. Finding the spectral decomposition.

We now generalize the process used to analyze \( \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \) to determine the spectral decomposition of an arbitrary linear map \( f : \mathbb{R}^2 \to \mathbb{R}^2 \). We break the process into a few steps.

**STEP 1.** Solve the equation \( \det(f - \lambda I) = 0 \) for \( \lambda \), where \( I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \) is the identity matrix. The eigenvalues of \( f \) are the solutions to this equation.

*Why does this work?* Suppose \( \lambda \) is an eigenvalue of \( f \). Then by definition, there exists a nonzero \( \vec{v} \) such that \( f(\vec{v}) = \lambda \vec{v} \). Just as in the Fibonacci example, this happens iff \( \exists \vec{v} \neq 0 \) such that \( (f - \lambda I)\vec{v} = 0 \). But this is occurs iff the map \( f - \lambda I \) is singular.

**STEP 2.** Solve the equation \( f(x,1) = (\lambda x, \lambda) \) for \( x \). Then \( \vec{v} := \begin{pmatrix} x \\ 1 \end{pmatrix} \) is an eigenvector corresponding to \( \lambda \).

*Why does this work?* We found an eigenvalue \( \lambda \) above, and we wish to find a corresponding eigenvector \( \vec{v} \). Since any rescaling of \( \vec{v} \) remains an eigenvector, we may as well rescale in such a way that \( \vec{v} = \begin{pmatrix} x \\ 1 \end{pmatrix} \). Now by definition, \( \vec{v} \) must satisfy the equation \( f(\vec{v}) = \lambda \vec{v} \).

**STEP 3.** Say the two eigenvalues of \( f \) are \( \lambda_1 \) and \( \lambda_2 \), with corresponding eigenvectors \( \vec{v}_1 = \begin{pmatrix} a \\ c \end{pmatrix} \) and \( \vec{v}_2 = \begin{pmatrix} b \\ d \end{pmatrix} \).

Let \( P := \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). Then \( f = P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P^{-1} \).

*Why is this? A good exercise!*

Before exploring specific examples, let’s try to predict what sorts of problems might arise in the steps above. In Step 1, we might only find a single eigenvalue; when this happens, this usually bodes ill for the spectral decomposition, as we shall see below. In Step 2, it’s possible that \( \vec{v}_1 \) is an eigenvector, in which case no renormalization would make it possess the form we described. Finally, in Step 3, we need to worry about the
possibility that \( P \) is not invertible. These fears are all justified, and some of these problems are fatal to the process. In fact, as we shall see, not every linear map admits a spectral decomposition. By contrast, every linear map admits a Singular Value Decomposition. (On the other hand, when a map does admit a spectral decomposition, it’s much easier to find than the SVD.)

Let’s explore a few representative examples.

\textbf{Ex. 1.} \( f = \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} \)

To find the eigenvalues, we first solve the equation

\[ \det(f - \lambda I) = 0 \]

for \( \lambda \). The LHS is

\[ \det \left( \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right) = \det \begin{pmatrix} 1 - \lambda & 3 \\ 5 & 3 - \lambda \end{pmatrix} = (1 - \lambda)(3 - \lambda) - 15. \]

Expanding this, setting equal to zero, and solving yields \( \lambda = -2, 6 \). Let’s set \( \lambda_1 := -2 \) and \( \lambda_2 := 6 \).

These are the eigenvalues.

Next, we find corresponding eigenvectors. We first solve the equation

\[ \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ 1 \end{pmatrix} = -2 \begin{pmatrix} x_1 \\ 1 \end{pmatrix} \]

From this we easily deduce that \( x = -1 \), whence our first eigenvector is \( \vec{v}_1 := \begin{pmatrix} -1 \\ 1 \end{pmatrix} \).

Similarly, solving

\[ \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ 1 \end{pmatrix} = 6 \begin{pmatrix} x_1 \\ 1 \end{pmatrix} \]

yields \( x = 3/5 \), whence \( \vec{v}_2 := \begin{pmatrix} 3/5 \\ 1 \end{pmatrix} \). If we wish, we can make this look nicer by rescaling it to \( \vec{v}_2 := \begin{pmatrix} 3/5 \\ 1 \end{pmatrix} \).

The final step of the process is to determine the change of basis map \( P \):\[
\begin{pmatrix} -1 & 3 \\ 1 & 5 \end{pmatrix}
\]

Thus, our spectral decomposition is

\[ \begin{pmatrix} 1 & 3 \\ 5 & 3 \end{pmatrix} = P \begin{pmatrix} -2 & 0 \\ 0 & 6 \end{pmatrix} P^{-1} \]

\textbf{Ex. 2.} \( R_{\pi/2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \)

As before, we begin by finding the eigenvalues of \( R_{\pi/2} \) via the equation

\[ \det(R_{\pi/2} - \lambda I) = 0. \]

This equation can be rewritten as

\[ \lambda^2 + 1 = 0, \]
so the eigenvalues of $R_{\pi/2}$ are $\lambda_1 = i$ and $\lambda_2 = -i$. Next, we find the corresponding eigenvectors. Write $\vec{v}_1 = \begin{pmatrix} x \\ 1 \end{pmatrix}$; we’re supposed to solve
\[
\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = i \begin{pmatrix} x \\ 1 \end{pmatrix},
\]
which immediately yields $x = i$. It follows that $\vec{v}_1 = \begin{pmatrix} i \\ 1 \end{pmatrix}$. A similar argument shows that $\vec{v}_2 = \begin{pmatrix} -i \\ 1 \end{pmatrix}$. Finally, let $P$ be the change-of-basis
\[
P = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}.
\]
Then we have the spectral decomposition
\[
R_{\pi/2} = P \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} P^{-1}.
\]
Note that we’ve successfully found a spectral decomposition of $R_{\pi/2}$, but only if we allow ourselves to use imaginary numbers. This is a bit odd, since the original function $R_{\pi/2}$ has nothing to do with imaginary numbers! This hints at a connection between rotations in $\mathbb{R}^2$ and complex numbers. On the other hand, a bit more thought shows that it’s not unreasonable that the spectral decomposition of a rotation should be unusual, since a rotation doesn’t stretch the plane in any direction.

**Ex. 3.** $g = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$

Following the above procedure, we find that the only eigenvalue of $g$ is $\lambda = 2$. Continuing along shows that the only eigenvectors of $g$ are scalar multiples of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. It follows that any change-of-basis matrix $P$ would not be invertible, which means that $g$ has no spectral decomposition. An alternative way to express this is that $g$ is not diagonalizable.

Note that we could have seen that $g$ wasn’t diagonalizable without solving for the eigenvalues. For, suppose $g$ did have a spectral decomposition. Since we know the only eigenvalue is 2, we would be able to write
\[
g = P \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} P^{-1}
\]
for some matrix $P$. But this would immediately imply $g = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, which isn’t the case! Put differently, we’ve just shown that the only diagonalizable matrix with both eigenvalues equal to 2 is $2I$. 