M2–1 Consider the sequence 1, 2, 5, 12, 29, ... where \( g_1 := 1 \), \( g_2 := 2 \), and \( g_{n+1} := 2g_n + g_{n-1} \) for all \( n \geq 2 \). The goal of this exercise is to adapt the method we used to find an explicit formula for the Fibonacci numbers to this sequence.

(a) Recall that \( \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \) generates the Fibonacci numbers. Find a matrix which generates the sequence \( g_n \). Prove that your matrix does so.

Claim. \( \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} g_{n+1} & g_n \\ g_n & g_{n-1} \end{pmatrix} \) for all \( n \geq 2 \).

**Proof.** We proceed by induction. Suppose that the claim holds true for some integer \( n \geq 2 \). I claim that the claim must continue to hold for \( n + 1 \). Indeed, we have

\[
\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^{n+1} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} g_{n+1} & g_n \\ g_n & g_{n-1} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2g_{n+1} + g_n & g_{n+1} \\ 2g_n + g_{n-1} & g_n \end{pmatrix}
\]

Using the recursive definition of \( g_n \), we deduce that

\[
\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^n = \begin{pmatrix} g_{n+1} & g_n \\ g_n & g_{n-1} \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}^{n+1} = \begin{pmatrix} g_{n+2} & g_{n+1} \\ g_{n+1} & g_n \end{pmatrix}
\]

A quick calculation shows that the claim holds for \( n = 2 \). The relation (*) then implies that the claim also holds for \( n = 3 \); applying (*) again implies the claim for \( n = 4 \); etc. Thus, the claim must hold for all \( n \geq 2 \). \( \square \)
(b) Use the matrix you found in (a) and the method from class to determine an explicit (i.e., non-recursive) formula for $g_n$. [If you are unable to solve part (a), this part of the problem will not be possible. In this case, instead determine a formula for the top-left entry of $\begin{pmatrix} 15 & 4 \\ 4 & 0 \end{pmatrix}^n$]

First we find the eigenvalues by solving the equation

$$\det \left( \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right) = 0.$$ 

Simplifying the left hand side yields $\lambda^2 - 2\lambda - 1 = 0$; applying the quadratic equation gives the eigenvalues:

$$\lambda_1 = 1 + \sqrt{2}, \quad \lambda_2 = 1 - \sqrt{2}.$$ 

Solving the equation

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda_1 \begin{pmatrix} x \\ y \end{pmatrix}$$

gives $x = \lambda_1 y$. Taking $y = 1$, we find an eigenvector corresponding to $\lambda_1$:

$$\vec{v}_1 = \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}.$$ 

Similarly, we find that the eigenvector corresponding to $\lambda_2$ is

$$\vec{v}_2 = \begin{pmatrix} \lambda_2 \\ 1 \end{pmatrix}.$$ 

Letting $P$ denote the change of basis matrix $\begin{pmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{pmatrix}$, we determine the spectral decomposition of $\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$:

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} = P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P^{-1}.$$ 

Some computation shows that

$$P^{-1} = \frac{1}{\lambda_1 - \lambda_2} \begin{pmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{pmatrix}.$$ 

We conclude that

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} = P \begin{pmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{pmatrix} P^{-1};$$

multiplying all three matrices together, we can determine all four entries of the resulting matrix. On the other hand, by (a) we know that the bottom left corner is $g_n$! Thus,

$$g_n = \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2} = \frac{(1 + \sqrt{2})^n - (1 - \sqrt{2})^n}{2\sqrt{2}}.$$
Given a linear map \( f : \mathbb{R}^2 \to \mathbb{R}^2 \), say with matrix \( f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). Define the function \( f^t : \mathbb{R}^2 \to \mathbb{R}^2 \) (called the \textit{transpose} of \( f \)) to be the linear map corresponding to the matrix \( \begin{pmatrix} a & c \\ b & d \end{pmatrix} \). For example, if \( f = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \), then \( f^t = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} \).

(a) Prove that for any two linear maps \( f, g : \mathbb{R}^2 \to \mathbb{R}^2 \), we have \((f \circ g)^t = g^t \circ f^t\).

\textit{Proof.} Since \( f, g \) are linear, we can write
\[
f = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} p & q \\ r & s \end{pmatrix}.
\]
Then
\[
(f \circ g)^t = \begin{pmatrix} ap + br & aq + bs \\ cp + dr & cq + ds \end{pmatrix}^t = \begin{pmatrix} ap + br & cp + dr \\ aq + bs & cq + ds \end{pmatrix} = \begin{pmatrix} p & r \\ q & s \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = g^t \circ f^t.
\]
\(\square\)

(b) Prove that \( R^{-1}_\theta = R^t_\theta \) for any \( \theta \).

\textit{Proof.}
\[
R^{-1}_\theta = \frac{1}{\cos^2 \theta + \sin^2 \theta} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}^t = R^t_\theta.
\]
\(\square\)

(c) Is it true that \( f^t = f^{-1} \) for all linear maps \( f : \mathbb{R}^2 \to \mathbb{R}^2 \)? If yes, prove it. If not, find a counterexample.

No, this is false. For example,
\[
\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix} \neq \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^t
\]

(d) Suppose the singular value decomposition of \( f \) is
\[
f = R_\alpha \begin{pmatrix} k \\ 0 \end{pmatrix} R_\beta.
\]
What is the singular value decomposition of the function \( f \circ f^t \)?

From part (a) of this question (as well as associativity of function composition), we see that
\[
\left( R_\alpha \begin{pmatrix} k \\ 0 \end{pmatrix} R_\beta \right)^t = R_\beta^t \circ \left( R_\alpha \begin{pmatrix} k \\ 0 \end{pmatrix} \right)^t = R_\beta^t \circ R_\alpha^t \circ \left( \begin{pmatrix} k \\ 0 \end{pmatrix} \right)^t.
\]

Applying part (b) to simplify this, we deduce
\[
f \circ f^t = R_\alpha \begin{pmatrix} k \\ 0 \end{pmatrix} R_\beta R_\beta^{-1} \begin{pmatrix} k \\ 0 \end{pmatrix} R_\alpha^{-1} = R_\alpha \begin{pmatrix} k^2 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & \ell^2 \end{pmatrix} R_{-\alpha}.
\]
(e) What’s the relationship between the singular values of \( f \) and its eigenvalues? Be as precise as you can.

This is an open-ended question, and there were a number of nice observations people made. Here are the three most common ones.

**Proposition 1.** The eigenvalues of \( f \circ f^t \) are the squares of the singular values of \( f \).

**Proof.** In part (d) of this problem we proved that if the SVD of \( f \) is 
\[
f = R_\alpha \begin{pmatrix} k & 0 \\ 0 & \ell \end{pmatrix} R_\beta,
\]
then
\[
f \circ f^t = R_\alpha \begin{pmatrix} k^2 & 0 \\ 0 & \ell^2 \end{pmatrix} R_\alpha^{-1}.
\]
Problem 6.3(a) implies that \( k^2 \) and \( \ell^2 \) are eigenvalues of \( f \circ f^t \). \( \square \)

Note that this result is useful for determining the singular values of \( f \): instead of finding them directly, we can first find the eigenvalues of \( f \circ f^t \) and then use this proposition.

A different observation was about the magnitudes of the eigenvalues and singular values:

**Proposition 2.** Suppose \( k \) and \( \ell \) are the singular values of \( f \), say with \( k \geq \ell \geq 0 \). Then \( \ell \leq |\lambda| \leq k \) for any eigenvalue \( \lambda \) of \( f \).

**Proof.** If \( \lambda \) is an eigenvalue of \( f \), then by definition there exists some eigenvector \( \vec{u}_0 \neq \mathbf{0} \) such that
\[
f(\vec{u}_0) = \lambda \vec{u}_0.
\]
Since any rescaling of an eigenvector is still an eigenvector, we may assume that \( \vec{u}_0 \) is a unit vector. In particular, \( |f(\vec{u}_0)| = |\lambda| \). Since \( k \) is the length of the major axis and \( \ell \) the length of the minor radius of the ellipse \( f(U) \), we see that
\[
k := \max_{|\vec{u}|=1} |f(\vec{u})| \quad \text{and} \quad \ell := \min_{|\vec{u}|=1} |f(\vec{u})|.
\]
This immediately implies the claim. \( \square \)

It turns out that the products of the eigenvalues and the singular values are related.

**Proposition 3.** Given a diagonalizable \( f \), let \( k, \ell \) be its singular values and \( \lambda_1, \lambda_2 \) be its eigenvalues. (If \( f \) only has a single eigenvalue, count it twice by setting \( \lambda_1 = \lambda_2 \).) Then \( k\ell = \lambda_1 \lambda_2 \).

**Proof.** Since \( f \) is diagonalizable, we can write
\[
f = P \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} P^{-1}
\]
for some invertible linear map \( P \). On the other hand, the SVD of \( f \) is
\[
f = R_\alpha \begin{pmatrix} k & 0 \\ 0 & \ell \end{pmatrix} R_\beta.
\]
It follows that
\[
\lambda_1 \lambda_2 = \det f = k \ell
\]
since \( \det P^{-1} = \frac{1}{\det P} \) and \( \det R_\theta = 1 \). \( \square \)