Midterm I - Answers

Econ 253 Spring 2001

You have 60 minutes to answer the following 6 questions. Each question is worth 20 points, so you should plan to spend an equal amount of time on each one. You may use a calculator and a single-sided sheet of paper with notes and formulas. Any collaboration is considered a violation of the honor code, as is discussing the exam with students in other sections who have not yet taken the exam. Good luck!

1. The following table describes the joint probability distribution of stock price/earnings ratios (PE) and their annual returns (R).

		R		
		0%	1%	5%
	1	0	0.1	0.1
PE	10	0.1	0	0.2
	100	0.1	0.4	0

Answer the following questions:

(a) What is the expected value of stock returns?

Answer: First we must calculate marginal probabilities:

		R			
		0%	1%	5%	
	1	0	0.1	0.1	0.2
PE	10	0.1	0	0.2	0.3
	100	0.1	0.4	0	0.5
		0.2	0.5	0.3	

The expected value of returns is $E(R) = \sum_i R_i f(R_i) = 0 \cdot 0.2 + 1 \cdot 0.5 + 5 \cdot 0.3 = 2\%$

(b) What is the expected value of stock returns given that the PE ratio is equal to 100?

Answer: First we have to calculate the conditional pdf: $P(R = 0|PE = 100) = \frac{P(R=1,PE=100)}{P(PE=100)} = 0.2$

$P(R = 1|PE = 100) = 0.8$

$P(R = 5|PE = 100) = 0.$

The expected return conditional on $PE = 100$ is $E(R|PE = 100) = \sum_i R_i f(R_i|PE = 100) = 0 \cdot 0.2 + 1 \cdot 0.8 + 5 \cdot 0 = 0.8\%$.

2. You need 18 computer memory chips to install in the motherboard of a PC. You order 20 memory chips because you know that 10% of all chips are defective.

   (a) What is the probability that all chips you ordered will work?

   Answer: The number of chips that will work is the binomial random variable with a probability of success 0.9. There is only one combination in which all 20 out of 20 chips will work $(\frac{20!}{20!0!})$. Thus, the probability that all 20 chips will work is $0.9^{20} = 0.12$.

   (b) What is the probability that exactly 18 chips will work?

   Answer: There are 190 combinations in which 18 out of 20 chips will work $(\frac{20!}{18!2!})$. The probability that any 18 chips will work and 2 won't is $0.9^{18}0.1^2$. Thus, the probability that exactly 18 chips will work is $190 * 0.9^{18}0.1^2 = 0.29$.

   (c) What is the probability that you will be able to make your computer work?

   Answer: You will be able to make your computer work if at least 18 chips work. That is if 18, 19 or 20 chips work. Events that 18, 19 and 20 chips work are mutually exclusive, hence the probability that you will be able to put your computer together is $P(X = 18) + P(X = 19) + P(X = 20) = 0.12 + 0.27 + 0.29 = 0.68$, since $P(X = 19) = 20 \cdot 0.9^{19}0.1 = 0.27$.

3. A random sample of 500 owners of single-family homes is drawn from the population of a city. Let the random variable X denote annual household income in thousands of dollars, and the random variable Y denote the value of a house, also in thousands of dollars. The following information is available:

$$n = 500 \qquad \sum_{i=1}^{n} x_i = 24,838 \qquad \sum_{i=1}^{n} = 107,226$$
$$\sum (x_i - \overline{x})^2 = 66,398 \qquad \sum (y_i - \overline{y})^2 = 1,398,308$$
$$\sum (x_i - \overline{x})(y_i - \overline{y}) = 194,293$$

Answer the following questions:

   (a) Compute the sample mean and standard deviation of the value of the houses in this sample. Do the same for household income.

   Answer: $\overline{x} = 49.68$, s_x=11.53, $\overline{y} = 214.45$, $s_y = 52.94$.

(b) Compute the correlation between income and house value.

Answer: The sample covariance is 389.36, therefore the correlation coefficient is 0.64.

(c) Construct a 95% confidence interval for the mean value of the houses. What assumptions do you have to make to do this?

Answer: If a house value is normally distributed with mean μ, then $\frac{\overline{y}-\mu}{s_y/\sqrt{n}}$ is distributed t_{499}. The 95% confidence interval is $\overline{y} - 1.96s/\sqrt{n}$, $\overline{y} + 1.96s/\sqrt{n}$, which is 209.81, 219.09.

4. Daily sales in a convenience store are normally independently distributed with an unknown mean μ and a variance σ^2. In January the store was open 28 days and average daily sales were $\overline{x}_J = \frac{1}{28}\sum_{i=1}^{28} x_i$. In February the store was open only 20 days and average daily sales were $\overline{x}_F = \frac{1}{20}\sum_{i=29}^{48} x_i$. Consider two estimators of the population mean μ:

$$\begin{aligned} \overline{x} &= \frac{1}{48}\sum_{i=1}^{48} x_i \\ \widetilde{x} &= \frac{1}{2}(\overline{x}_J + \overline{x}_F) \end{aligned}$$

Thus, $\overline{x}$ is the sample mean over all observations, while $\widetilde{x}$ is the average of the two averages. Answer the following questions:

(a) Are both estimators unbiased?

Answer: Sample mean $\overline{x}$ is an unbiased estimator of the population mean, $E(\overline{x}) = \mu$. $E(\widetilde{x}) = \frac{1}{2}(E(\overline{x}_J) + E(\overline{x}_F)) = \mu$. Hence, both estimators are unbiased.

(b) Which estimator is more efficient?

Answer: The variance of sample mean $\overline{x}$ is $\frac{\sigma^2}{48}$. The $var(\widetilde{x})$ is $\frac{1}{4}(var(\overline{x}_J) + var(\overline{x}_F))$ since $\overline{x}_J$ and $\overline{x}_F$ are independent. $var(\overline{x}_J) = \frac{\sigma^2}{28}$ and $var(\overline{x}_F) = \frac{\sigma^2}{20}$. Hence, $var(\widetilde{x}) = \frac{1}{4}\left(\frac{\sigma^2}{28} + \frac{\sigma^2}{20}\right) = \frac{(28+20)\sigma^2}{4\cdot 20\cdot 28} = \frac{3\sigma^2}{140}$ which is greater than $\frac{\sigma^2}{48}$. Thus, $\overline{x}$ is more efficient than $\widetilde{x}$.

5. A credit card company surveyed 225 of its customers and found that their average income is \$35,000 with a sample standard deviation of \$5,102.

(a) Construct a 95% confidence interval for population mean of income.

Answer: The confidence interval is $(\overline{x} - t_{(0.975),n-1}s/\sqrt{n}, \overline{x} + t_{(0.975),n-1}s/\sqrt{n})$. In our case, $(35,000 - 1.96 \cdot 5102/\sqrt{225}, \quad 35,000 + 1.96 \cdot 5102/\sqrt{225})$ which equals $(\$34,333, \ \$35,667)$.

(b) How many customers should the company survey so that the 95% interval is within $\pm\$100$ of the population mean?

Answer: With $n = 225$ the 95% confidence interval is too wide. The width of the confidence interval is $\pm 1.96 s/\sqrt{n}$, therefore we need n such that $1.96 \cdot 5102/\sqrt{n} = 100$. Solving this equation yields $n = 10,000$. The company has to survey at least $10,000$ customers to get the population mean with a $\pm\$100$ precision.

6. Suppose that X denotes annual incomes, in thousands of dollars, and that X is normally distributed with mean 26 and variance 36. A random sample of 25 persons is drawn from the group. Answer the following questions:

(a) What is the probability that the average income is between 25 and 29?

Answer: Let $\overline{X}$ be the average income, $\overline{X} \sim N(\mu, \sigma^2/n) = N(26, 1.44)$. We need $P(25 \leq \overline{X} \leq 29)$. If we subtract the mean and divide by the standard deviation, the resulting random variable (Z) is distributed standard normal. Therefore, the probability is equal to:

$$P\left(\frac{25-26}{1.2} \leq Z \leq \frac{29-26}{1.2}\right) = P(-0.83 \leq Z \leq 2.5)$$

Using the symmetry of the distribution, we have

$$\begin{aligned} P(-0.83 \leq Z \leq 2.5) &= P(0 \leq Z \leq 0.83) + P(0 \leq Z \leq 2.5) \\ &= 0.2967 + 0.4938 = 0.7905 \end{aligned}$$

(b) What is the probability that the sample variance will be greater than 54.62?

Answer:

$$\begin{aligned} P(s^2 > 54.62) &= \ ? \\ &= P((n-1)s^2/\sigma^2 > (n-1)54.62/\sigma^2) \\ &= P((n-1)s^2/\sigma^2 > 24 \cdot 54.62/36) \\ &= P((n-1)s^2/\sigma^2 > 36.41) \end{aligned}$$

We know that $(n-1)s^2/\sigma^2 \sim \chi^2_{24}$, finding 36.41 inside the χ^2 table for 24 degrees of freedom indicates that $P(s^2 > 54.62) = 0.05$.