

Debt Collateralization, Capital Structure, and Maximal Leverage

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Abstract

We study the effects of allowing risky debt to be used as collateral in a general equilibrium model with heterogeneous agents and collateralized financial contracts. With debt collateralization, investors switch to using exclusively high-leverage contracts for every investment they choose (issuing risky debt when possible). High-leverage positions maximize the ability of contracts to serve as collateral, expanding the set of state-contingencies created from collateralized debt. We provide conditions under which debt collateralization will increase the price of the underlying asset. Our results also apply to variations in capital structure since many capital structures implicitly provide the ability to use debt contracts as collateral.

Keywords: Leverage, incomplete markets, asset prices, default, securitized markets, asset-backed securities, collateralized debt obligations.

JEL classification: D52, D53, G11, G12.

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1 Introduction

An essential feature of many securitized markets is the explicit or implicit ability to use debt contracts as collateral to issue new financial promises. In using debt as collateral, risky assets can be tranced into securities with state contingencies quite different from the underlying asset or from simple debt and equity. Such features of securitized markets significantly contributed to the growth of the market for leveraged buyouts ([Shivdasani and Wang, 2011](#)) and subprime mortgages (via Asset-backed securities (ABS) and collateralized debt obligations (CDOs)). We argue that one reason for this expansion is that these securitized markets, by using debt as collateral to issue other promises, vastly increased the set of state-contingent payoffs available to trade. These innovations allowed investors, explicitly or implicitly, to choose leverage decisions that would maximize the ability for assets to serve as collateral for multiple levels of promises. We show that allowing debt to be used as collateral endogenously increases leverage in the economy as investors switch to issuing exclusively high-leverage risky contracts.

We use a general equilibrium model featuring heterogeneous agents and collateralized financial contracts following [Geanakoplos \(1997, 2003\)](#). Our main analysis considers the interaction of two key frictions. First, we suppose that collateral is the only means of enforcing promises, with lenders seizing collateral that has been agreed upon in advance by contract. Second, we suppose that investors are limited to making non-contingent promises, so markets are incomplete. As a result, there is a meaningful role for using debt contracts as collateral. We consider a model with multiple states of uncertainty so that in an economy with debt contracts, agents trade risky and risk-free debt in equilibrium. We then allow agents to use debt contracts as collateral to back new financial contracts, a process we call *debt collateralization*. In equilibrium agents use *risky* debt as collateral to issue new promises, which changes the state-contingent properties of risky debt.

While it is well understood that default can create state-contingent securities when incomplete markets restrict contracts to non-contingent promises ([Zame, 1993](#)), debt collateralization does not *merely* mechanically expand the set of contingencies via default. Instead, in equilibrium investors make decisions to isolate only a subset of contingent payoffs rather than capturing the full set of contingent payoffs. We show that with debt collateralization investors switch to using only the highest-leverage promises available for the assets or contracts in which they invest. Using

maximal leverage creates new securities that can be further collateralized (i.e., leveraged) by “downstream investors” to the maximal degree; investing otherwise creates securities with fewer opportunities for collateralization and also fewer opportunities to create state-contingencies. Thus, only those state-contingent payoffs that maximize further collateralization “downstream” occur in equilibrium, and payoffs created by issuing risk-free promises on “upstream” assets do not occur.

Allowing debt to back debt (to back debt, *ad infinitum*) increases collateral values, increasing leverage in each contract; each “level of debt collateralization” reinforces these effects. With complete collateralization, equilibrium features a “pyramiding arrangement” of investors lending to downstream investors by issuing promises that are used as collateral to issue further promises.¹ This arrangement can be implemented with contingent claims defined by senior-subordinated capital structures. Our results suggest that one motivating factor for senior-subordinated capital structures is to provide a way to stretch scarce collateral.

The result that agents issue contracts that maximize downstream collateral opportunities holds even when agents can trade a full set of state-contingent contracts that must be backed by collateral. In this case, collateralization is *still* important for creating contingencies because cross-netting frictions prevent an asset from serving as collateral for multiple contracts so that markets are not complete. Allowing contingent contracts to back further contingent promises extends the collateral capacity of the underlying asset and increases the degree to which the asset’s payoffs can be split. The process of collateralization in the presence of cross-netting frictions can complete markets.

We show that debt collateralization has important implications for risk premia, debt prices, and asset prices. First, increases in economy-wide leverage on the original risky asset can be driven by financial innovations in debt collateralization, and not only by changes in fundamental risk or beliefs (Fostel and Geanakoplos, 2012b; Simsek, 2013). Second, we show that the prices of risky debt always increase (risk premia decrease) because debt contracts now have collateral value. Third, debt collateralization affects asset prices through both a collateral channel, and a required return channel. When debt backed by the asset can serve as collateral, the collateral value of the asset increases, which puts upward pressure on the asset price. However, investors in the asset now

¹Nonetheless, debt collateralization does not complete markets because the set of contingencies remains limited (i.e., does not recover Arrow-Debreu securities) and the set of fundamental assets that can be used to issue contracts may remain limited. Complete markets would require contracts like credit default swaps and for all assets to serve as collateral (Fostel and Geanakoplos, 2012a) as well as sufficient supply of collateral (Gottardi and Kubler, 2015).

have the more attractive alternative of investing in debt with leverage—this required return effect exerts downward pressure on the asset price. We characterize sufficient conditions under which the collateral effect dominates and debt collateralization increases asset prices.

1.1 Related Literature

Our paper follows the model of collateral equilibrium developed in [Geanakoplos \(1997, 2003\)](#) and [Geanakoplos and Zame \(2014\)](#), and is closely related to the literature on collateral and financial innovation ([Fostel and Geanakoplos, 2008, 2012a,b, 2015, 2016](#)). This literature uses binomial models to explain asset prices and investment, and defines the financial environment as the set of assets that can serve as collateral and the set of promises that can be made with existing collateral. Debt collateralization, or “pyramiding” to use the term introduced by [Geanakoplos \(1997\)](#), expands the set of assets that can be used as collateral, fitting directly into this definition of financial environment. Our main contribution is characterizing the equilibrium pyramiding structure, together with asset pricing implications, in the model of [Fostel and Geanakoplos \(2012b\)](#).

[Geanakoplos and Zame \(2013, 2014\)](#) discuss how using promises to back further promises (what they call pyramiding and what we are calling debt collateralization given our restriction to debt) can potentially allow the market to achieve efficient allocations, though the central finding of [Geanakoplos and Zame \(2013\)](#) is that even with pyramiding, equilibrium is robustly inefficient. The central result of our analysis is that, when investors are restricted to debt contracts, the set of state-contingent payoffs that arise in equilibrium are those created when investors use maximum leverage for their investments. Thus, not all possible state-contingencies are traded, but only those that correspond to maximal leverage because these trades maximize the collateral value of all assets and derivative debt contracts.

Few papers study debt collateralization, or pyramiding, in equilibrium. [Gottardi and Kubler \(2015\)](#) implicitly assume that all financial securities serve as collateral. Provided the financial markets are sufficiently rich in terms of the specification of payoffs and of collateral requirements, any Arrow-Debreu equilibrium allocation with limited pledgeability can also be attained at a collateral-constrained financial market equilibrium and debt pyramiding can replicate tranching. In contrast to this rich environment, we focus our analysis on non-contingent debt and show that

similar results emerge when state-contingencies are created via default. [Geerolf \(2015\)](#) studies an economy with a continuum of states and a continuum of agents with differing point-beliefs about the asset payoff. A continuum of contracts are traded in equilibrium, and with pyramiding the asset price increases with each layer of pyramiding, the measure of contracts traded decreases, and the distribution of leverage changes.

While these results are closely related to ours, there are important distinctions. In [Geerolf \(2015\)](#), agents' disagreements are of the form of point-expectations about the asset's value, implying that agents trade debt they perceive to be risk-free. With pyramiding, agents switch to making larger promises, which are perceived to be risk-free by the buyers, and interest rates adjust to clear supply and demand, not to compensate for risk ("risk-free" promises are collateral for other risk-free promises). In our setting, interest rates compensate for default risk because agents use risky debt as collateral. We prove that with debt collateralization agents use maximal leverage on the assets in which they invest—agents switch to using contracts with the highest possible level of risk—and economy-wide margins decrease because the composition of leverage changes as more investors issue risky contracts. Critically, in our setting agents make larger promises because the downstream valuation of risk changes, precisely because buyers of risky debt can leverage their debt position to create objectively risk-free debt for investors who demand it. Additionally, our results apply to allowing agents to trade a full set of contingent contracts.

Several papers study collateral equilibrium with multiple states. [Simsek \(2013\)](#) uses a model with a continuum of states to study belief disagreements, and conjectures that equilibrium in multi-state models will feature a pyramiding arrangement when debt contracts can be used as collateral. We prove that this conjecture holds only when the *maximum* level of securitization has been reached. [Toda \(2015\)](#) shows that demand for safe assets, to hedge and insure idiosyncratic risks, lead investors to take maximum leverage when collateralized loans are securitized into pools of ABS, and [Phelan and Toda \(2019\)](#) study the consequences of cross-country margin heterogeneity for international capital flows and risk sharing. These papers focus on the welfare consequences of maximum leverage and securitization. [Araujo et al. \(2012\)](#) examine the effects of default and collateral on risk sharing. [Gong and Phelan \(2017\)](#) study how expanding the sets of assets that can serve as collateral affects the basis between risky bonds and credit default swaps.

Our results relate to the literature on how securitized markets create safe and liquid assets (see

Gorton and Metrick, 2009), and we show that this process increases the supply of both risky and safe debt and the overall level of leverage and volatility increase. Cao (2010, 2017) and Cao and Nie (2017) study how collateral constraints and incomplete markets affect asset price volatility and amplification (see also Brumm et al., 2015). Shen et al. (2014) propose a collateral view of financial innovation driven by the cross-netting friction. In our model, debt collateralization and innovative capital structures are ways of stretching collateral, which is similar to their insight that financial innovation is a response to scarce collateral (see also Gottardi et al., 2019, regarding collateral re-use). Dang et al. (2011) study how debt collateralization can alleviate asymmetric information problems by creating information-insensitive securities, and they show that the optimal financial instrument is debt backed by debt. Finally, Rampini and Viswanathan (2013) also argue that asset tangibility and collateral requirements determine firms capital structure, and their analysis focuses on firm decisions to lease versus buying capital, with implications for investment and risk management.

2 General Equilibrium Model with Collateral

This section presents the basic general equilibrium model with collateralized borrowing and characterizes the potential contracts traded in equilibrium in a general setting.

2.1 The Model

To simplify the analysis and the exposition, we consider a multi-state extension of Geanakoplos (2003) as found in Fostel and Geanakoplos (2012b).

Time, Assets, and Households

We consider a two-period, N -state general equilibrium model with time $t = 0, 1$. Uncertainty is represented by a tree with a node s_0 at $t = 0$ and N states $n \in \mathcal{N} = \{1, \dots, N\}$ at $t = 1$.

There are two fundamental assets, a risk-free asset X and a risky asset Y , which produce dividends of the consumption good at time 1. For a generic asset Z , let d_n^Z be the dividend of asset Z in state n . We normalize $d_n^X = 1$ for all n , and $d_n^Y = s_n$, where $s_1 < s_2 < \dots < s_N$ (states are

ordered so higher n implies higher dividend payout), and we normalize $s_N = 1$.

We suppose that agents are uniformly distributed on $\mathcal{H} = (0, 1)$, that is they are described by Lebesgue measure. (We will use the terms “agents” and “investors” interchangeably.) Agents are risk-neutral and have linear utility in consumption c at time 1. Each agent $h \in (0, 1)$ assigns subjective probability $\gamma_n(h)$ to the state n , and beliefs $\gamma_n(h)$ are continuous in h . The expected utility of agent h is

$$U^h(c) = \sum_{n=1}^N \gamma_n(h) c_n,$$

where c_n is consumption in state n . At $t = 0$, each investor is endowed with (e^X, e^Y) units of assets X and Y .

To ensure that in equilibrium investors’ positions are sorted by their level of optimism, we suppose hazard rate dominance (see also [Simsek, 2013](#); [Phelan, 2015](#)):

$$\text{For all } n \in \{1, \dots, N-1\}, \text{ the ratio } \frac{\gamma_n(h)}{\sum_{k=n}^N \gamma_k(h)} \text{ is strictly decreasing in } h. \quad (\text{A1})$$

This condition implies that $\frac{\sum_{k>n} \gamma_k(h)}{\sum_{k \geq n} \gamma_k(h)}$ is strictly increasing in h , which means more optimistic agents are increasingly optimistic about states above a threshold state n . Investors with higher h have uniformly higher marginal utility for consumption in states in which the asset payoff is higher (i.e., they are uniformly more optimistic). This setup is equivalent to a model with finitely many heterogeneous risk-averse agents, where endowments and preferences are such that marginal utilities or “hedging needs” are monotonic and uniformly increasing by state.

Financial Contracts and Collateral

The heart of our analysis involves contracts and collateral. We explicitly incorporate repayment enforceability problems, and we suppose that collateral acts as the only enforcement mechanism. Agents trade financial contracts at $t = 0$. A financial contract $j = (A^j, C^j)$, consists of a promise $A^j = (A_n^j)_{n \in \mathcal{N}}$ of payment in terms of the consumption good at $t = 1$, and an asset C^j serving as collateral backing the promise. The lender has the right to seize as much of the collateral as was promised, but no more. Therefore, upon maturity, the financial contract yields $\min\{A_n^j, d_n^{C^j}\}$ in state n . Agents must own collateral in order to make promises. Let J be the set of all possible

financial contracts. Each contract $j \in J$ trades for a price π_j .

Our analysis first considers non-contingent debt contracts (Sections 4.2 and 5 introduce contingent contracts). We introduce multiple levels of debt collateralization inductively. Level-0 debt contracts are promises using the risky asset Y as collateral. Without loss of generality we normalize the collateral to one unit of Y , and let J^0 denote the set of promises backed by one unit of Y . A promise $j_n^0 = (s_n, Y) \in J^0$, which promises to pay s_n at time 1 and uses Y as collateral, delivers $\min\{s_n, s_k\}$ in the state k . Note that $j_1^0 = (s_1, Y)$ is risk-free debt because it delivers s_1 in every state.

We allow level-0 debt contracts in J^0 to be used as collateral to issue further non-contingent promises.

Definition 1. We say the *first level of debt collateralization* is the creation of promises j_n^1 using $j_k^0 \in J^0$ as collateral. Denote the set of contracts at the first level of debt collateralization by J^1 . We write $j_n^1(j_k^0) = (s_n, j_k^0)$ to denote the *debt contract* that is traded when an agent holds j_k^0 as collateral and promises to pay s_n . We denote the act of holding j_k^0 and selling j_n^1 by j_k^0/j_n^1 .

For a contract j_k to be meaningful collateral for a promise s_n it must be that $s_k > s_n$ because otherwise the payoff to j_k would always be less than the promise (and equality would render the new promise redundant). Thus, in what follows we will only consider when agents use meaningful collateral to make new promises, restricting our attention to contracts $j_n^1(j_k)$ with $k > n$. Given this restriction, the payoffs to $j_n^1(j_k)$ are the same for every $k > n$, and so we can denote the price of a contract $j_n^1(j_k)$ by π_n^1 .

In general, level L debt collateralization is to promise a non-contingent payment using a level $L - 1$ debt as collateral.

Definition 2. We say the *L -th level of debt collateralization* is the creation of promises j_n^L using $j_k^{L-1} \in J^{L-1}$ as collateral, where $1 < n < N - L$ and $1 < k < N - L + 1$. Denote the set of contracts at the L -th level of debt collateralization by J^L . We write $j_n^L(j_k^{L-1}) = (s_n, j_k^{L-1})$ to denote the debt contract that is traded when an agent holds j_k^{L-1} as collateral and promises to pay s_n . We denote the act of holding j_k^{L-1} and selling j_n^L by j_k^{L-1}/j_n^L . Again, we must have $n < k$.

With L levels of collateralization, the set of financial contracts is given by $J = J^0 \cup J^1 \cup \dots \cup J^L$. Thus, each additional level of collateralization involves the creation of new bonds and allows all

previously existing, risky bonds to be purchased with leverage. So long as the backing collateral is meaningful, given the monotonicity of payoffs for debt contracts, the payoff of any contract is defined by the promise. Since the payoff depends on n and not on k , we use π_n^L to denote the price of any debt security $j_n^L(j_k^{L-1}) \in J^L$ with $k > n$. Note that for all k, l , the contract promising s_1 backed by j_k^{l-1} delivers s_1 in every state (it is risk-free debt).

We denote contract holdings of $j \in J$ by φ_j , where $\varphi_j > 0$ denote *sales* and $\varphi_j < 0$ denote *purchases*. The sale of a contract corresponds to borrowing the sale price and the purchase of a promise is equivalent to lending the price in return for the promise. A position of $\varphi_j > 0$ units of a contract requires ownership of φ_j units of the collateral, whereas the purchase of such contracts does not require ownership of the collateral.

The financial environment in our model (the set of contracts J) is the set of assets used as collateral or the permissible promises that can be backed by the same collateral. Debt collateralization expands the set of contracts in J . We take the financial environment as exogenous (see [Dang et al., 2011](#); [Gennaioli et al., 2013](#); [Gorton and Ordoñez, 2014](#), for informational explanations for why financial markets may *decrease* the available set of assets serving as collateral). The assumed financial structures allow us to focus on the abilities to leverage and securitize assets in the most straightforward setting without loss of generality. The cash flows produced when investors issue contracts directly against assets could also correspond to financial assets produced by financial intermediaries or to securities issued by firms as part of their capital structure.

Budget Set

Without loss of generality, we normalize the price of risk-free asset X to be 1 in all states of the world, making X the numeraire good (since there is no consumption in the initial period, the price of X is arbitrary at $t = 0$). We let p denote the price of the risky asset Y . Given asset and contract prices at time 0, each agent decides how much X and Y he holds and trades contracts j to maximize

utility, subject to the budget set

$$B^h(p, \pi) = \left\{ (x, y, \varphi, (c_n)_{n \in \mathcal{N}}) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^J \times \mathbb{R}_+^N : \right. \\ \left. (x - 1) + p(y - 1) \leq \sum_{j \in J} \varphi_j \pi_j, \right. \quad (1)$$

$$\left. \sum_{j \in J^0} \max\{0, \varphi_j\} \leq y, \right. \quad (2)$$

$$\left. \sum_{j=j'_n(j'_k)^{-1} \in J^l} \max\{0, \varphi_{j'_n(j'_k)^{-1}}\} \leq \varphi_{j'_k} \forall l \in 1, \dots, L, \forall 1 \leq n < k \leq N, \right. \quad (3)$$

$$\left. c_n = x + yd_n^Y - \sum_{j \in J} \varphi_j \min\{A_n^j, d_n^{C^j}\} \right\}. \quad (4)$$

Equation (1) states that expenditures on assets (purchased or sold) cannot be greater than the resources borrowed by selling contracts. Equation (2) is the collateral constraint for debt backed by Y , requiring that agents must hold sufficient assets to collateralize the contracts they sell. Equation (3) is the collateral constraint for contracts backed by the risky asset, and for contracts backed by debt, up to L levels which is a parameter of the financial environment. Equation (4) states that in the final states, consumption must equal dividends of the assets held minus debt repayment. Recall that a positive φ_j denotes that the agent is selling a contract or borrowing π_j , while a negative φ_j denotes that the agent is buying the contract or lending π_j . Thus there is no sign constraint on φ_j . Additionally, short selling of fundamental assets is not possible ($y \geq 0$ and $x \geq 0$).

Collateral Equilibrium

Definition 3. A *Collateral Equilibrium* in this economy is a price of risky asset Y , contract prices, asset purchases, contract trades, and consumption decisions all by agents, $((p, \pi), (x^h, y^h, \varphi^h, (c^h)_{h \in \mathcal{H}})) \in (\mathbb{R}_+ \times \mathbb{R}_+^J) \times (\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^J \times \mathbb{R}_+^N)^{\mathcal{H}}$, such that

1. $\int_0^1 x^h dh = 1$,
2. $\int_0^1 y^h dh = 1$,
3. $\int_0^1 \varphi_j^h dh = 0, \forall j \in J$,
4. $(x^h, y^h, \varphi^h, c^h) \in B^h(p, \pi), \forall h$,
5. $(x, y, \varphi, c) \in B^h(p, \pi) \Rightarrow U^h(c) \leq U^h(c^h), \forall h$.

Conditions 1 and 2 are the asset market clearing conditions for X and Y at time 0 and condition 3 is the market clearing condition for financial contracts. Condition 4 requires that all portfolio and consumption bundles satisfy agents' budget sets, and condition 5 requires that agents maximize their expected utility given their budget sets. By the same arguments made in [Geanakoplos and Zame \(2014\)](#), equilibrium in this model exists under the assumptions made thus far.

2.2 Discussion of the Financial Environment

Using level-0 contracts as collateral is meaningful whenever assets can back only a single contract at a time, even if contracts can be fully state-contingent (see Section 4.2). Our main results generalize to when agents can trade contingent contracts: if possible, agents issue only contracts that can be used as collateral further downstream. The degree to which contract collateralization is redundant or not depends on whether assets can back multiple contracts simultaneously.

Our main analysis considers when agents are restricted to non-contingent contracts (debt) in order to study the behavior of leverage and to illustrate the role of debt collateralization specifically in creating contingencies. In reality, agents may be restricted to non-contingent promises because of un-modeled informational frictions, or because markets are segmented and some investors are restricted to buying “tier-1” securities.² Leverage and debt collateralization are mechanisms that create state-contingent payoffs from underlying non-contingent contracts *without* violating the informational friction (they depend on collateral seizure and limited repayment enforceability). Financial markets can create state-contingent contracts in the presence of these informational frictions via debt collateralization.

In addition, state-contingent contracts may be available, but agents may not be able to use an asset as collateral to back multiple promises (i.e., no tranching), even when doing so would still guarantee repayment. [Geanakoplos and Zame \(2014\)](#) show that equilibrium may be endogenously incomplete when collateral is scarce (agents may trade debt contracts even when Arrow securities are available because debt contracts economize on collateral). [Shen et al. \(2014\)](#) show that financial innovations are likely to occur in such a setting. Our results apply to environments with richer

²For examples relating to securitization see [DeMarzo \(2005\)](#); [Pagano and Volpin \(2012\)](#); [Friewald et al. \(2015\)](#). [Mada and Soubra \(1991\)](#) show that nonextremal securities (debt and equity rather than “Arrow Securities”) may be optimal when securities must be marketed at a cost. [Lemmon et al. \(2014\)](#) provide evidence that one value of securitization (for nonfinancial firms) is providing access to segmented markets.

contracts and cross-netting frictions. Contracts are meaningful collateral precisely when the risky asset cannot back multiple contracts at once. (Senior-subordinated capital structures allow an asset to simultaneously back multiple state-contingent contracts.) Thus, our restriction to debt reflects some combination of informational frictions limiting state-contingencies together with some degree of cross-netting frictions.

3 A Model with Three States

We now focus on a 3-state economy in order to more carefully characterize the equilibrium and to provide intuition for the economic forces determining investors' positions. Uncertainty is represented by a tree with a root s_0 at $t = 0$ and three states of nature denoted U, M, D at time 1. With a slight abuse of notation we let M, D , be the dividends in states M, D with $D < M < 1$, and the dividend is 1 in U . Figure 1 shows asset payoffs. To simplify exposition, going forward we also set asset endowments to 1, i.e., $(e^X, e^Y) = (1, 1)$.³ Note that assumption A1 on beliefs means that $\gamma_U(h) + \gamma_M(h)$ and $\frac{\gamma_U(h)}{\gamma_U(h) + \gamma_M(h)}$ are increasing in h . High h investors believe that state D is unlikely and that, conditional on the state being at least M , state U is relatively likely.

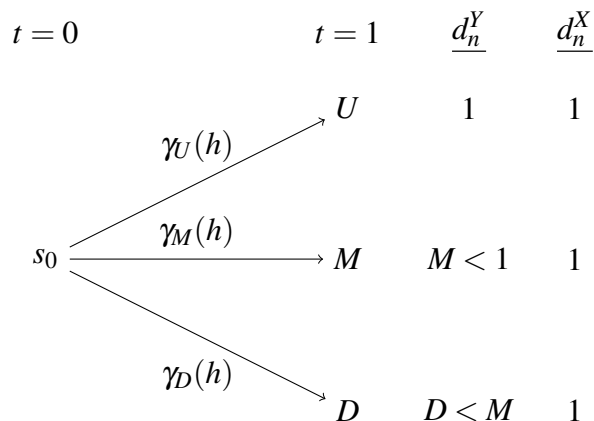


Figure 1: Payoff tree of assets X and Y in three-state world.

We characterize equilibrium with leverage only (when agents can trade debt backed by Y) and with debt collateralization (when agents can also trade debt backed by debt). In the leverage-only economy, agents can issue non-contingent promises using the asset Y as collateral. With debt

³None of our theoretical or qualitative results depend on the endowment choice.

collateralization, contracts $j^0 \in J^0$ can also serve as collateral. All proofs are in Appendix A.

3.1 Leverage-only Economy with 3 States

As shown by [Fostel and Geanakoplos \(2012b\)](#), in equilibrium with debt two contracts are traded: a risk-free promise j_D promising D and a risky promise j_M promising M , with prices π_D and π_M . The interest rate on j_D is zero ($\pi_D = D$) because it is a risk-free promise. However, the delivery of j_M depends on the realization of the state at time 1 and j_M is therefore risky; j_M pays (M, M, D) . This means that any agent issuing the promise j_M can only borrow $\pi_M < M$. Thus, the interest rate for j_M is strictly positive, defined by $i_M = \frac{M}{\pi_M} - 1$, and is endogenously determined in equilibrium. We refer to changes in the interest rate as changes in the *risk premium* for the debt contract.

In equilibrium there are three marginal investors h_M, h_D, h_J . Agents $h > h_M$ will sell their endowment of X , buy the asset Y , and promise M (issue j_M) for every unit of the asset bought.⁴ These agents receive state-contingent payoffs $(1 - M, 0, 0)$, equivalent to an Arrow U . Agents $h \in (h_D, h_M)$ will sell their endowment of X and buy the risky asset, promising D against every asset bought. These agents receive state-contingent payoffs $(1 - D, M - D, 0)$, with payoffs in U and M . Agents $h \in (h_J, h_D)$ will sell their endowment of X and Y and buy j_M (effectively lending to agents $h > h_M$). Agents $h < h_J$ will sell their endowment of Y and buy both risk-free assets X and contracts j_D backed by the risky asset (these two are equivalent). Figure 2 illustrates the equilibrium regime. It is easy to see how the assumption on beliefs implies this ordering of investors.

Agents $h > h_M$ are “maximally leveraged” in the sense that making a larger promise would simply result in a transfer of resources to lenders in U , the state in which the asset has the maximum payoff. Agents can choose to promise more to attain additional leverage—they can issue any promise j —but $j > M$ is unattractive to borrowers. Fundamentally, any contract $j > M$ has the same delivery as j_M in states M and D (because of default against the asset’s payoff) and delivers more only in state U . While U is the state that investors $h > h_M$ believe to be comparatively the most likely to happen, the larger promise in U is priced by more pessimistic agents. Hence, a promise $j > M$ would result in raising less than the value of the promise. Agents $h \in (h_D, h_M)$,

⁴Since the marginal agent has measure zero, to simplify notation we will use strict inequalities when referencing the marginal agent.

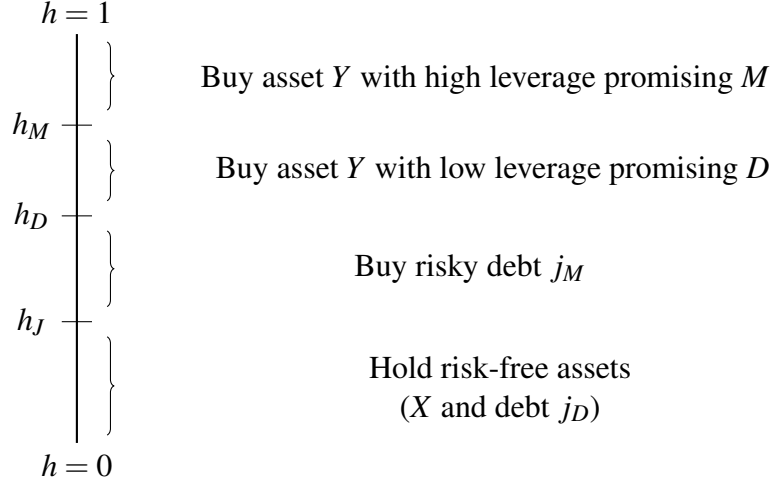


Figure 2: Equilibrium regime with leverage in static 3-state model.

promising D against each unit of the asset, are not maximally leveraged because promising M changes the delivery to borrowers in both states U and M .

Equilibrium is described by the following set of equations. Agent h_M is indifferent between buying Y with high leverage promising M , and buying asset with low leverage promising D ,

$$\frac{\gamma_U(h_M)(1-M)}{p-\pi_M} = \frac{\gamma_U(h_M)(1-D) + \gamma_M(h_M)(M-D)}{p-D}. \quad (5)$$

Agent h_D is indifferent between buying Y with leverage promising D , and holding risky debt j_M ,

$$\frac{\gamma_U(h_D)(1-D) + \gamma_M(h_D)(M-D)}{p-D} = \frac{(1-\gamma_D(h_D))M + \gamma_D(h_D)D}{\pi_M}. \quad (6)$$

Agent h_J is indifferent between holding risky debt j_M and holding risk-free assets (X or risk-free debt),

$$\frac{\gamma_U(h_J)M + \gamma_M(h_J)M + \gamma_D(h_J)D}{\pi_M} = 1. \quad (7)$$

Market clearing for the risky asset Y requires

$$(1-h_M)\frac{1+p}{p-\pi_M} + (h_M-h_D)\frac{1+p}{p-D} = 1, \quad (8)$$

and market clearing for the risky debt j_M requires

$$(1 - h_M) \frac{1 + p}{p - \pi_M} = (h_D - h_J) \frac{1 + p}{\pi_M}. \quad (9)$$

Equation (8) states that the agents buying the risky asset, $h \in (h_D, 1)$, will spend all of their endowment, $(1 + p)$, to purchase the risky asset, which costs price p , borrowing either π_M or D to leverage their purchases, and that the demand is equal to the supply of the risky asset, 1. Equation (9) states that the amount of risky debt demanded by agents $h \in (h_M, 1)$ is equal to the amount of risky debt supplied by agents $h \in (h_J, h_D)$.

3.2 Economy with Debt Collateralization

We now suppose agents can also trade contracts of the form $j_\ell^1 = (\ell, j_M)$, i.e., $C^j = j_M$. This contract specifies a non-contingent promise (ℓ, ℓ, ℓ) backed by the risky debt j_M acting as collateral. The restriction to j_M is without loss of generality.⁵ The payoff to j_ℓ^1 is $\min\{\ell, d_s^{j_M}\}$, the minimum of the promise ℓ and the payoff of the debt contract j_M . The budget set now includes the constraint $\sum_{j \in J^1} \max\{0, \varphi_j\} \leq \varphi_{j_M}$ in addition to the collateral constraint in (2). That is, they must hold sufficient positions in j_M to issue contracts backed by j_M . We denote equilibrium variables with debt collateralization by a ‘hat’ ($\hat{\cdot}$) to distinguish them from their counterparts with leverage only.

Consider how this expansion of the financial environment affects the ability to create state-contingent securities. For concreteness, let Y have payoffs $M = 0.3$ and let $D = 0.1$. Buying the risky asset with leverage and promising M splits the asset’s cash flows into risky debt and an “Arrow U .” Buying the risky asset and promising D splits the risky asset’s cash flows into risk-free debt and payoffs in U and M .

$$d^Y = \begin{pmatrix} 1 \\ 0.3 \\ 0.1 \end{pmatrix} \rightarrow \begin{pmatrix} 0.3 \\ 0.3 \\ 0.1 \end{pmatrix} + \begin{pmatrix} 0.7 \\ 0 \\ 0 \end{pmatrix}, \quad d^Y = \begin{pmatrix} 1 \\ 0.3 \\ 0.1 \end{pmatrix} \rightarrow \begin{pmatrix} 0.1 \\ 0.1 \\ 0.1 \end{pmatrix} + \begin{pmatrix} 0.9 \\ 0.2 \\ 0 \end{pmatrix}.$$

With debt collateralization, the *risky debt* can also be split into risk-free debt and payoffs in U and

⁵We could let any contract $j \in J^0$ serve as collateral; however, we show that in equilibrium only j_M will be traded and thus only j_M will serve as collateral. Making a non-contingent promised backed by j_D , which is non-contingent, is redundant, and using j_U is equivalent to using Y .

M . Note that the act of holding j_M and selling the contract j_D^1 is equivalent to buying j_M with leverage promising D , yielding a payoff of $(M - D, M - D, 0)$, i.e., $(0.2, 0.2, 0)$ in our example. Our first result is that any investor buying risky debt will choose to use leverage in this way.

Lemma 1. *Suppose that in equilibrium agents are able to collateralize debt. Then every agent holding risky debt will maximally leverage their purchases of risky debt. That is, all agents holding j_M will sell the promise $j_D^1 = (D, j_M)$.*

The intuition is straightforward. In the leverage economy, only the marginal agent investing in risky debt thinks the debt is priced to exactly compensate for risk, while every other agent thinks the expected payoff is higher than implied by the price and thus would like to leverage their investment in the debt. Since agents investing in risky debt can leverage their purchases, all else equal the demand for risky debt increases, which decreases the risk premium on the risky debt. Promising D maximally leverages the investment in j_M ; any agent that is not willing to maximally leverage their investment in j_M will be priced out by those who are.

When agents and leverage risky debt, demand for risky debt increases and increases the supply of safe assets. As a result, the marginal buyer of risky debt will be more optimistic, increasing the price of risky debt.

Proposition 1. *Suppose that in equilibrium agents are able to collateralize debt. Then, the price of risky debt increases.*

Critically, when risky debt can be used as collateral, *in equilibrium no agent chooses to leverage Y by promising risk-free debt*—no investor chooses the payoff $(0.9, 0.2, 0)$ —which is stated in the following lemmas.

Lemma 2. *Let agents be allowed to collateralize debt. Then, every agent holding the risky asset will maximally leverage their purchases of the risky asset. In other words, every agent holding the risky asset will promise M .*

The intuition for Lemma 2 is that promising M creates a debt contract that can be used as collateral, while promising D does not. Additionally, debt collateralization decreases the risk premium of risky debt, increasing the amount of leverage agents get from risky debt. As a result, it becomes more attractive for investors to use Y to issue the risky debt (which has a higher price),

rather than issuing risk-free debt, which can also be issued by owners of the risky debt. The general equilibrium consequences imply that any investor who is not willing to buy Y and promise M finds it more attractive to leverage the risky debt j_M rather than to buy Y and promise D . In other words, Y is priced so that the only efficient investment is to use a high level of leverage, and so investors who desire a low level of leverage will choose to buy a different asset. Thus, the set of state-contingent payoffs associated with buying Y with low leverage are priced so that no investor chooses those payoffs.

The key insight for our result is that the price of any asset is a sum of the payoff value and the collateral value. Allowing a debt contract to be used as collateral increases its price—it now has a collateral value—which increases the value to buying the risky asset and issuing that debt contract. Because only the risky asset will back risky debt in equilibrium (the risky debt will back risk-free debt in equilibrium), the collateral value of the risky debt, in effect, gets imparted to the risky asset. Using the risky asset to issue risk-free debt is “inefficient.” Instead, by issuing risky debt against the asset, the risky asset can be used to back *both* risky debt and risk-free debt, where the risk-free debt has been issued against the risky debt. This process creates a new security with collateral value (risky debt), while using the asset to issue risk-free debt does not.

Proposition 2. *In equilibrium, there exist two marginal buyers \hat{h}_M and \hat{h}_J such that all $h \in (\hat{h}_M, \hat{h}_J)$ will hold risky debt with maximal leverage (promise D); all $h < \hat{h}_J$ will hold risk-free debt and X , and all $h > \hat{h}_M$ will hold the risky asset with maximal leverage (promise M).*

The proposition characterizes equilibrium in the 3-state model and follows directly from the previous two lemmas and the fact that marginal utilities/optimism is strictly and monotonically increasing in h . Figure 3 illustrates the equilibrium regimes with debt collateralization and with leverage. This result is analogous to Geerolf (2015), in which equilibrium with pyramiding produces the same ordering of lending in the economy with a continuum of states. Importantly, in our result the threshold promises are defined by the discrete payoffs of the states and the ordering of investors follows from valuations of payoffs in different states (either tolerance for risk or subjective probabilities of default), with debt prices compensating for risk. The qualitative break in the equilibrium regime in our model corresponds to changes in the sets of state contingent payoffs agents trade. Our result for maximal leverage would hold even if agents had some degree of risk-

sharing needs so long as marginal utilities of agents are monotonic with dividends.⁶

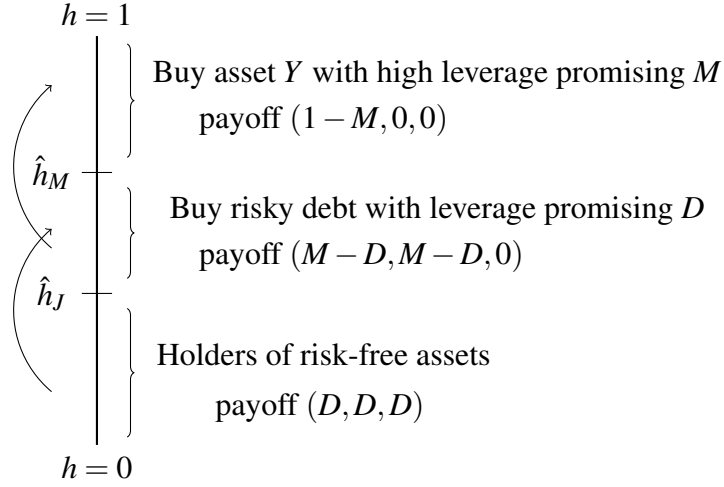


Figure 3: Equilibrium with debt collateralization in static 3-state model.

Thus, equilibrium is characterized by the following equations. Agent \hat{h}_M is indifferent between holding the risky asset with high leverage promising M , and the risky debt with leverage,

$$\frac{\gamma_U(\hat{h}_M)(1-M)}{\hat{p} - \hat{\pi}_M} = \frac{\gamma_U(\hat{h}_M)(M-D) + \gamma_M(\hat{h}_M)(M-D)}{\hat{\pi}_M - D}. \quad (10)$$

In equilibrium both of these investment options are preferred over holding Y with low leverage (promising D). Agent \hat{h}_J is indifferent between holding the risky debt with leverage and holding risk-free assets,

$$\frac{\gamma_U(\hat{h}_J)(M-D) + \gamma_M(\hat{h}_J)(M-D)}{\hat{\pi}_M - D} = 1. \quad (11)$$

Market clearing for the risky asset Y requires

$$\frac{(1 - \hat{h}_M)(1 + \hat{p})}{\hat{p} - \hat{\pi}_M} = 1, \quad (12)$$

⁶We could reproduce the distribution of marginal utilities we get from differences in prior probabilities by instead assuming common probabilities, strictly concave utilities, and by allocating endowments of consumption goods appropriately. An implication is that our results continue to hold (weakly) whether there are more agents than states or whether there are more states than agents. Our results continue to hold when marginal utilities are endogenous so long as there are appropriate bounds on risk aversion and endowments so that even with endogenous portfolio choices, optimists remain uniformly optimistic after accounting for changes in marginal utilities (see [Phelan, 2015](#), for an analysis in a two-agent economy); see also the example in Appendix B.3.

and market clearing for risk-free debt requires

$$\hat{h}_J(1 + \hat{p}) = 1 + D. \quad (13)$$

Collateralizing risky debt thus serves two purposes: it isolates upside payoffs to agents buying risky debt with leverage, and it creates risk-free debt for more pessimistic agents, increasing the supply of risk-free securities.

3.3 Asset Pricing

The effect of debt collateralization on the price of the risky asset is somewhat ambiguous because there are two forces affecting the price. There is a collateral effect, which raises the asset price, and a required return effect, which may decrease the asset price.

Let R and \hat{R} denote the alternative return according to the most pessimistic investor who maximally leverages the asset in the leverage economy and the debt collateralization economy:

$$R = \frac{\gamma_U(h_M)(1 - D) + \gamma_M(h_M)(M - D)}{p - D}, \quad \hat{R} = \frac{\gamma_U(\hat{h}_M)(M - D) + \gamma_M(\hat{h}_M)(M - D)}{\hat{\pi}_M - D},$$

which are taken from equations (5) and (10). It therefore follows that we can write the asset prices as

$$p = \pi + \frac{\gamma_U(h_M)(1 - M)}{R}, \quad \hat{p} = \hat{\pi} + \frac{\gamma_U(\hat{h}_M)(1 - M)}{\hat{R}}.$$

The ‘‘collateral effect’’ implies that debt collateralization increases the collateral value of the risky asset because it can now be used to issue a contract (risky debt) that can serve as collateral ($\pi < \hat{\pi}$). This force increases the price of the risky asset and endogenously increases leverage in the economy. The ‘‘required return effect’’ implies that the required return for investing in the risky asset may increase because alternative investments have become more attractive, namely, investing in *risky debt* with leverage so that generally $R < \hat{R}$. In the leverage economy, the most optimistic agent h_M compares the return to Y with high leverage to the return to Y with low leverage. In the debt collateralization economy, the most optimistic agent \hat{h}_M compares the return to Y with high leverage to the return to risky debt with leverage, and in the debt collateralization economy this investment is strictly preferred to buying Y with low leverage. The required return force tends to

decrease the price of the risky asset.

With debt collateralization, $\hat{h}_M < h_M$ because every agent buying Y makes the risky promise; the marginal investor buying Y and promising M is less optimistic and so the price of Y could fall. Debt collateralization would decrease the asset price if (i) risky debt prices do not increase by much (i.e., $\hat{\pi}$ near π), (ii) the marginal investor becomes much less optimistic about U (i.e., $\gamma_U(h_M) \gg \gamma_U(\hat{h}_M)$), and (iii) the perceived return on leveraged debt is more attractive than the return on Y with low leverage. For a wide range of parameters it appears that debt collateralization increases the asset price (Appendix B.1) because the primary effect of debt collateralization is to increase the price of risky debt. However, Appendix B.2 provides an example where the price p decreases with debt collateralization because the collateral effect is small. This result is in contrast to Geerolf (2015), where pyramiding strictly increases prices.

We can provide some restrictive sufficient conditions under which the collateral effect dominates the return effect so that debt collateralization will increase prices. We require three conditions. First, belief heterogeneity among “pessimists” is greater than among “optimists”. Denote the hazard rates by $f_U(h) = \frac{\gamma_U(h)}{\gamma_U(h) + \gamma_M(h)}$ and $f_M(h) = \frac{\gamma_U(h) + \gamma_M(h)}{\gamma_U(h) + \gamma_M(h) + \gamma_D(h)}$. We require

$$f_U, f_M \text{ are concave.} \tag{A2}$$

Second, optimism about the down state *not* occurring must increase faster than the optimism about the conditional likelihood of the up state.

$$\text{For all } h \geq h', \quad f_U(h) - f_U(h') \leq f_M(h) - f_M(h') \tag{A3}$$

As an example, constant hazard rates for each investor (i.e., $f_U(h) = f_M(h)$ for all h) satisfies this condition.

Third, the fraction of buyers using high leverage in the leverage economy must be sufficiently high, which implies that \hat{h}_M does not differ too much from h_M and collateralization sufficiently expands the supply of safe debt. Let η denote the fraction of Y purchased by investors promising M (high leverage) in the leverage equilibrium. Then we can state the following proposition.

Proposition 3. *Suppose $\eta > \frac{(1-M)^2}{(1-M)^2 + (M-D)D}$ and that beliefs satisfy A2, A3. Then $\hat{p} > p$.*

Here is the logic for the result. First, the lower bound on η ensures that h_M is not too different from \hat{h}_M , since the fraction of the risky asset bought with high leverage goes from η to 1 in the leverage-only to debt-collateralization economy. Additionally, the supply of safe assets increases from $1 + (1 - \eta)D$, in the leverage-only economy, to $(1 + D)$ in the debt-collateralization economy. The condition on η therefore also guarantees that \hat{h}_J is much larger than h_J so that the marginal buyer of safe assets is more optimistic. Second, conditions A2 and A3 together imply a sufficiently large increase in the price of risky debt. Condition A3 states that increasing h results in a faster increase in optimism about states U and M together than for state U alone, and concavity of beliefs (A2) ensures that there is more heterogeneity among pessimists than among optimists, meaning that the increase in optimism from h_J to \hat{h}_J is sufficiently larger than the decrease in optimism from h_M to \hat{h}_M . As a result, the collateral effect dominates.

Additionally, we can isolate the collateral effect by considering an economy that simultaneously contains multiple assets, one that can be leveraged and one that can be used for debt collateralization. Then investors have access to all investment options and so the different leveraged investments will have common required returns. In this case, the collateral effect from debt collateralization will increase the asset price.

Proposition 4. *Consider an economy with risky assets Y and Z with identical dividends but debt backed by Z cannot be used as collateral (Z can be leveraged), while debt backed by Y can be used as collateral. Then in equilibrium the price of Y exceeds the price of Z .*

Because Y and Z are available to investors at the same time, the required return for any investor applies equally to both assets and so the required return force does not differentially affect Y over Z . But the risky promise backed by Y has collateral value, while the promise backed by Z does not, and thus the risky promise backed by Y has a higher price. As a result, Y must also have a higher price since it is used to issue a more valuable contract.

In reality not every financial contract can be used as collateral to issue further contracts. Perhaps debt collateralization is prevalent in one market, but not necessarily in others. (Consider how the mortgage market is often the vanguard of financial innovation.) To the extent that investors may have access to assets and financial contracts with differential degrees of collateralizability, investment opportunities will have common required returns but debt collateralization will isolate

the collateral effect. We, therefore, suspect that the setting in Proposition 4 is an empirically realistic setting.

3.4 Numerical Example

A numerical example is helpful to suggest what happens to prices and economy-wide margins. We roughly “calibrate” the 3-state model so that the move from leverage-only to debt collateralization explains the following moments: we target economy-wide average margins with leverage to be 15% and with debt collateralization to be 5%, and we target risky debt spreads to be 3.9% with leverage and 1.6% with debt collateralization.⁷ (Of course many other changes occurred pre-crisis, not just the innovation of debt collateralization.) We parametrize marginal utilities of the form $\gamma_U(h) = h^\zeta$ and $\gamma_M(h) = h^\zeta(1 - h^\zeta)$, with $\zeta > 0$.⁸ Thus we choose parameters M , D , and ζ to match the four moments. Our calibration yields $M = 0.93$, $D = 0.81$, $\zeta = 6.5$.⁹ Appendix B.1 discusses parameter robustness.

Table 1 compares the equilibria with leverage and with debt collateralization (“DC”). While our calibration targets economy-wide average margins, the model solution is able to show why margins fall. Economy-wide average margins decrease for two reasons: all agents who buy the risky asset use the low margin (high leverage) strategy, and the risky margin (buying the asset with j_M) decreases because the risky debt price increases by relatively more than the asset price p . In this example (and across a wide range of parameters), the first effect is much larger.

The asset price increases by a modest 0.7 percent. Across a range of parametrizations, the model typically delivers modest increases in p . In our numerical simulations, the price of risky debt always increases, and the price of the risky asset increases in most cases. This result is in line with evidence by [Kaplan et al. \(2015\)](#), who quantitatively assess the contributions of changes in mortgage margins, productivity, and expectations about future house prices to explain house prices

⁷[Fostel and Geanakoplos \(2012a\)](#) show that for subprime mortgages from 2000–2008, average margins decreased from 12% to 3% in 2006 and then increased to roughly 18% by end of 2007. Pre crisis 10 year Baa corporate bond credit spreads ranged from 3.9% to roughly 1.6% through 2007, which we use as a rough measure of financing spreads.

⁸We show that these utilities satisfy assumption A1 in Lemma 5 in the Appendix.

⁹An alternative, attractive parametrization is to set payoffs to $M = 0.9$ and $D = 0.65$: the middle payoff corresponds to a mild recession for firms or a bad-but-typical decrease in house prices; the down payoff is a deep recession or a dramatic (35%) decrease in house prices. We then choose beliefs so that risky spreads and margin changes correspond roughly to levels over the early 2000s, yielding $\zeta = 2$. In this case, introducing debt collateralization, average margins decrease from 30.27% to 8.7%, spreads decrease from 3.88% to 2.45%, and the price increases by 1.16%.

Table 1: Equilibrium with Leverage and with Debt Collateralization

Prices and Marginal Investors			Interest Spread and Margins		
	Leverage	DC ($\hat{\cdot}$)		Leverage	DC ($\hat{\cdot}$)
p	0.9542	0.9608 \uparrow			
π_M	0.9014	0.9103 \uparrow	i_M	3.17 %	2.17% \downarrow
h_M	0.9984	0.9742 \downarrow	Risky Margin	5.53%	5.26% \downarrow
h_D	0.9289	–	Safe Margin	15.11%	–
h_J	0.9021	0.9231 \uparrow	Average Margin	14.78%	5.26% \downarrow

during the housing boom and bust, and find that house prices are explained primarily by changes in expectations about future appreciation, not by margins. Thus, our model is best understood as a model of margins and leverage.

Dynamic Extension The static model illustrates that debt collateralization leads to agents making larger promises, increasing the leverage in the economy. In Appendix C we consider a dynamic extension of the 3-state model in order to study the effect of debt collateralization on price crashes and volatility. The maximal leverage result has several important implications for economy-wide margins and asset price levels and volatility (crashes). First, debt collateralization exacerbates the Leverage Cycle (Geanakoplos, 2003, 2010), amplifying price fluctuations and creating more price volatility than occur with leverage alone. Higher leverage increases the risky asset’s collateral value, which fluctuates in response to news about fundamentals. Second, higher leverage endogenously increases defaults after bad news. Accordingly, our analysis explains how financial innovations in CDO, LBO, and similar markets, can lead to credit expansions and potentially higher volatility.

4 Characterizing Equilibrium in the General Model

We characterize the set of contracts potentially traded in equilibrium in the general setting with N states and L levels of collateralization. The main result of this section is that the possible set of investment options chosen in equilibrium decreases with more levels of collateralization. We first consider when agents can trade only debt contracts. In this case, only higher-leverage strategies remain with more levels of collateralization. We then consider when agents can trade a complete set of contingent contracts. Our results with contingent contracts provide a generalization of the

maximum leverage results with debt.

4.1 Economy with Debt Contracts

When only debt contracts in J^0 can be traded, agents can buy the risky asset leveraged with any promise s_1, \dots, s_N by selling the promise $j_n = (s_n, Y)$. We let Y/j_n denote the act of holding Y and selling the debt contract j_n . Following [Araujo et al. \(2012\)](#) and [Fostel and Geanakoplos \(2012b\)](#), in the leverage economy agents will do one of the following in equilibrium:

1. hold Y/j_n , where $1 \leq n \leq N - 1$,
2. hold risky debt j_n with $2 \leq n \leq N - 1$,
3. hold risk-free debt j_1 or the risk-free asset X .

Debt collateralization will allow the contracts traded in the leverage economy to be used as collateral, and as a result the set of debt contracts traded will endogenously change.

Our main result is that every level of debt collateralization increases the minimum promise made by agents buying the asset, and with “complete collateralization”—when any existing risky debt contract can be used as collateral—agents make the maximum (natural) promise available for every investment, risky asset or risky debt. With more than 3 states, multiple *risky* contracts will typically be traded in equilibrium. When agents can use these initial debt contracts as collateral, in equilibrium some agents will invest in risky debt contracts and make *risky* promises. These second-level debt contracts can potentially be used as collateral to make further promises. Equilibrium thus depends on how many “levels of debt” can be used as collateral.

Proposition 5. *Consider an economy in which, when agents can leverage, $N - 1$ contracts are traded in equilibrium. In any equilibrium, there exists an equivalent equilibrium such that at the L -th level of debt collateralization, at most the following leveraged positions exist in the economy*

1. Y/j_n , where $L < n < N$
2. j_m^l/j_k^{l+1} , where $0 \leq l < L$, $L - l < m < N - l$, $L - l \leq k < m$
3. j_ℓ^L , where $1 \leq \ell < N - L$.

Additionally, more optimistic investors invest in assets with larger face values, and within each asset-class investors are ordered by the amount of leverage they use.

This result is a generalization of the three-state environment and the intuition is similar.

Each level of collateralization increases the collateral value of new promises and of every debt contract that could already be used as collateral. As collateralization increases, more debt contracts have collateral value, as do the “upstream” debt contracts that can back those promises. As a result, when a security can be used to back promises that serve as collateral L times, making a smaller promise than stipulated by the proposition would not maximize the collateral value of debt contracts. Thus, investors make the largest promise that maximizes the collateral value of “downstream” promises.

We state a few implications of the proposition to provide more meaning. Corollary 1 explicitly states that debt collateralization decreases the number of low-level leverage strategies, and Corollary 2 states that with maximal debt collateralization, only the highest leverage positions remain in equilibrium, which corresponds to the conjecture in [Simsek \(2013\)](#) that in multi-state models when debt contracts can be used as collateral, equilibrium will feature a pyramiding arrangement; in other words the conjecture in [Simsek \(2013\)](#) holds at the maximal level of collateralization. By simple accounting, there can be at most $N - 2$ levels of debt collateralization.

Corollary 1. *With each additional level of debt collateralization, there is one fewer marginal buyer of the risky asset Y , and thus one fewer “low level” of leverage used to buy the risky asset.*

Corollary 2 (Pyramiding Arrangement). *Consider the continuum of agents in the economy. At the maximum $N - 2$ levels of debt collateralization, the interval $(0, 1)$ is broken up into N sub-intervals, denoted $(0, \hat{a}_1), (\hat{a}_1, \hat{a}_2), \dots, (\hat{a}_{N-1}, 1)$. The first interval consists of agents holding safe assets while the last interval, $(\hat{a}_{N-1}, 1)$ consists entirely of agents holding Y / j_{N-1}^0 . In general, the k^{th} interval, where $N > k > 1$, consists of agents holding $j_k^{N-k} / j_{k-1}^{N+1-k}$. In other words, every level of agents in the economy is lending directly to the level above and maximally leveraging the asset or contract in which they invest.*

The corollaries follow immediately from Proposition 5. In the pyramiding arrangement investors are maximally leveraged: every investor makes the largest promise (from among the discrete set of states), given the asset or contract in which they invest.

Our maximal leverage result follows because belief heterogeneity concerns upside states, with increasing optimism implying a greater desire to concentrate payoffs in upside states. Maximal leverage need not occur if the nature of heterogeneity changes. For example, maximal leverage

need not occur if disagreements were primarily about downside states so that pessimists want to concentrate payoffs in the lowest states (see [Simsek, 2013](#)), because issuing debt does nothing to isolate payoffs in the lowest states. Additionally, maximal leverage may not occur if disagreements were about “tails,” not just upside payoffs. If high h investors value payoffs in extrema states, maximal leverage would isolate payoffs in the upside tails but not in the downside tail.¹⁰

4.2 Economy with Contingent Contracts

We now suppose that contracts can be state contingent. Agents can issue contracts j with any set of promised payoffs A^j . Accordingly, we now suppose that both fundamental assets X and Y can serve as collateral for level-0 contracts. The risk-free asset X is now meaningful collateral since it can back contingent promises; analogous results also hold if only Y can serve as collateral.

We define *contract collateralization* with contingent contracts just as we did for debt. A level-0 contract promises payoffs A^j backed either by X or Y . Since contracts are state-contingent, without loss of generality we can restrict promises to paying no more than the value of the backing collateral in n , so that a contract will deliver A_n^j in state n .

Definition 4. We say the L -th level of contract collateralization is the creation of contracts j^L using $k^{L-1} \in J^{L-1}$ as collateral. Denote the set of contracts at the L -th level of collateralization by J^L . We write $j^L(k^{L-1}) = (A^{j^L}, k^{L-1})$ to denote the contract that is traded when an agent holds $k^{L-1} \in J^{L-1}$ as collateral and promises to pay $A_n^{j^L}$ in state n . The contract delivers $\min\{A_n^{j^L}, A_n^{k^{L-1}}\}$ in state n .

Note that an economy with level-0 contracts only cannot implement an Arrow-Debreu equilibrium because an asset X or Y can back only one contract at a time (see [Geanakoplos and Zame, 2014](#), markets are not complete owing to collateral constraints). In the 3-state economy, an agent can hold an “Arrow- U ” by buying Y and issuing a contract that promises $(0, M, D)$, but that means some other agent would hold $(0, M, D)$ in equilibrium. Generally, if a contract pays in $K \leq N$ states, then the issuer of the contract retains payments in at least $N - K$ states. Collateral constraints *require* that

¹⁰Each of these examples of beliefs disagreement also correspond to economies with heterogeneity in risk-aversion or endowments. In the first case, agents could be risk averse with endowments heterogeneously skewed over downside states (analogous to the example in Appendix B.3). In the second case, agents could be risk averse with heterogeneous variance of endowments.

in equilibrium some agents must hold “bundles” of Arrow-Debreu securities, which is not required with complete markets—in other words, collateral constraints prevent the complete splitting of asset payoffs into Arrow-Debreu securities. Because of this, allowing level-0 contingent contracts to serve as collateral is not redundant in equilibrium precisely because it increases the collateral capacity of the underlying asset. Contract collateralization effectively allows the asset to serve as collateral for *multiple* contracts—the asset directly backs the level-0 contract, and indirectly backs a level-1 contract.

We show that first, if a contract pays in multiple states and can be used as collateral, then agents *will* use this contract as collateral. Second, agents will only issue level-0 contracts that pay in multiple states because Arrow-Debreu securities paying in a single state are not meaningful collateral.

Lemma 3. *Suppose in equilibrium an agent holds a contract that pays out in $K \geq 2$ states and that the financial environment allows this contract to serve as collateral. Then the agent will use the contract as collateral to issue some other contract.*

The intuition for this result is that an agent that prefers consumption in K states over consumption in only a single state among those K (say state k) must have a high valuation for the other $K - 1$ states given the market price for consumption in k . But such an agent would therefore be willing to sell the contract k to increase consumption in the other $K - 1$ states.

Lemma 4. *Suppose level-0 contracts can be used as collateral. Then in equilibrium any agent buying X or Y will issue a contract that pays in more than one state. No agent will use X or Y to issue a contract that cannot serve as collateral.*

The intuition for this result is similar to the maximal leverage result for debt collateralization: in equilibrium, agents take actions to maximize the ability of X or Y to serve as collateral. Using X or Y to issue a contract that pays in only one state exhausts the collateral capacity of X or Y as neither the contract nor (trivially) X or Y can serve as further collateral. But using X or Y to issue a contract that pays in $N - 1$ complementary states creates the exact same payoffs, but the multi-state contract can serve as collateral, extending the collateralizability of X or Y . In the 3-state economy an Arrow- U can be created using Y as collateral either by issuing an Arrow- U directly, or by issuing the promise $(0, M, D)$ and retaining the dividend in U . With contract collateralization,

agents will trade only the latter strategy, since the promise $(0, M, D)$ can be used as collateral to issue another promise, while issuing an Arrow- U does not allow further collateral possibilities.

Proposition 6. *Any equilibrium with only level-0 contracts is essentially different from an equilibrium in which level-0 contracts can serve as collateral. Contract collateralization changes the payoffs that agents hold.*

This proposition follows directly from the previous lemmas. Contract collateralization plays a meaningful role even when agents are allowed to make state-contingent promises. With debt contracts, collateralization was important for creating state-contingent payoffs. Similarly, with contingent contracts, collateralization is *still* important for creating contingencies because collateral is scarce as a result of cross-netting frictions. An asset can back only a single contingent promise at a time, and equilibrium does not achieve the complete markets outcome as the assets' payoffs cannot be completely split into Arrow-Debreu securities. Allowing contracts to back further promises expands the ability of the underlying asset to serve as collateral, which enables more flexibility to split the asset's payoffs in equilibrium.¹¹

The next proposition generalizes the above results for L levels of collateralization, providing the analog for Proposition 5 with debt contracts.

Proposition 7. *Suppose contracts can be state contingent but no asset or contract can directly serve as collateral for more than one contract. At L levels of contract collateralization:*

1. *Investors holding X or Y will issue level-0 contracts that pay in K states, with $L < K < N$.*
2. *Any level- ℓ contract traded in equilibrium will pay in least $L + 1 - \ell$ states. For $\ell < L$, level- ℓ contracts will be used as collateral to issue level- $\ell + 1$ contracts that pay in K states, with $K > L - \ell$.*
3. *Any level- ℓ contract that pays in $K \leq L + 1 - \ell$ states is priced by Arrow-Debreu securities.*
4. *Any Arrow-Debreu equilibrium can be implemented with $N - 2$ levels of contract collateralization.*

This result serves as a building block to the result of [Gottardi and Kubler \(2015\)](#) that with sufficiently rich contract and collateral space, any Arrow-Debreu equilibrium allocation with limited

¹¹Consider the 3-state economy. Optimists could hold Y and issue a contract paying $(0, M, D)$ to moderates, who could use that contract as collateral to issue an Arrow- D paying $(0, 0, D)$ to pessimists. In this way, Y is able to effectively support two contracts, an Arrow- M and an Arrow- D . However, the implementation is not unique (consider having pessimists hold Y and issue $(1, M, 0)$ to moderates, who then use that contract to issue $(1, 0, 0)$ to optimists); for this reason, there is not a corresponding way to define "maximal leverage" with contingent contracts.

pledgeability can also be attained at a collateral-constrained financial market equilibrium if contracts can be used to back other contracts *ad infinitum*. Since in our economy all endowments are capitalized as assets at $t = 0$, which can serve as collateral, our economy features sufficient supply of collateral so that an Arrow-Debreu equilibrium with limited pledgeability is a standard Arrow-Debreu equilibrium (see [Gottardi and Kubler, 2015](#), for the distinction when collateral is scarce). With enough levels of collateralization, the payoff of X or Y can eventually be separated into Arrow-Debreu securities. Since equilibrium with level-0 contracts alone does not implement Arrow-Debreu, contract collateralization is required—and therefore meaningful—and with more states more levels are required to implement Arrow-Debreu. Just as $N - 2$ levels of debt collateralization were needed to achieve maximal leverage, cross-netting frictions require $N - 2$ levels of contingent contract collateralization to achieve a complete market equilibrium.

5 Tranching and Capital Structure

Tranching refers to the process of using collateral to back multiple promises of different types. Senior-subordinated capital structures define tranches with realized payoffs determined by the seniority of the tranche. Critically, subordinated tranches (and subordinated capital) are equivalent to leveraged positions in risky debt backed by equity tranches, giving investors the *implicit* ability to use debt as collateral. We show the equivalence between tranching and debt collateralization formally in the N -state model.

5.1 Theoretical Analysis of Tranching

Consider the N -state model. Suppose the asset Y can be split by a financial intermediary into the following tranches: T_1, \dots, T_N where T_1 pays s_1 in all states of the world, and for $k > 1$ T_k pays $s_k - s_{k-1}$ when $n \geq k$ and 0 otherwise. That is, one unit of the risky asset Y can be used to

simultaneously back multiple promises, creating the following tranches:

$$\begin{aligned}
T_N &: (s_N - s_{N-1}, 0, 0, \dots, 0), \\
T_{N-1} &: (s_{N-1} - s_{N-2}, s_{N-1} - s_{N-2}, 0, \dots, 0), \\
&\vdots \qquad \qquad \qquad \vdots \\
T_2 &: (s_2 - s_1, s_2 - s_1, \dots, s_2 - s_1, 0), \\
T_1 &: (s_1, s_1, \dots, s_1).
\end{aligned}$$

Note that $T_1 + T_2 + \dots + T_N = Y$. We refer to this financial structure as *senior-subordinated tranching* to emphasize the state-contingency is defined according to a senior-subordinated capital structure (complete tranching would refer to the creation of Arrow securities, not just paying zero in down states). In this economy, investors buy and sell the tranches listed above rather than trading the risky asset Y (though they can exactly replicate Y by buying all the tranches). Each investor must hold a non-negative quantity of each tranche. We refer to equilibrium as the senior-subordinated tranching equilibrium. This yields the following result (with formal conditions in the appendix).

Proposition 8. *The senior-subordinated tranching equilibrium is equivalent to equilibrium with complete debt collateralization. That is, there exists a bijective mapping of assets and prices from the debt collateralization equilibrium to the senior-subordinated tranching equilibrium such that the buyers of assets remain the same.*

While the result follows essentially from accounting, the result is important: tranching and debt collateralization have an essential equivalence in terms of the state-contingent promises they create to maximize collateral values.

For the intuition for this result, consider a typical ABS deal, which consists of a pool of mortgages (collateral) supporting senior, mezzanine, and equity/residual securities. The equity security behaves like a leveraged position in the collateral, with the payoff declining “linearly” with the value of the collateral and paying zero when the collateral falls below a certain level. The senior security behaves like debt, making a predetermined payoff unless the collateral value falls below a certain threshold, at which point the payoff declines linearly to zero only when the collateral

is worth zero. The subordinated, or mezzanine, security, however, behaves like a leveraged debt position. For sufficient values of collateral the subordinated security gets the predetermined payoff (there is not additional upside as with a leveraged position in the collateral), but gets nothing if the value of the collateral is low (like a leveraged position). In fact, the subordinated tranches are leveraged positions in the debt implicitly “issued” by the equity tranche.¹²

In reality financial innovation includes forms of both tranching and debt collateralization. Subprime mortgage pools have been used to create tranches of different seniority. Each tranche of the asset-backed security (“ABS”) pays different amounts depending on the aggregate value of the mortgage pool (i.e., in different states of the world). A typical ABS deal tranches a pool of mortgages into 4 or 5 rated bonds and a residual, or equity, tranche. These tranches (typically the mezzanine bonds) are then be pooled together to serve as collateral for a CDO, which would issue another 4-5 bonds. And the process continues as the tranches from the CDO are collateralized into a CDO-squared. Each stage includes both tranching and collateralization of existing debt securities. Because mortgage pools do contain idiosyncratic risk, pooling tranches together to diversify this risk is an important step of the securitization process.

Informational or agency frictions (e.g., risk retention) may limit contract contingencies or cross-netting. Any of these limitations will have implications for the levels of collateralization that would occur in equilibrium. The degree of collateralization is clearly endogenous, depending on the financial sector’s ability to track and clear payments backed to the L -th degree and the need for diversification (or retention) at every level of pooling.

6 Conclusion

When agents have the ability to use risky debt backed by a risky asset as collateral for other financial promises, agents use exclusively maximal leverage in equilibrium. Debt collateralization expands the set of possible contingent payoffs in the economy, and maximal leverage maximizes the ability of assets to serve as collateral, and thus providing a way of stretching scarce collateral.

¹²Layered capital structures are essentially “CDOs” with different collateral. Examples go back to unit trusts in the 1920s, the “unit trust of unit trusts” created by Goldman Sachs in 1928, Trust Preferred (“TruPS”) CDOs, and, more prevalent, structured leveraged buyouts (“LBOs”). Similarly, securitized second-lien mortgages (see Bear Stearns Second Lien Trust 2007) created tranches in debt that were part of a complex capital structure financing housing (Chambers et al., 2011).

This shift in the set of state-contingent payoffs traded in the economy decreases margins on the risky asset (increases leverage), decreases the risk premia for risky debt, and generally increases the price of the risky asset. Our results offer important empirical implications for economy-wide margins, risk-premia, and asset prices.

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Appendices for Online Publication

The appendix is organized as follows. Section A contains proofs for results. Section B contains additional analyses in the static model. Section C presents a dynamic analysis in a 3-period model. Section D presents empirical and testable implications based on the results in our static and dynamic analyses.

A Proofs

To economize on notation, we denote level-0 debt contracts simply by j .

Proof of Lemma 1. Suppose for contradiction that there exists an h_i who prefers to hold the risky debt with some amount of leverage L , $0 \leq L < D$, less than the maximum. Since $L < D$ it is risk-free and thus $\hat{\pi}^L = L$. The marginal utilities from investing in j_M against promise L , from investing

in j_M against promise D , and from holding risk-free assets are:

$$\text{debt with leverage } L: \frac{(\gamma_U(h_i) + \gamma_M(h_i))(M - L) + \gamma_D(h_i)(D - L)}{\hat{\pi}_M - L} \quad (14)$$

$$\text{debt with leverage } D: \frac{\gamma_U(h_i)(M - D) + \gamma_M(h_i)(M - D)}{\hat{\pi}_M - D} \quad (15)$$

$$\text{risk-free asset: } 1. \quad (16)$$

Since by assumption h_i strictly prefers the first option, it must be the case that (14) > (15) and (14) > (16). That is, the investor is optimistic enough to prefer the risky debt to risk-free debt but not so optimistic as to want zero payoff in D . Hence,

$$\frac{(\gamma_U(h_i) + \gamma_M(h_i))(M - L) + \gamma_D(h_i)(D - L)}{\hat{\pi}_M - L} > \frac{\gamma_U(h_i)(M - D) + \gamma_M(h_i)(M - D)}{\hat{\pi}_M - D}, \quad (17)$$

$$\frac{(\gamma_U(h_i) + \gamma_M(h_i))(M - L) + (1 - \gamma_U(h_i) - \gamma_M(h_i))(D - L)}{\hat{\pi}_M - L} > 1 \quad (18)$$

Simplifying 17, we obtain

$$\hat{\pi}_M - (\gamma_U(h_i) + \gamma_M(h_i))M - \gamma_D(h_i)D > 0 \implies \hat{\pi}_M > (\gamma_U(h_i) + \gamma_M(h_i))M + \gamma_D(h_i)D$$

Simplifying 18, we obtain

$$\hat{\pi}_M - \gamma_D(h_i)D - (\gamma_U(h_i) + \gamma_M(h_i))M < 0 \implies \hat{\pi}_M < \gamma_D(h_i)D + (\gamma_U(h_i) + \gamma_M(h_i))M$$

Note that the above gives us $\hat{\pi}_M > \hat{\pi}_M$. This is a contradiction so long as any of the inequalities are strict. Given our strict monotonicity assumptions on beliefs/marginal utilities, if the above set of inequalities are weak for any agent (i.e., equalities), then they are strict inequalities for every other agent. Thus, in equilibrium, all agents holding risky debt (but potentially a measure zero) will do so with maximal leverage. \square

Proof of Proposition 1. Market clearing for the risk-free asset in the leverage and DC economies

are given by

$$h_J(1+p) = 1 + \left(\frac{(h_M - h_D)(1+p)}{p-D} \right) D < 1+D, \quad \hat{h}_J(1+\hat{p}) = 1+D$$

This implies that $\hat{h}_J(1+\hat{p}) > h_J(1+p)$ so either $\hat{h}_J > h_J$ or $\hat{p} > p$.

Suppose $\hat{h}_J > h_J$. The marginal buyer pricing the risky debt in the leverage economy has

$$\pi = \gamma_U(h_J)M + \gamma_M(h_J)M + \gamma_D(h_J)D$$

The marginal buyer in the DC economy has

$$\hat{\pi} - D = \gamma_U(\hat{h}_J)(M-D) + \gamma_M(\hat{h}_J)(M-D) \implies \hat{\pi} = \gamma_U(\hat{h}_J)M + \gamma_M(\hat{h}_J)M + \gamma_D(\hat{h}_J)D$$

So $\hat{h}_J > h_J \iff \hat{\pi} > \pi$. Now suppose for contradiction that $\hat{p} > p$ but $\hat{\pi} < \pi$. Then,

$$\frac{\hat{p}-D}{\hat{p}-\hat{\pi}} < \frac{p-D}{p-\pi}$$

Consider the marginal buyer h_M in the leverage economy who is indifferent between buying the asset with high or low leverage. This agent is defined by

$$\frac{\gamma_U(h_M)(1-M)}{p-\pi} = \frac{\gamma_U(h_M)(1-D) + \gamma_M(h_M)(M-D)}{p-D} \implies \frac{\hat{p}-D}{\hat{p}-\hat{\pi}} < \frac{\gamma_U(h_M)(1-D) + \gamma_M(h_M)(M-D)}{\gamma_U(h_M)(1-M)}$$

So, under the prices in the DC, this marginal buyer would strictly prefer to NOT buy the asset with high leverage, implying $\hat{h}_M > h_M$. Now, combining this with the market clearing for the risky asset in the leverage and DC economies imply

$$\frac{(1-h_M)(1+\hat{p})}{\hat{p}-\hat{\pi}} > \frac{(1-\hat{h}_M)(1+\hat{p})}{\hat{p}-\hat{\pi}} = 1 > \frac{(1-h_M)(1+p)}{p-\pi}$$

where the first inequality follows from $\hat{h}_M > h_M$, the equality follows from the market clearing in the DC economy, and the last inequality follows from market clearing in the leverage economy.

Simplifying,

$$\frac{(1+\hat{p})}{\hat{p}-\hat{\pi}} > \frac{(1+p)}{p-\pi}$$

But this contradicts $\hat{p} > p$ and $\hat{\pi} < \pi$. □

Proof of Lemma 2. In equilibrium, each unit of the leveraged risky asset must be backed by one unit of debt, either risk-free or risky and leveraged. By previous lemma, we have shown that all agents holding risky debt will be maximally leveraged. We therefore know that agents holding the risky asset must either be leveraged against state D or state M and not something in-between.

Suppose for contradiction that there is some agent h_i who prefers to hold the risky asset leveraged against state D and the price of debt is D . That is, the investor is optimistic enough to prefer the risky asset with low leverage to the leveraged risky debt, but not so optimistic as to want to maximally leverage the asset and get zero payoff in M . Note that returns from investment strategies are:

$$\text{Marginal utility from risky asset with debt } D : \frac{\gamma_U(h_i)(1-D) + \gamma_M(h_i)(M-D)}{\hat{p}-D} \quad (19)$$

$$\text{Marginal utility from risky asset with debt } M : \frac{\gamma_U(h_i)(1-M)}{\hat{p}-\hat{\pi}_M} \quad (20)$$

$$\text{Marginal utility from risky debt : } \frac{(\gamma_U(h_i) + \gamma_M(h_i))(M-D)}{\hat{\pi}_M-D} \quad (21)$$

Since by assumption h_i strictly prefers the first option, it must be the case that (19) > (20) and (19) > (21). That is,

$$\frac{\gamma_U(h_i)(1-D) + \gamma_M(h_i)(M-D)}{\hat{p}-D} > \frac{\gamma_U(h_i)(1-M)}{\hat{p}-\hat{\pi}_M} \quad (22)$$

$$\frac{\gamma_U(h_i)(1-D) + \gamma_M(h_i)(M-D)}{\hat{p}-D} > \frac{(\gamma_U(h_i) + \gamma_M(h_i))(M-D)}{\hat{\pi}_M-D} \quad (23)$$

Simplifying 22, we obtain that

$$\gamma_M(h_i)(M\hat{p} + D\hat{\pi}_M - M\hat{\pi}_M - D\hat{p}) > \gamma_U(h_i)(\hat{\pi}_M + MD + D\hat{p} - D - D\hat{\pi}_M - M\hat{p})$$

Simplifying 23, we obtain

$$\gamma_U(h_i)(\hat{\pi}_M + MD + D\hat{p} - D - D\hat{\pi}_M - M\hat{p}) + \gamma_M(h_i)(M\hat{\pi}_M + D\hat{p} - D\hat{\pi}_M - M\hat{p}) > 0$$

For convenience, let

$$\alpha := \gamma_M(h_i)(M\hat{p} + D\hat{\pi}_M - M\hat{\pi}_M - D\hat{p}), \quad \beta := \gamma_U(h_i)(\hat{\pi}_M + MD + D\hat{p} - D - d\hat{\pi}_M - M\hat{p})$$

Notice that the above two equations simplify to $\alpha > \beta$ and $\beta - \alpha > 0$. Again, given our monotonicity assumptions on beliefs/utilities, these inequalities must be strict for all except a measure zero of investors. Hence, this is clearly a contradiction since we cannot have both $\alpha > \beta$ and $\beta > \alpha$. Thus, all investors holding the risky asset will be maximally leveraged against state M . \square

Proof of Proposition 3. First, suppose for contradiction that $\hat{p} \leq p$. From market clearing for safe assets, we have

$$h_J(1+p) \leq 1+D = \hat{h}_J(1+\hat{p}),$$

and so $\hat{p} \leq p$ implies $\hat{h}_J > h_J$, which also means that $\hat{\pi} > \pi$. If $\hat{p} \leq p$ then we must also have $\hat{h}_M < h_M$, so that altogether $h_J < \hat{h}_J < \hat{h}_M < h_M$.

Second, using equations (10) and (11) for the marginal buyers, we can write the asset price with debt collateralization as

$$\hat{p} = \hat{\pi} + (1-M) \frac{\gamma_U(\hat{h}_M)}{\gamma_U(\hat{h}_M) + \gamma_M(\hat{h}_M)} (\gamma_U(\hat{h}_J) + \gamma_M(\hat{h}_J)). \quad (24)$$

Rearranging equation (5) for the marginal investor h_M in the leverage economy, we have

$$\begin{aligned} \frac{\gamma_U(1-M)}{p-\pi} &= \frac{\gamma_U(1-M) + (\gamma_U + \gamma_D)(M-D)}{p-D}, \\ (p-D)\gamma_U(1-M) &= (p-\pi)\gamma_U(1-M) + (p-\pi)(\gamma_U + \gamma_D)(M-D), \\ p-\pi &= \frac{\gamma_U(h_M)}{\gamma_U(h_M) + \gamma_M(h_M)} \left(\frac{1-M}{M-D} \right) (\pi-D), \end{aligned}$$

which together with equation (7) gives

$$p = \pi + (1-M) \frac{\gamma_U(h_M)}{\gamma_U(h_M) + \gamma_M(h_M)} (\gamma_U(h_J) + \gamma_M(h_J)). \quad (25)$$

Let $f_U(h) = \frac{\gamma_U(h)}{\gamma_U(h) + \gamma_M(h)}$ and $f_M(h) = \gamma_U(h) + \gamma_M(h)$ denote the hazard ratios for states U and

M . Then we can combine equations (24) and (25) to

$$\hat{p} - p = (M - D)(f_M(\hat{h}_J) - f_M(h_J)) + (1 - M)(f_U(\hat{h}_M)f_M(\hat{h}_J) - f_U(h_M)f_M(h_J)). \quad (26)$$

By assumption A1, $f_U(h)$ and $f_M(h)$ are increasing so that $f_M(\hat{h}_J) > f_M(h_J)$ and $f_U(h_M) > f_U(\hat{h}_M)$. To show that $\hat{p} > p$, therefore, requires showing that

$$(M - D)(f_M(\hat{h}_J) - f_M(h_J)) > (1 - M)(f_U(h_M) - f_U(\hat{h}_M)).$$

By our assumptions on the concavity of the hazard rates, and the ordering of the marginal buyers, it is sufficient to show that

$$(M - D)(\hat{h}_J - h_J) > (1 - M)(h_M - \hat{h}_M)$$

We proceed by providing bounds on $\hat{h}_J - h_J$ and $h_M - \hat{h}_M$. From market clearing for risk-free assets, we have

$$\hat{h}_J - h_J > \frac{\eta D}{1 + \hat{p}},$$

and from market clearing for risky assets, we have

$$h_M - \hat{h}_M < \frac{p - \eta(p - \pi) - \hat{\pi}}{1 + \hat{p}} < \frac{(p - \pi)(1 - \eta)}{1 + \hat{p}}.$$

By assumption A2 and A3, we therefore have:

$$\begin{aligned} \frac{(h_M - \hat{h}_M)}{(p - \pi)(1 - \eta)} &< \frac{(\hat{h}_J - h_J)}{\eta D} \\ \implies \frac{(h_M - \hat{h}_M)}{(\hat{h}_J - h_J)} &< \frac{(p - \pi)(1 - \eta)}{\eta D} \end{aligned}$$

Note that we have $p - \pi < 1 - M$. Combining this with our assumption on η , we get

$$\frac{(p - \pi)(1 - \eta)}{\eta D} < \frac{(1 - M)(1 - \eta)}{\eta D} < \frac{M - D}{1 - M}$$

which proves that $\hat{p} > p$ from our earlier inequality.

□

Proof of Proposition 4. Denote the prices of Y and Z by \hat{p} and p . Denote risky debt promising M backed by Y and Z by \hat{j}_M and j_M , and denote their prices by $\hat{\pi}_M$ and π_M respectively. Since \hat{j}_M can be used as collateral while j_M cannot, it must be that $\hat{\pi}_M > \pi_M$ in equilibrium. Note that buying Y and promising M yields the same payoffs as buying Z and promising M —namely, $1 - M$ in state U and zero otherwise. In there is a marginal buyer \hat{h}_M willing to hold an Arrow U . Since both investment strategies yield identical payoffs, they must have the same prices, i.e., $\hat{p} - \hat{\pi}_M = p - \pi_M$. Using $\hat{\pi}_M > \pi_M$ we have that $\hat{p} > p$. □

Lemma 5. *In the 3-state model, the beliefs $\gamma_U(h) = h^\zeta$, $\gamma_M(h) = h^\zeta(1 - h^\zeta)$ satisfy assumption A1.*

Proof. To satisfy A1, we need to have $\gamma_D(h)$ and $\frac{\gamma_M}{\gamma_U + \gamma_M}$ strictly decreasing in h . We do this by taking derivatives:

$$\gamma'_D(h) = 2\zeta h^{2\zeta-1} - 2\zeta h^{\zeta-1} = 2\zeta h^{\zeta-1}(h^\zeta - 1)$$

The derivative has one zero which occurs at $h = 1$. Between the interval $(0, 1)$, the derivative is strictly negative. Similarly,

$$\begin{aligned} \frac{\gamma_M}{\gamma_U + \gamma_M} &= \frac{h^\zeta(1 - h^\zeta)}{2h^\zeta - h^{2\zeta}} = \frac{1 - h^\zeta}{2 - h^\zeta} \\ \implies \frac{d}{dh} \left[\frac{\gamma_M}{\gamma_U + \gamma_M} \right] &= \frac{(1 - h^\zeta)\zeta h^{\zeta-1} - (2 - h^\zeta)\zeta h^{\zeta-1}}{(2 - h^\zeta)^2} = -\frac{\zeta h^{\zeta-1}}{(2 - h^\zeta)^2} \end{aligned}$$

Since $\zeta > 0$, the above is negative. □

A.1 Proof of Proposition 5

We proceed with the proof by induction, and break the proof into the following two parts. (1) The equilibrium at the first level of collateralization. (2) Equilibrium at the L^{th} level of collateralization.

A.1.1 Equilibrium at the first level of collateralization

To prove the base case, we will show that agents will hold one of the following assets in equilibrium:

(i) Y/j_n , where $2 \leq n \leq N - 1$; (ii) j_n/j_ℓ^1 , where $2 \leq n \leq N - 1$, $1 \leq \ell < i$; (iii) j_ℓ^1 , where

$1 \leq \ell \leq N - 2$. That is, agents will hold the risky asset, Y , leveraged with promises s_2, \dots, s_{N-1} ; the risky debt contract, j_n (backed by the risky asset), leveraged with promise s_ℓ with $\ell < n$; or a debt security j_k^1 ($1 \leq k \leq N - 2$), which is backed by risky debt.

Note that the j_k^1 contracts are just securities created in the first round of debt collateralization. Apart from this, the only difference from equilibrium with leverage is that all risky debt contracts backed directly by Y are now bought with leverage, and no agent holds Y , leveraged against $n = 1$. Thus, to prove the base case, it suffices to prove the following two lemmas:

Lemma 6. *In the first level of collateralization, no agent will hold Y/j_1 .*

Proof of Lemma 6. The intuition for this lemma is nearly identical to the intuition for lemmas 1 and 2. Suppose for contradiction that some agent, h prefers to hold Y/j_1 . Then, it must be the case that the expected return of holding Y/j_1 is greater than holding Y/j_{N-1} . This implies that we must have

$$\frac{\sum_{i=1}^N \gamma_i(h)(s_i - s_1)}{\hat{p} - \hat{\pi}_1} > \frac{\gamma_N(h)(s_N - s_{N-1})}{\hat{p} - \hat{\pi}_{N-1}} \quad (27)$$

Rearranging and simplifying, we obtain

$$\sum_{i=1}^{N-1} \gamma_i(h)(s_i - s_1)(\hat{p} - \hat{\pi}_{N-1}) + \gamma_N(h)[(s_N - s_1)(\hat{p} - \hat{\pi}_{N-1}) - (s_1 - s_{N-1})(\hat{p} - \hat{\pi}_{N-1})] > 0. \quad (28)$$

Furthermore, we know that the expected return of holding Y/j_1 is greater than holding j_{N-1}/j_1^1 , which gives

$$\frac{\sum_{i=1}^N \gamma_i(h)(s_i - s_1)}{\hat{p} - \hat{\pi}_1} > \frac{\sum_{i=1}^{N-1} \gamma_i(h)(s_i - s_1) + \gamma_N(h)(s_{N-1} - s_1)}{\hat{\pi}_{N-1} - \hat{\pi}_1^1} \quad (29)$$

Rearranging and simplifying, we obtain

$$\sum_{i=1}^{N-1} \gamma_i(h)(s_i - s_1)(\hat{\pi}_{N-1} - \hat{p}) + \gamma_N(h)[(s_N - s_1)(\hat{\pi}_{N-1} - \hat{\pi}_1^1) - (s_{N-1} - s_1)(\hat{p} - \hat{\pi}_1^1)] > 0 \quad (30)$$

A quick check will assure readers that equations (28) and (30) provide a contradiction because the expressions to the left of the $>$ sign are additive inverses and therefore cannot be both strictly greater than 0.

□

Lemma 7. *In the first level of collateralization, any agent buying the promise j_ℓ with $\ell > 1$ will also sell a promise j_k^1 with $1 \leq k < \ell$.*

Proof of Lemma 7. Now suppose that some agent h prefers to hold a promise j_ℓ with $\ell > 1$, but not sell a debt security. Then, it must be the case that the expected return of holding j_ℓ^0 is greater than the expected return of holding j_ℓ^0/j_1^1 . That is,

$$\frac{\sum_{i=1}^{\ell-1} \gamma_i(h)(s_i) + \sum_{i=\ell}^N \gamma_i(h)(s_\ell)}{\hat{\pi}_\ell^0} > \frac{\sum_{i=1}^{\ell-1} \gamma_i(h)(s_i - s_1) + \sum_{i=\ell}^N \gamma_i(h)(s_\ell - s_1)}{\hat{\pi}_\ell^0 - \hat{\pi}_1^1} \quad (31)$$

Note that $\hat{\pi}_1^1 = s_1$, since j_1^1 promises s_1 in all states and is therefore risk-free debt. Thus, rearranging and simplifying 31, we obtain

$$s_1 \left(\hat{\pi}_\ell^0 - \sum_{i=1}^{\ell-1} \gamma_i(h)(s_i) - \sum_{i=\ell}^N \gamma_i(h)(s_\ell) \right) > 0 \implies \hat{\pi}_\ell^0 - \sum_{i=1}^{\ell-1} \gamma_i(h)(s_i) - \sum_{i=\ell}^N \gamma_i(h)(s_\ell) > 0$$

We also know that the expected return of holding j_ℓ^0 must be greater than holding the risk-free asset. Consequently,

$$\frac{\sum_{i=1}^{\ell-1} \gamma_i(h)(s_i) + \sum_{i=\ell}^N \gamma_i(h)(s_\ell)}{\hat{\pi}_\ell^0} > 1 \quad (32)$$

Rearranging and simplifying 32, we obtain

$$\sum_{i=1}^{\ell-1} \gamma_i(h)(s_i) + \sum_{i=\ell}^N \gamma_i(h)(s_\ell) > \hat{\pi}_\ell^0 \implies \sum_{i=1}^{\ell-1} \gamma_i(h)(s_i) + \sum_{i=\ell}^N \gamma_i(h)(s_\ell) - \hat{\pi}_\ell^0 > 0$$

The above clearly cannot happen because we have that the two equations are additive inverses of each other and therefore cannot both be strictly greater than 0. Thus, no agent holding a risky debt contract will prefer to hold the contract unleveraged.

□

A.1.2 Induction Hypothesis

We now assume that the proposition holds for all levels of collateralization T with $T < L$. Specifically, this means that the proposition holds with $L - 1$ levels of collateralization. Looking at this level, we have that agents will hold one of the following assets in equilibrium: (i) Y/j_i , with $L - 1 < i \leq N - 1$, (ii) j_ℓ^l/j_k^{l+1} , with $0 \leq l < L - 1$, $L - 1 - l < \ell < N - l$, and $L - 1 - l \leq k < j$, (iii) j_ℓ^{L-1} , with $1 \leq \ell < N - L + 1$.

A.1.3 Equilibrium at the L^{th} level of collateralization

At the L^{th} level of collateralization, we allow all agents holding j_ℓ^{L-1} (with $1 \leq \ell < N - L + 1$) to sell the promise $j_n^L(j_\ell^{L-1}) = (s_n, j_\ell^{L-1})$ where $1 \leq p < \ell$.

We will prove the following: (i) No agent holds Y/j_M . This implies that the asset j_M/j_{M-1}^1 no longer exists; (ii) No agent holding the debt security j_ℓ^{L-1} with $1 < \ell < N - L + 1$ will do so without leveraged; (iii) No agent will hold j_ℓ^l/j_{L-1-l}^{l+1} , for $0 \leq l < L - 1$ and $L - 1 - l < \ell < N - l$. This implies that all $j_{L-1-l}^{l+1}/A_{L-2-l}^{l+2}$ no longer exist in equilibrium.

Note that the above are the changes between the $L - 1$ and L^{th} levels of collateralization given by the proposition. We break up the proof into three lemmas, corresponding to the three claims listed above.

Lemma 8. *At the L^{th} level of collateralization, no agent will hold Y/j_L .*

Proof of Lemma 8. Suppose for contradiction that some investor h wants to hold Y/j_L . Then the leveraged expected return to this asset must be strictly greater than the expected return to holding Y/j_{N-1} . This means that

$$\frac{\sum_{i=L}^N \gamma_i(h)(s_i - s_L)}{\hat{p} - \hat{\pi}_L} > \frac{\gamma_N(h)(s_N - s_{N-1})}{\hat{p} - \hat{\pi}_{N-1}}. \quad (33)$$

Rearranging and simplifying the above, we obtain

$$\sum_{i=L}^{N-1} \gamma_i(h)(s_i - s_L)(\hat{p} - \hat{\pi}_{N-1}) + [\gamma_N(h)(s_N - s_L)(\hat{p} - \hat{\pi}_{N-1}) - (s_N - s_{N-1})(\hat{p} - \hat{\pi}_L)] > 0 \quad (34)$$

Additionally, holding Y/j_L must have a higher expected return than holding j_{N-1}/j_L^1 . Note

that at the L^{th} level of collateralization, both j_L and j_L^1 are fully securitized so they have the same price. That is $\hat{\pi}_L = \hat{\pi}_L^1$.

$$\frac{\sum_{i=L}^N \gamma_i(h)(s_i - s_L)}{\hat{p} - \hat{\pi}_L} > \frac{\sum_{i=L}^{N-1} \gamma_i(h)(s_i - s_L) + \gamma_N(h)(s_{N-1} - s_L)}{\hat{\pi}_{N-1} - \hat{\pi}_L} \quad (35)$$

Rearranging and simplifying, we have

$$\sum_{i=L}^{N-1} \gamma_i(h)(s_i - s_L)(\hat{\pi}_{N-1} - \hat{p}) + \gamma_N(h)[(s_N - s_L)(\hat{\pi}_{N-1} - \hat{\pi}_L) - (s_{N-1} - s_L)(\hat{p} - \hat{\pi}_L)] > 0 \quad (36)$$

The expressions on the left side of the $>$ sign in equations (34) and (36) are additive inverses, and therefore cannot both be strictly greater than 0. Thus, we have a contradiction and no agent will hold Y/j_L .

□

Lemma 9. *At the L^{th} level of collateralization, every agent holding j_i^{L-1} with $1 < i < N - L + 1$ will sell a promise j_m^L , where $1 \leq m < i$.*

Proof of Lemma 9. Suppose that there exists an agent, h , holding j_i^{L-1} with $1 < \ell < N - L + 1$ and prefers not to sell any promises. Then, it must be the case that the expected return of holding j_ℓ^{L-1} is greater than the expected return of holding j_ℓ^{L-1}/j_1^L . That is,

$$\frac{\sum_{i=1}^{\ell-1} \gamma_i(h)(s_i) + \sum_{i=\ell}^N \gamma_i(h)(s_\ell)}{\hat{\pi}_\ell^{L-1}} > \frac{\sum_{i=1}^{\ell-1} \gamma_i(h)(s_i - s_1) + \sum_{i=\ell}^N \gamma_i(h)(s_\ell - s_1)}{\hat{\pi}_\ell^{L-1} - \hat{\pi}_1^L} \quad (37)$$

Note that $\hat{\pi}_1^L = s_1$, since j_1^L promises s_1 in all states and is therefore risk-free debt. Thus, rearranging and simplifying 37, we obtain

$$s_1 \left(\hat{\pi}_\ell^{L-1} - \sum_{i=1}^{\ell-1} \gamma_i(h)(s_i) - \sum_{i=\ell}^N \gamma_i(h)(s_\ell) \right) > 0 \implies \hat{\pi}_\ell^{L-1} - \sum_{i=1}^{\ell-1} \gamma_i(h)(s_i) - \sum_{i=\ell}^N \gamma_i(h)(s_\ell) > 0$$

We also know that the expected return of holding j_ℓ^{L-1} must be greater than holding the risk-free asset. Consequently,

$$\frac{\sum_{i=1}^{\ell-1} \gamma_i(h)(s_i) + \sum_{i=\ell}^N \gamma_i(h)(s_\ell)}{\hat{\pi}_\ell^{L-1}} > 1 \quad (38)$$

Rearranging and simplifying 38, we obtain

$$\sum_{i=1}^{\ell-1} \gamma_i(h)(s_i) + \sum_{i=\ell}^N \gamma_i(h)(s_\ell) > \hat{\pi}_\ell^{L-1} \implies \sum_{i=1}^{\ell-1} \gamma_i(h)(s_i) + \sum_{i=\ell}^N \gamma_i(h)(s_\ell) - \hat{\pi}_\ell^{L-1} > 0$$

The above clearly cannot happen because we have that the two equations are additive inverses of each other and therefore cannot both be strictly greater than 0. Thus, no agent holding a risky debt contract will prefer to hold the contract unleveraged at the L^{th} level of collateralization. \square

Lemma 10. *At the L^{th} level of collateralization, for all $0 \leq l < L-1$, no agent will hold j_k^l / j_{L-1-l}^{l+1} , where $L-l \leq k < N-l$.*

Proof of Lemma 10. Suppose for contradiction that there exist some agent h who prefers to be in the position stated above. Then, it must be the case that the is greater than the expected return of holding j_k^l / j_{L-1-l}^{l+1} . Thus,

$$\frac{\sum_{i=L-l}^{k-1} \gamma_i(h)(s_i - s_{L-1-l}) + \sum_{i=k}^N \gamma_i(h)(s_k - s_{L-1-l})}{\hat{\pi}_k^l - \hat{\pi}_{L-1-l}^{l+1}} > \frac{\sum_{i=L-l}^{k-1} \gamma_i(h)(s_i - s_{L-l}) + \sum_{i=k}^N \gamma_i(h)(s_k - s_{L-l})}{\hat{\pi}_k^l - \hat{\pi}_{L-l}^{l+1}}. \quad (39)$$

We rearrange and simplify the above to obtain

$$\sum_{i=L-l}^{k-1} \gamma_i(h)\Omega + \sum_{i=k}^N \gamma_i(h)\Psi > 0 \quad (40)$$

where

$$\Omega := (s_i - s_{L-1-l})(\hat{\pi}_k^l - \hat{\pi}_{L-1-l}^{l+1}) - (s_i - s_{L-l})(\hat{\pi}_k^l - \hat{\pi}_{L-l}^{l+1}) \quad (41)$$

and

$$\Psi := (s_k - s_{L-1-l})(\hat{\pi}_k^l - \hat{\pi}_{L-l}^{l+1}) - (s_k - s_{L-l})(\hat{\pi}_k^l - \hat{\pi}_{L-1-l}^{l+1}) \quad (42)$$

Furthermore, it must also be the case that the expected return of holding j_k^l/j_{L-1-l}^{l+1} is greater than the expected return from holding $j_{L-l}^{l+1}/j_{L-1-l}^{l+2}$. It is important to note here that the price of j_{L-1-l}^{l+2} is the same as the price of j_{L-1-l}^{l+1} because at the L^{th} level of collateralization, both have been securitized to the exact same degree, so the two have the same value. Thus, abusing notation, we can write $\hat{\pi}_{L-1-l}^{l+2} = \hat{\pi}_{L-1-l}^{l+1}$. This give us

$$\frac{\sum_{i=L-l}^{k-1} \gamma_i(h)(s_i - s_{L-1-l}) + \sum_{i=k}^N \gamma_i(h)(s_k - s_{L-1-l})}{\hat{\pi}_k^l - \hat{\pi}_{L-1-l}^{l+1}} > \frac{\sum_{i=L-l}^N \gamma_i(h)(s_{L-l} - s_{L-1-l})}{\hat{\pi}_{L-l}^{l+1} - \hat{\pi}_{L-1-l}^{l+1}} \quad (43)$$

Rearranging and simplifying the above inequality, we obtain

$$\sum_{L-l}^{k-1} \gamma_i(h)\Upsilon + \sum_{i=k}^N \gamma_i(h)\Phi > 0, \quad (44)$$

where

$$\Upsilon := (s_i - s_{L-1-l})(\hat{\pi}_{L-l}^{l+1} - \hat{\pi}_{L-1-l}^{l+1}) - (s_{L-l} - s_{L-1-l})(\hat{\pi}_k^l - \hat{\pi}_{L-1-l}^{l+1}), \quad (45)$$

and

$$\Phi := (s_k - s_{L-1-l})(\hat{\pi}_{L-l}^{l+1} - \hat{\pi}_{L-1-l}^{l+1}) - (s_{L-l} - s_{L-1-l})(\hat{\pi}_k^l - \hat{\pi}_{L-1-l}^{l+1}). \quad (46)$$

A quick check will assure the readers that $\Upsilon = -\Omega, \Phi = -\Psi$, a contradiction, meaning equations (40) and (44) cannot both be true. Thus, no agent will hold j_k^l/j_{L-1-l}^{l+1} , where $0 \leq l < L-1$ and $L-l \leq k < N-l$.

□

A.2 Proofs for Contingent Contract Collateralization

Proof of Lemma 3. For contradiction, suppose otherwise. Consider a contract j with price π_j that pays in $K \geq 2$ states. Without loss of generality, suppose it pays s_k in state k . Let k denote the contract that pays s_k in state k , and let π_k denote the price of k . Denote the delivery of contract j by Del_j , and let $\gamma^h \cdot Del_j$ denote the expected payoff of the contract j according to investor h .

Consider an agent h_i whose preferred investment is the contract j (we can assume the preference is strict except for an agent of measure zero). Then it must be that

$$\frac{\gamma^{h_i} \cdot Del_j}{\pi_j} > \frac{\gamma_k^{h_i} s_k}{\pi_k}. \quad (47)$$

If the agent is unwilling to use the contract j as collateral, then the agent prefers the return to buying j over the return to buying j and using it to issue the contract k :

$$\frac{\gamma^{h_i} \cdot Del_j}{\pi_j} > \frac{\gamma^{h_i} \cdot Del_j - \gamma_k^{h_i} s_k}{\pi_j - \pi_k}, \quad (48)$$

where the RHS denotes the return to using j to issue the contract paying in k .

Simplifying equation (48) yields

$$\gamma_k^{h_i} s_k \pi_j > \gamma^{h_i} \cdot Del_j \pi_k,$$

but equation (47) implies

$$\gamma_k^{h_i} s_k \pi_j < \gamma^{h_i} \cdot Del_j \pi_k,$$

which is a contradiction. Thus, if an agent's preferred investment is j , the agent must be willing to use j as collateral to issue some contract k . (From the equations above, at most an agent of measure zero would be indifferent.)

□

Proof of Lemma 4. Pick any state k and denote by k the contract that delivers s_k in state k . Consider a contract j that pays the asset Y dividend s_n in all remaining states $n \neq k$. Any agent can replicate the payoffs to j by buying Y and using it as collateral to issue k . However, the contract j can be used as collateral, so by Proposition 4, the contract paying in $N - 1$ states must cost more than

buying Y and selling the one-state contract: $\pi^j > p - \pi^k$. Similarly, the contract k that pays in state k can be replicated by buying Y and selling the j contract that pays in $N - 1$ states. From the previous inequality, it follows that $p - \pi^j < \pi^k$, so it is strictly cheaper to buy Y and sell the $N - 1$ state contract j . It follows that no agent will want to buy the contract that pays in only one state, and such a contract backed directly by Y will not be traded in equilibrium.

The same argument applies to using X as collateral to back a contract that pays in just one state. □

Proof of Proposition 7. We proceed by induction.

At the first level of contract collateralization, level 0 contracts must pay in at least 2 states. This is the result from Lemma 4. Now consider a level 0 contract that pays in exactly 2 states. By Lemma 3, this contract will be used to issue a level 1 contract that pays in 1 state, which is an Arrow-Debreu (AD) security. The agent buying the level 0-contract and selling the level-1 contract obtains payment in exactly one state, so this is also an Arrow-Debreu security. The price of the level-0 contract is then equal to the sum of price of these two positions, and is thus priced by Arrow-Debreu securities.

Now suppose the statement of the proposition is true at L levels of contract collateralization.

Consider going from L levels of collateralization to $L + 1$ levels of contract collateralization. By the inductive step, a level-0 contract that pays in $L + 1$ states can be priced by AD securities. But since there are $L + 1$ levels of collateralization, a level-0 contract that pays in $L + 2$ states can be used to issue a contract that pays in $L + 1$ states, and there are enough levels of collateralization so that the level-1, $L + 1$ state contract can be split into the exact same payoffs as achieved in L levels collateralization. Therefore, the level-1 $L + 1$ state contract is priced by AD securities. Since, the price of the $L + 2$ state level-0 contract is exactly the price of the $L + 1$ state level-1 contract and the remainder (an AD security), it follows that the $L + 2$ state level-0 contract is priced by AD securities.

Suppose an agent strictly prefers to use X or Y to issue a level-0 contract that pays in $L + 1$ states, denoted $s = \{s_i\}_{i_1, \dots, i_{L+1}}$, where the i indicates the states where payments are promised. Then, the expected payoff to the agent of such a position is strictly greater than selling a level 0 contract that pays in $s' = s \cup \{s_m\}$, or buying a contract that pays only in s_m (which is an AD security). Let γ_σ denote the marginal utility for the agent of consumption in states σ and π_s^0 denote

the price of the the level-0 contract paying in states s . So,

$$\frac{\gamma_{\mathcal{N}\setminus s}}{p - \pi_s^0} > \frac{\gamma_{\mathcal{N}\setminus s'}}{p - \pi_{s'}^0}, \quad \frac{\gamma_{\mathcal{N}\setminus s}}{p - \pi_s^0} > \frac{\gamma_{s'\setminus s}}{\pi_{s'\setminus s}}$$

Rearranging, we see that the first and second inequalities can be written as

$$\gamma_{\mathcal{N}\setminus s}(p - \pi_{s'}^0) > \gamma_{\mathcal{N}\setminus s'}(p - \pi_s^0), \quad \gamma_{\mathcal{N}\setminus s}(\pi_{s'\setminus s}) > \gamma_{s'\setminus s}(p - \pi_s^0)$$

Since the contracts are priced by AD securities, $\pi_{s'}^0 = \pi_{s'\setminus s} + \pi_s^0$. Additionally $\gamma_{\mathcal{N}\setminus s} = \gamma_{\mathcal{N}\setminus s'} + \gamma_{s'\setminus s}$. Thus, adding the two inequalities together yields

$$\gamma_{\mathcal{N}\setminus s}(p - \pi_s^0) > \gamma_{\mathcal{N}\setminus s}(p - \pi_s^0)$$

which is a contradiction

Similarly, suppose that an agent prefers to use a level $\ell < L + 1$ contract to issue a contract paying in exactly $L + 1 - \ell$ states (where from the induction step, no agent will want to issue contracts paying in a smaller number of states). Again, denote the $L + 1 - \ell$ states by s . The expected payoff must be strictly greater than selling a contract that pays in $s' = s \cup \{s_m\}$ or buying the contract paying in only s_m . Suppose also that the payoff set of the level ℓ contract is Σ .

$$\frac{\gamma_{\Sigma\setminus s}}{\pi_\sigma^\ell - \pi_s^{\ell+1}} > \frac{\gamma_{\mathcal{N}\setminus s'}}{\pi_\sigma^\ell - \pi_{s'}^{\ell+1}}, \quad \frac{\gamma_{\Sigma\setminus s}}{\pi_\sigma^\ell - \pi_s^{\ell+1}} > \frac{\gamma_{s'\setminus s}}{\pi_{s'\setminus s}}$$

By the induction step, both the level $\ell + 1$ contract paying in $L + 1 - \ell$ states and the one paying in $L + 2 - \ell$ states can be priced by AD securities, so $\pi_{s'\setminus s} + \pi_s^{\ell+1} = \pi_{s'}^{\ell+1}$. Rearranging the two inequalities and summing them yields the contradiction, as above.

It follows that at $L = N - 2$ levels of collateralization is the most needed. Any contract that pays off in $N - 1$ states or less are priced by AD securities. Agents buying either the X or Y asset can use it to issue a contract that pays in $N - 1$ states, keeping the payoff in one state—which is an AD security. Therefore, the X and Y assets are priced by AD securities, and there are enough levels of collateralization to completely split the payment of the assets into AD securities.

□

A.3 Proof for Tranching

The full statement of Proposition 8: *The senior-subordinated tranching equilibrium is equivalent to equilibrium with complete debt collateralization. That is, there exists a bijective mapping of assets and prices from the debt collateralization equilibrium to the senior-subordinated tranching equilibrium such that the buyers of assets remain the same. Specifically,*

1. Any agent buying Y/j_{N-1} (collateralization) will buy T_N (senior-subordinated tranching).
2. Any agent holding j_n^l/j_{n-1}^{l+1} with $N > n > 1$ (collateralization) will buy T_n (senior-subordinated tranching).
3. Any agent holding j_1^l (collateralization) will buy T_1 (down-tranching).
4. Letting q_N denote the price of T_N , senior-subordinated tranching equilibrium will have (i) $q_N = \hat{p} - \hat{\pi}_{N-1}$, (ii) $q_n = \hat{\pi}_n^{N-n-1} - \hat{\pi}_{n-1}^{N-n}$, (iii) $q_1 = \hat{\pi}_1^{N-2} = s_1$, where \hat{p} and $\hat{\pi}_j^k$, are the equilibrium prices for the asset and debt securities in the complete collateralization equilibrium, respectively.

Proof of Proposition 8. This follows because the expected return of holding T_n in the senior-subordinated tranching equilibrium is the same as holding j_n^l/j_{n-1}^{l+1} in the collateralization equilibrium, when $N > n > 1$. Similarly, the expected return of Y/j_{N-1} is identical to that of T_N ; the expected return to holding q_1 is exactly the return of j_1^{N-2} □

B Extensions in the Static Model

B.1 Parameter Robustness and Comparative Statics

Robustly, debt collateralization decreases average margins by shifting agents to high-leverage contracts, decreases margins on high-leverage contracts, decreases spreads, and increases the asset price. We consider a wide range of payoffs pairs, with $M \in (0.2, 0.95)$ and $D \in (0.05, 0.95 * M)$, and a broad range of belief parameterizations for $\zeta = 1/3, 1, 3$. Our results—that debt collateralization decreases margins, increases prices, and decreases spreads—hold for every combination. (Our results extend for ζ outside the range, but we omit the figures.)

Figures 4-6 display how moving from leverage to debt collateralization affects prices, the economy-wide average margin, and the interest rate i_M on the risky debt. The figures plot these

changes for pairs of payoffs (M, D) . The figures respectively use a belief parameterization with $\zeta = 3, 1, \frac{1}{3}$. For each of these cases, for every pair (M, D) , debt collateralization increases asset prices, decreases economy-wide average margins, and decreases the interest rate on risky debt. The effects on prices and spreads are larger when ζ is larger.

Marginal Utilities/Beliefs Because it is the least intuitive of the parameters, we discuss the role of the marginal utilities/belief parameterization in greater detail. The parameter ζ determines the relative frequency of optimists and pessimists in the economy; equivalently, the frequency of pessimists can be interpreted as the relative demand for assets that pay in bad states (negative-beta assets), perhaps from hedging needs or from risk aversion. High ζ corresponds to relatively more pessimists and low ζ to more optimists (with $\zeta > 1$, γ 's are convex; $\zeta < 1$, concave). Increasing ζ , has the following consequences: (i) in both financial environments risky spreads increase and risky and safe margins decrease; (ii) moving from leverage to debt collateralization results in a bigger change in spreads; (iii) average margins under leverage converge to the safe margin (for low ζ (many optimists), more agents use risky margins); (iv) moving from leverage to debt collateralization, margins decrease by a greater amount; (v) moving from leverage to debt collateralization, the percent change in p increases (nonlinearly) with ζ so that p increases by more when ζ is high.

Comparative Statics Additionally, the model provides several interesting comparative statics for each parameter. Figure 7 plots comparative statics for prices, spreads, and margins. Figures 7a and 7b show how varying the down-payoff D affects the change in prices and spreads. We fix $M = 0.9$ (results are robust to varying M) and plot results for three values of ζ . Debt collateralization always increases asset prices and decreases debt spreads, and the effects are greatest when D is small and when ζ is large. The effect from D is intuitive: as D increases, the risk-free promise becomes closer to the risky promise, and thus the value from using the risky promise as collateral diminishes (the risk-free debt is already good for leverage). Figure 7c shows how varying the middle payoff M affects changes in economy-wide average margins, setting $D = 0.8$ (again, results are robust to varying D). As M increases the decrease in average margins when moving to debt collateralization becomes much greater, precisely because the margin on the risky debt decreases as M increases,

and average margins decrease because investors shift toward using the risky debt instead of the risk-free debt.

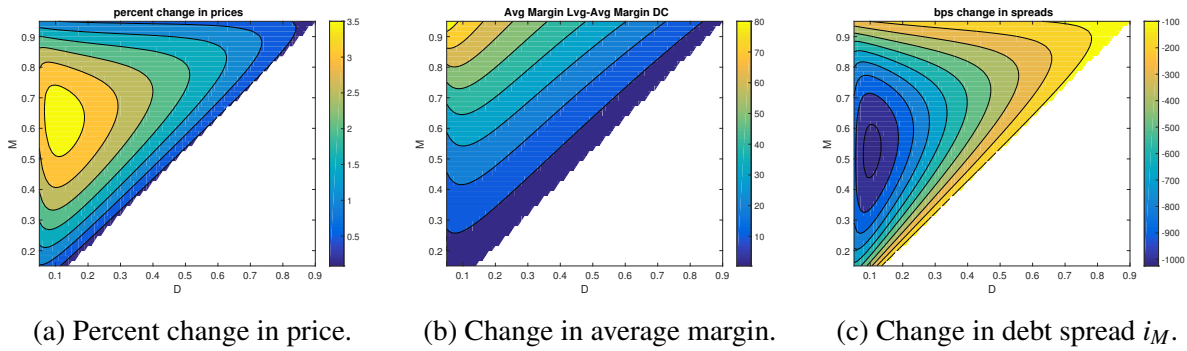


Figure 4: Equilibrium changes moving from leverage to debt collateralization. Beliefs exponent $\zeta = 3$.

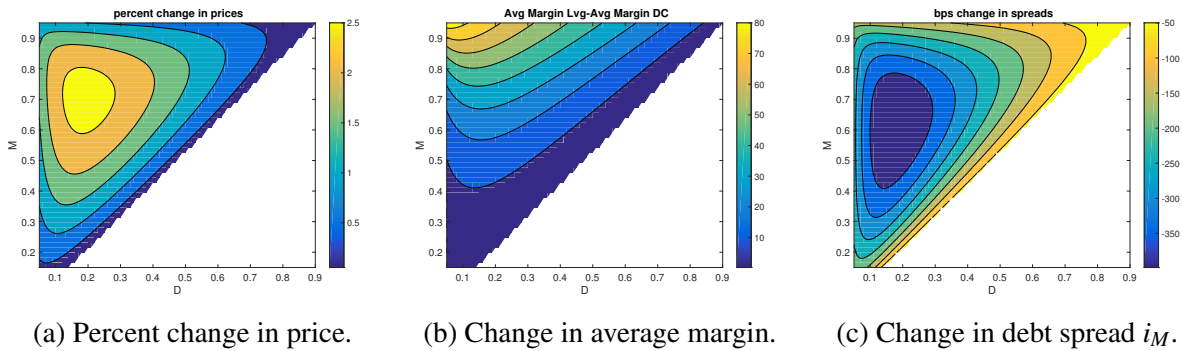


Figure 5: Equilibrium changes moving from leverage to debt collateralization. Beliefs exponent $\zeta = 1$.

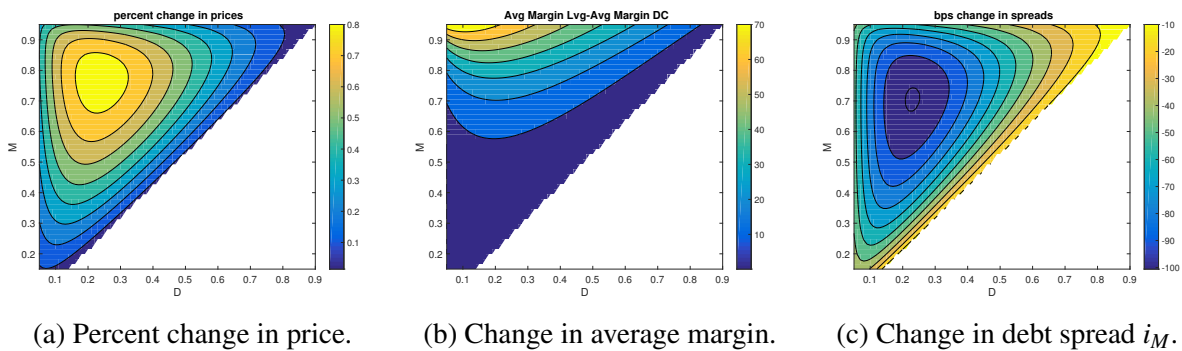
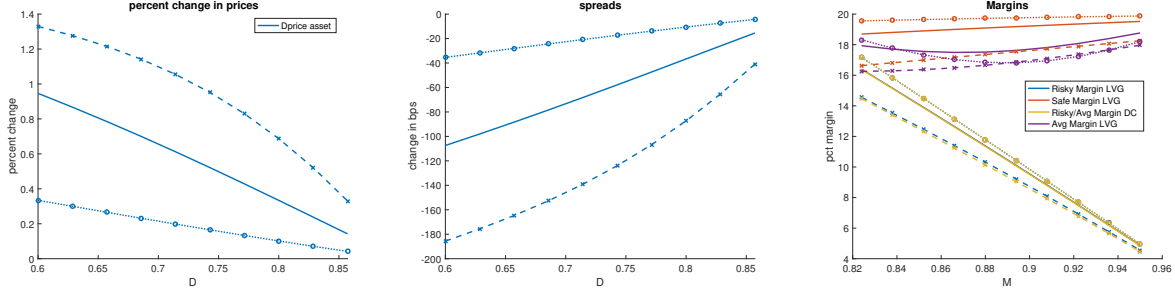


Figure 6: Equilibrium changes moving from leverage to debt collateralization. Beliefs exponent $\zeta = 1/3$.



(a) Percent change in price, (b) Change in debt spread i_M , (c) Change in average margin, varying D ; $M = 0.9$. varying D ; $M = 0.9$. varying M ; $D = 0.8$.

Figure 7: Comparative statics. Plain line is $\zeta = 1$; ‘-x’ is $\zeta = 3$; ‘x’ is $\zeta = 1/3$.

B.2 Debt Collateralization Does Not Necessarily Increase Prices

We demonstrate that prices do not necessarily increase with the introduction of securitization when beliefs are weakly monotonic. In this example, the beliefs of the marginal agents under each regime are very different—there is a discontinuous jump in beliefs. In contrast, in the examples in the robustness exercises with beliefs defined by $\gamma = h^\zeta$, beliefs vary smoothly (and not very much) over the relevant range where marginal buyers may fall.

As before, we let $M = 0.3$ and $D = 0.1$, be the payouts of asset Y in states M and D . We also define the following marginal agents: $h_1 = 0.6$, $h_2 = 0.65$, $h_3 = 0.69$. Beliefs are given as follows:

$$\text{For } h \leq h_1: \gamma_U(h) = 1 - (1 - h_1)^2, \gamma_M(h) = h_1(1 - h_1)^2, \gamma_D(h) = (1 - h_1)^3.$$

$$\text{For } h \in (h_1, h_2): \gamma_U(h) = 1 - (1 - h)^2, \gamma_M(h) = h(1 - h)^2, \gamma_D(h) = (1 - h)^3.$$

$$\text{For } h \in [h_2, h_3): \gamma_U(h) = 1 - (1 - h_2)^2, \gamma_M(h) = h_2(1 - h_2)^2, \gamma_D(h) = (1 - h_2)^3.$$

$$\text{Finally, for } h > h_3: \gamma_U(h) = 1 - (1 - (h - (h_3 - h_2)))^2, \gamma_M(h) = (h - (h_3 - h_2))(1 - (h - (h_3 - h_2)))^2, \gamma_D(h) = (1 - (h - (h_3 - h_2)))^3.$$

Equilibrium in the economy with leverage has price $p = 0.894$ while equilibrium in the debt collateralization economy has price $\hat{p} = 0.888$.¹³ Introducing debt collateralization in this case causes the price to *decrease*. The reason is that the leveraged return on debt has increased sufficiently, which increases the required return for investing in the risky asset, decreasing its price.

¹³Furthermore, $h_M = 0.715, h_D = 0.669, h_J = 0.534, \pi_M = 0.287$ and $\hat{h}_M = 0.682, \hat{h}_J = 0.583, \hat{\pi}_M = 0.287$.

B.3 A Numerical Example With 3 Agents

We provide an example with 3 agents (not the only possible one) that replicates the numerical example with a continuum of agents. There are 3 agents with log-utility over consumption in period-1, $u_i(c) = \log(c)$, and the three states be equiprobable. Agents are endowed with units of the safe asset X and the risky asset Y , and endowments are as follows: $x^1 = 0.212$, $x^2 = 0.185$, and $x^3 = 0.603$ units of the safe asset X , and the risky asset Y is endowed entirely to agent 1, $y_0^1 = 1$. Agents have future endowments given by: $e^1 = (14.41, 38.18, 100)$, $e^2 = (100, 1, 100)$, and $e^3 = (3.25, 66.67, 100)$. Assets are priced according to the standard marginal analysis using the marginal utilities of the buyers. Two contracts, j_M and j_D , continue to be traded in equilibrium.

Given these parameters, in equilibrium with leverage, agent 3 buys all of the risky asset and issues contracts promising M and D ; agent 2 buys risky contract M ; and agent 1 buys contracts M and D and all of the safe asset X . In equilibrium with debt collateralization, agent 3 buys the asset leveraged with M ; agent 2 buys the risky debt leveraged with D ; agent 1 buys safe assets. Coincidentally, the asset prices p and \hat{p} and debt prices π_M and $\hat{\pi}_M$ are exactly the same as in the economy in Section 3.

C Price Volatility in a Dynamic Model

In this section we examine how debt collateralization affects volatility and default. [Geanakoplos \(2003, 2010\)](#) demonstrate that using an asset as collateral creates a “Leverage Cycle” in which asset prices become more volatile because of fluctuations in the asset’s collateral value and the distribution of investors’ wealth.

C.1 Setup in the Dynamic Model

We consider a dynamic variation of the model in Section 3 with three periods, $t = 0, 1, 2$ following [Geanakoplos \(2003, 2010\)](#). Uncertainty in the payoffs of Y is represented by a tree $S = \{0, U, M, D, UU, MU, MD, DU, DD\}$, illustrated in Figure 8. The asset pays only at $t = 2$ with payoffs d_s^Y . To simplify, we will normalize the asset payoffs so that $d_{UU}^Y = d_{MU}^Y = d_{DU}^Y = 1$. Thus the possible “down payoffs” of the asset are d_{MD}^Y and $d_{DD}^Y < d_{MD}^Y$. In other words, the payoff

tree is binary at $t = 1$ with a worse possible realization at state M than at D , and at $t = 0$ there is uncertainty about what the minimum possible asset payoff will be.

The risky asset Y has price p_0 at $t = 0$ and prices p_M and p_D in states M and D in $t = 1$. (In state U the price is trivially 1.) Just as before, we first look at an economy where leverage is the only financial innovation and then move on to explore the consequences of debt collateralization. Endogenously, all financial contracts traded in equilibrium are one-period contracts.

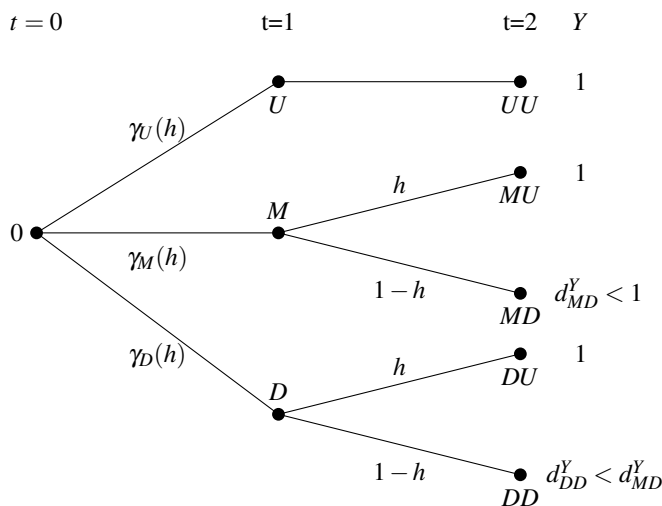


Figure 8: Payoff tree for risky asset in dynamic three-state model.

C.2 The Dynamic Economy with Leverage

With leverage, the dynamic equilibrium is essentially different from the static equilibrium because of the interaction between prices and leverage across time. However, the equilibrium regimes in each state resemble the equilibrium regime in the static economy of Section 3. The dynamic equilibrium with leverage is as follows.

In equilibrium, at time 0 there are three marginal agents, h_{M0} , h_{D0} , and h_{J0} . Agents $h > h_{M0}$ buy the risky asset and promise p_M (i.e., they sell the contract j_{p_M}), which is a risky promise (the contract j_{p_M} delivers $p_D < p_M$ in state D); agents $h \in (h_{D0}, h_{M0})$ buy the risky asset and promise p_D (i.e., they sell the contract j_{p_D}), which is a risk-free promise; agents $h \in (h_{J0}, h_{D0})$ buy the risky debt j_{p_M} ; and agents $h < h_{J0}$ buy risk-free asset X and risk-free debt j_{p_D} . Unlike in a binomial economy, there is a possibility of default in the down state D because agents $h \in (h_{M0}, 1)$ cannot

pay off the entirety of their debt, having promised p_M when the asset is only worth $p_D < p_M$. We denote the price of the risky debt j_{p_M} by π_0 , which has interest rate $i_0 = \frac{p_M}{\pi_0} - 1$.

At time 1, agents receive news about the economy, borrowers repay their debts (margin calls occur) and the remaining agents trade assets and issue new promises. Because the economy is binomial at time 1, in equilibrium agents trade only risk-free contracts. In equilibrium there is one marginal investor in each state, with the remaining optimistic investors buying the risky asset against the maximal risk-free promise possible given the state. Thus, in state M there is a marginal investor h_{MM} . Investors $h > h_{M0}$ have zero wealth after repaying their promise. Investors $h \in (h_{MM}, h_{M0})$ buy the risky asset and promise M , which is the minimum payoff at $t = 2$. Investors $h < h_{MM}$ buy risk-free assets. In state D there is one marginal investor h_{DD} . Investors $h > h_{D0}$ have zero wealth after repaying their promise. Investors $h \in (h_{DD}, h_{D0})$ buy the risky asset and promise D , which is the minimum payoff at $t = 2$. Investors $h < h_{DD}$ buy risk-free assets.¹⁴

It is instructive to compare the 3-state model with leverage to the standard binomial model. Compared to the binomial model, crashes in this economy are larger. One reason the crash in D is so large is that investors who bought the risky debt are receiving less than the face value, and less than they invested. This “default mechanism” depresses p_D because remaining investors have less wealth. Larger crashes occur precisely because the three-state model with collateralization has more bankrupt agents at time 1 in the down state when compared to the two-state models.¹⁵

C.3 The Dynamic Economy with Debt Collateralization

Given our results in the previous section, we can easily characterize equilibrium in the dynamic model with debt collateralization. In equilibrium there are two marginal agents at time 0, \hat{h}_{M0} , and \hat{h}_{J0} . Agents $h > \hat{h}_{M0}$ buy the risky asset and promise p_M ; agents $h \in (\hat{h}_{J0}, \hat{h}_{M0})$ buy the risky debt with promise p_M and use it as collateral to promise p_D ; and agents $h < \hat{h}_{J0}$ buy the risk-free asset X and the risk-free debt (with promise p_D). In equilibrium, at time 1 there is one marginal investor in each state as discussed previously. Notice that if the economy is in state M at time 1, then agents

¹⁴Since we do not know the positions of h_{MM} and h_{DD} relative to the marginal investors at time 0, there are several possible equilibrium cases. These cases, as well as the equations defining equilibrium, are listed in Appendix C.5.

¹⁵We isolate the impact of the default mechanism in Appendix C.4 by considering a surprise bailout in $s = D$ to replace the wealth lost to default. Appendix C.4 also compares the 3-state dynamic model to corresponding binomial models and shows that in each case the price crash in the down state of the 3-state world is greater than the price crash in any of the two-state models and that the biggest price crash is obtained in the case of debt collateralization.

$h \in (\hat{h}_{M0}, 1)$ will be bankrupt; if the economy is in state D at time 1, agents $h \in (\hat{h}_{J0}, 1)$ will be bankrupt.¹⁶

We solve the system numerically with $\zeta = 2$, and payoffs $d_{MD}^Y = 0.3$ and $d_{DD}^Y = 0.1$. Table 2 gives the equilibrium with debt collateralization and compares it to the equilibrium with leverage. The price crash in states M and D are larger with debt collateralization. Note that the “default mechanism” is effectively much greater with debt collateralization since all debt in the economy is fully collateralized and leveraged. Rather than being poorer in the down state, agents holding risky debt will be completely out of the market. In this case, debt collateralization leads to even more volatility since the agents buying the asset in the down state will be more pessimistic.

Table 2: Dynamic Equilibrium with Debt Collateralization and with Leverage

	Leverage	Collateralization ($\hat{\cdot}$)
p_0	0.805	0.814 \uparrow
p_M	0.663	0.659 \downarrow
p_D	0.434	0.431 \downarrow
π_0	0.595	0.611 \uparrow
h_{M0}	0.963	0.888 \downarrow
h_{D0}	0.823	–
h_{J0}	0.719	0.789 \uparrow
h_{MM}	0.720	0.717 \downarrow
h_{DD}	0.610	0.606 \downarrow
M Crash	17.58 %	19.02% \uparrow
D Crash	46.00 %	47.09% \uparrow

Crashes in states M and D increase by 1.44% and 1.09%. Our result that debt collateralization increases volatility is closely related to previous work studying collateral values and volatility.¹⁷ We have shown that debt collateralization increases the collateral value of debt contracts and of the risky asset, and as a result, asset price volatility increases because debt collateralization increases fluctuations in both collateral values and the distribution of wealth.¹⁸

¹⁶Note that we do not know the position of \hat{h}_{MM} relative to the positions of the other marginal investors at time 0, but we do know the relative position of \hat{h}_{DD} . The equations defining equilibrium are in Appendix C.5.

¹⁷Fostel and Geanakoplos (2012a) show that asset price volatility increases when agents can tranche assets. Tranching increases the collateral value of the risky asset, and in a dynamic setting the “Tranching Cycle” exhibits larger fluctuations in collateral values and in the distribution of wealth.

¹⁸In general, debt collateralization improves welfare for pessimistic agents, whose wealth increases while the price of risk-free assets remains the same, but decreases welfare for optimistic agents because the risky asset and risky debt

The model is highly nonlinear and so it is difficult to characterize in general how debt collateralization affects price volatility. Thus, for comparative dynamics we focus on how the model parameters affect how volatility changes (how the M and D crashes change) when we introduce debt collateralization. Robustly, the D crash increases with debt collateralization. However, when beliefs at $t = 0$ are very convex, the M crash may decrease with debt collateralization. Hence, for a wide range of parameters (which almost surely includes the empirically relevant cases), debt collateralization increases volatility and fat tails.

The complete results are in the following sections, but we briefly summarize the results here. When beliefs are concave ($\zeta < 1$) indicating a larger proportion of optimism, the changes in both crashes are larger for larger D . Furthermore, the change in both crashes are smaller for smaller ζ ; with more optimism, price crashes are already large and debt collateralization does little to amplify fluctuations. When beliefs are convex ($\zeta > 1$) indicating a larger proportion of pessimism, debt collateralization can *decrease* the crash in the M state for moderate levels of M but *increase* the crash for high M . For high M the price increases ($p_M > p_0$), so the price increase in state M would be muted. Changes in crashes can also be much larger (over 3%).

C.4 Default in the Dynamic 3-state Model

This section isolates the role of default in the dynamic model in two ways. First, it considers a surprise injection of wealth to bailout agents who lost money to default. Second, it maps the 3-state model onto binomial models, in which there is no default, and compares equilibrium in each case.

We solve our model numerically with $\gamma_U(h) = h^2$, $\gamma_M(h) = h^2(1 - h^2)$, and payoffs $d_{MD}^Y = 0.3$ and $d_{DD}^Y = 0.1$. For these parameters, h_J holds the risk-free asset in state M and the risky asset in state D . Notice that the probability of down states (M or D) is $1 - h$, similar to parametrization in

are more expensive. It is not necessarily the case in this model that increased volatility is bad for welfare. Given the stylized nature of this model, it is worth considering the effects of volatility in a richer model. Our model leaves out many important factors, such as production, investment, cash flow problems, agency issues, and bankruptcy costs, just to name a few. [Geanakoplos \(2010\)](#) discusses a number of these issues in greater detail. [Phelan \(2016\)](#) uses a richer model to show that financial leverage creates a pecuniary externality through excess volatility, resulting in an economy with higher likelihood of crises and recessions and lower welfare. One can suspect that incorporating the mechanisms of our paper into a richer framework with any of the aforementioned issues would result in welfare losses from increased volatility.

binomial models. The marginal investors and prices in equilibrium are:

$$\begin{aligned} h_{M0} &= 0.963, & h_{D0} &= 0.823, & h_{J0} &= 0.719, & p_0 &= 0.805, & \pi_0 &= 0.595, \\ h_{MM} &= 0.720, & h_{DD} &= 0.610, & p_M &= 0.663, & p_D &= 0.434, & i_0 &= 2.25\%. \end{aligned}$$

The percent price drop, given by $1 - \frac{p_s}{p}$, is 17.6% in state M , and 46% in state D .

C.4.1 The Default Mechanism

We demonstrate the impact that default has on asset prices at time 1 by considering the counterfactual scenario. We now suppose that in the down state, holders of risky debt receive an unexpected, exogenous wealth increase at time 1: suppose that the holders of risky debt are compensated the difference between p_M and p_D . This wealth shock is unexpected; it does not change the equilibrium at time 0 and only at time 1. In the down-state, we now have $h \in (h_{J0}, h_{D0})$ holding p_M units of wealth and $h \in (0, h_{J0})$ holding $1 + p_0$ units of wealth. The marginal investor and market clearing equations defining equilibrium are:

$$\frac{\gamma_U(h_{DD})(1 - d_{DD}^Y)}{p_D - d_{DD}^Y} = 1, \text{ and } \frac{(h_{D0} - h_{J0}) \left(\frac{1+p}{\pi_0} \right) p_M}{p_D - d_{DD}^Y} + \frac{(h_{J0} - h_{DD})(1 + p_0)}{p_D - d_{DD}^Y} = 1.$$

Using the same specifications as before, as well as the results for $h_{D0}, h_{J0}, \pi_0, p_0$ and p_M from the previous subsection, we find that $h_{DD} = 0.598, p_D = 0.638$, down crash = 33.08%.

Without this injection of cash, $p_D = 0.607$ and the crash was 36.33%. The wealth increase, offsetting the default mechanism, increases the asset price in D and lowers the volatility. However, it is important to note that this result only occurs if agents do not expect the wealth shock. If agents anticipated the wealth increase at time $t = 0$, then the increase in the p_D price will lead to higher margins at the initial time period and the expectation that all debt is actually risk-free.

C.4.2 Comparison With Dynamic Two-State Model

We can compare the volatility in the dynamic three-period model to the dynamic 2-period model in several ways. We can normalize the expected payoff of the asset, normalize beliefs in the up state, as well as normalize the beliefs in the downstate.

We first consider normalization by the expected payoff of the asset. We use $\gamma_U(h) = h$ and $\gamma_M(h) = h(1-h)$ for the probabilities in the three-state model. We want to set the belief of the upstate, $\xi(h)$, in the two-state model so that for every agent, the ultimate expected payout of the risky asset is the same in the two models. That is,

$$h + h^2(1-h) + h(1-h)^2 + h(1-h)^2\alpha d_{DD}^Y + (1-h)^3 d_{DD}^Y = \xi(h) + (1-\xi(h))\xi(h) + (1-\xi(h))^2 d_{DD}^Y,$$

where $d_{MD}^Y = \alpha d_{DD}^Y$. Solving the above, we find that

$$\varphi_i = 1 + \frac{\sqrt{(d_{DD}^Y - 1)(1-h)^2(d_{DD}^Y - 1 + (\alpha - 1)d_{DD}^Y h)}}{d_{DD}^Y - 1}.$$

Letting $\alpha = 3$, we can solve for equilibrium in the two-state dynamic model.

When we normalize the belief about the up state, we have $\xi(h) = \gamma_U(h) = h$. Alternatively, if we normalize the belief about the down state, we need $1 - \xi(h) = \gamma_D(h)$. In summary, we obtain the following differences by normalization.

	Dynamic 3-State		Dynamic 2-State		
	Collateralized debt	No Collateralization	Expected Payout	Belief Up	Belief Down
p_0	.970	0.9531	0.9470	0.9351	0.9876
p_D	.602	0.6069	0.6339	0.6116	0.7314
crash	37.91%	36.33%	33.06%	34.593 %	25.94%

Note that for every normalization, the price crash in the down state of the three state world is greater than the price crash in any of the two-state models and that the biggest price crash is obtained in the case of debt collateralization. Furthermore, the price of the asset in the down state is lowest in the three-state world with debt collateralization. This phenomenon occurs precisely because the three-state model with collateralization has more bankrupt agents at time 1 in the down state when compared to the two-state models. Thus, the agents who are left to buy the asset are more pessimistic and do not value the asset as highly.

C.4.3 Comparative Dynamics

Figures 9-10 show the percent change in the crashes in the M and D states for (M, D) pairs. When ζ is not too high, both crashes are larger with debt collateralization. However, for high ζ ($\zeta \geq 3.2$ in

this case), the crash in the M -state could decrease with debt collateralization; the D -crash increases. While debt collateralization often increases both crashes, but not always, we have not been able to solve for a set of parameters where the D crash decreases.

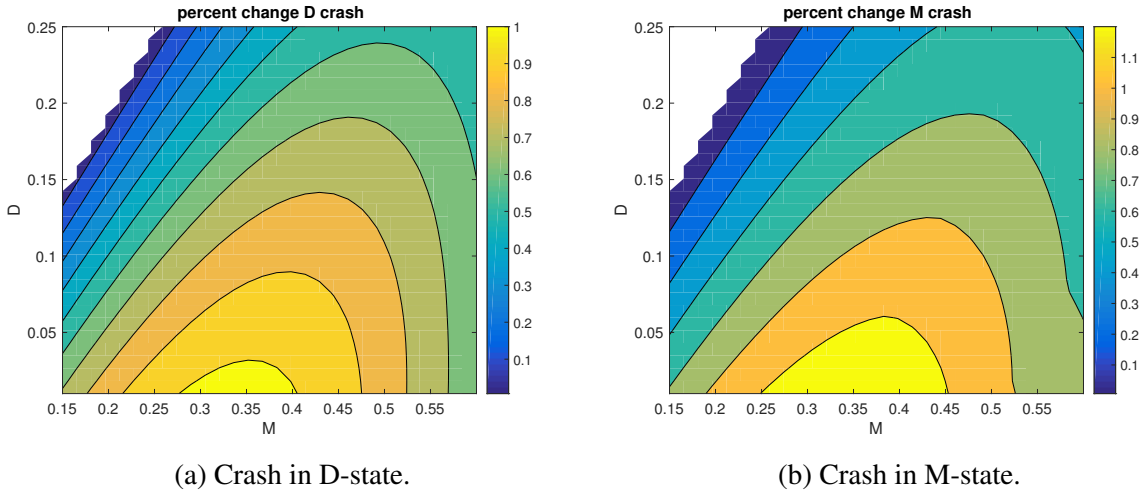


Figure 9: Percent Changes in crashes moving from leverage to debt collateralization. Beliefs exponent $\zeta = 1$.

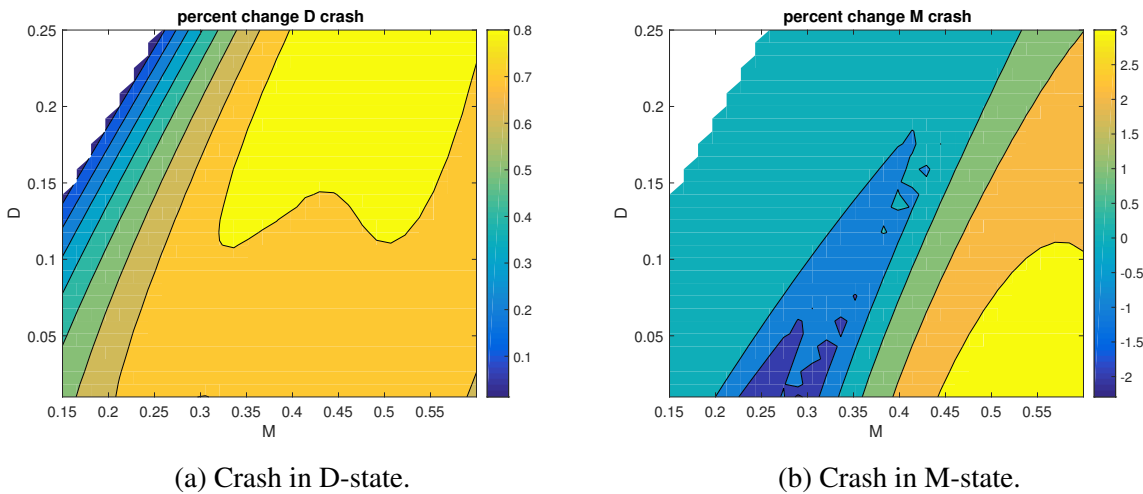


Figure 10: Percent changes in crashes moving from leverage to debt collateralization. Beliefs exponent $\zeta = 3.2$.

C.5 Equilibrium Conditions in the Dynamic Model

C.5.1 Equilibrium Conditions with Leverage: Section C.2

For a few marginal investors, we have no doubt about their course of action. The equations defining them are as follows.

Marginal Investors, known

h_{M0} : indifferent between leveraging against p_M and p_D at time $t = 0$. If at time $t = 1$ we are in state D , then h_{M0} is no longer in the market. If at $t = 1$ we are in state M , h_{M0} will choose to hold the risky asset because he is the most optimistic investor in the market. Thus, we have

$$\frac{\gamma_U(h_{M0})(1 - p_M)}{p_0 - \pi_0} = \frac{\gamma_U(h_{M0})(1 - p_D)}{p_0 - p_D} + \left(\frac{\gamma_M(h_{M0})(p_M - p_D)}{p_0 - p_D} \right) \left(\frac{\gamma_U(h_{M0})(1 - d_{MD}^Y)}{p_M - d_{MD}^Y} \right).$$

The above equates the marginal utility divided by payoff of holding the risky asset leveraged against p_M and the marginal utility divided by the payoff of holding the asset leveraged against p_D .

h_{MM} : Indifferent between risky asset and riskless asset given the realization of state M at $t = 1$. Since this marginal investor only exists at time $t = 1$ in state M . There is no ambiguity.

$$\frac{\gamma_U(h_{MM})(1 - d_{MD}^Y)}{p_M - d_{MD}^Y} = 1$$

h_{DD} : Indifferent between risky asset and riskless asset given the realization of state D at $t = 1$. h_{DD} also only exists at time $t = 1$ in state D .

$$\frac{\gamma_U(h_{DD})(1 - d_{DD}^Y)}{p_D - d_{DD}^Y} = 1$$

Marginal Investors, unknown

h_{D0} : indifferent between leveraging against p_D and holding risky debt at time $t = 0$. If at time $t = 1$ the world is in state D , h_{D0} will choose to hold the risky asset. But, at time $t = 1$ in state M , h_{D0} can either choose to hold the risky asset or the risk-free asset. Let h_{D0} hold the risky asset. To simplify notation, let $\gamma_U^{D0} = \gamma_U(h_{D0})$, $\gamma_M^{D0} = \gamma_M(h_{D0})$ and $\gamma_D^{D0} = \gamma_D(h_{D0})$.

Then, equating payoffs (multiplied by continuation values, we have:

$$\begin{aligned} & \frac{\gamma_U^{D0}(1-p_D)}{p_0-p_D} + \left(\frac{\gamma_M^{D0}(p_M-p_D)}{p_0-p_D} \right) \left(\frac{\gamma_U^{D0}(1-d_{MD}^Y)}{p_M-d_{MD}^Y} \right) \\ &= \frac{\gamma_U^{D0}p_M}{\pi_0} + \left(\frac{\gamma_M^{D0}(p_M)}{\pi_0} \right) \left(\frac{\gamma_U^{D0}(1-d_{MD}^Y)}{p_M-d_{MD}^Y} \right) + \left(\frac{\gamma_D^{D0}p_D}{\pi_0} \right) \left(\frac{\gamma_U^{D0}(1-d_{DD}^Y)}{p_D-d_{DD}^Y} \right) \end{aligned}$$

When h_{D0} holds the risk-free asset, we have:

$$\frac{\gamma_U^{D0}(1-p_D) + \gamma_M^{D0}(p_M-p_D)}{p-p_D} = \frac{(\gamma_U^{D0} + \gamma_M^{D0})p_M}{\pi_0} + \left(\frac{\gamma_D^{D0}p_D}{\pi_0} \right) \left(\frac{\gamma_U^{D0}(1-d_{DD}^Y)}{p_D-d_{DD}^Y} \right)$$

h_{J0} : indifferent between holding risky debt and holding riskless asset at time $t = 0$. There are several possibilities for this agent. At time $t = 1$ in state M , the agent can either hold the risk-free or risky asset. At time $t = 1$ in state D , the agent can either hold the risk-free or risky asset. To simplify notation, let $\gamma_U^{\pi 0} = \gamma_U(h_{J0})$, $\gamma_M^{\pi 0} = \gamma_M(h_{J0})$ and $\gamma_D^{\pi 0} = \gamma_D(h_{J0})$. Thus, we have four possible equations defining this agent:

M safe, D safe.

$$\frac{(\gamma_U^{\pi 0} + \gamma_M^{\pi 0})p_M + (\gamma_D^{\pi 0})p_D}{\pi_0} = 1$$

M risky, D risky.

$$\begin{aligned} & \frac{\gamma_U^{\pi 0}p_M}{\pi_0} + \left(\frac{\gamma_M^{\pi 0}p_M}{\pi_0} \right) \left(\frac{\gamma_U^{\pi 0}(1-d_{MD}^Y)}{p_M-d_{MD}^Y} \right) + \left(\frac{\gamma_D^{\pi 0}p_D}{\pi_0} \right) \left(\frac{\gamma_U^{\pi 0}(1-d_{DD}^Y)}{p_D-d_{DD}^Y} \right) \\ &= \gamma_U^{\pi 0} + \gamma_M^{\pi 0} \left(\frac{\gamma_U^{\pi 0}(1-d_{MD}^Y)}{p_M-d_{MD}^Y} \right) + \gamma_D^{\pi 0} \left(\frac{\gamma_U^{\pi 0}(1-d_{DD}^Y)}{p_D-d_{DD}^Y} \right) \end{aligned}$$

M safe, D risky.

$$\frac{(\gamma_U^{\pi 0} + \gamma_M^{\pi 0})p_M}{\pi_0} + \left(\frac{\gamma_D^{\pi 0}p_D}{\pi_0} \right) \left(\frac{\gamma_U^{\pi 0}(1-d_{DD}^Y)}{p_D-d_{DD}^Y} \right) = \gamma_U^{\pi 0} + \gamma_M^{\pi 0} + \gamma_D^{\pi 0} \left(\frac{\gamma_U^{\pi 0}(1-d_{DD}^Y)}{p_D-d_{DD}^Y} \right)$$

M risky, D safe.

$$\frac{\gamma_U^{\pi 0}p_M}{\pi_0} + \left(\frac{\gamma_M^{\pi 0}p_M}{\pi_0} \right) \left(\frac{\gamma_U^{\pi 0}(1-d_{MD}^Y)}{p_M-d_{MD}^Y} \right) + \frac{\gamma_D^{\pi 0}p_D}{\pi_0} = \gamma_U^{\pi 0} + \gamma_M^{\pi 0} \left(\frac{\gamma_U^{\pi 0}(1-d_{MD}^Y)}{p_M-d_{MD}^Y} \right) + \gamma_D^{\pi 0}$$

The known market clearing conditions are for the asset and debt at time 0:

Time $t = 0$, Asset:

$$(1 - h_{M0}) \frac{(1 + p_0)}{(p_0 - \pi_0)} + (h_{M0} - h_{D0}) \frac{(1 + p_0)}{(p_0 - p_D)} = 1$$

Time $t = 0$, Risky Debt:

$$(1 - h_{M0}) \frac{(1 + p_0)}{(p_0 - \pi_0)} = (h_{D0} - h_{J0}) \frac{(1 + p_0)}{\pi_0}$$

Market Clearing, unknown

Time $t = 1$, State M Asset: We are unsure whether $h_{MM} \in (0, h_{J0})$, $h_{MM} \in (h_{J0}, h_{D0})$, or $h_{MM} \in (h_{D0}, h_{M0})$. This issue can be resolved by considering all cases, solving for equilibrium, and checking that h_{MM} is indeed in the specified interval. $h_{MM} \in (0, h_{J0})$:

$$\frac{(h_{M0} - h_{D0}) \left(\frac{1+p_0}{p_0-p_D} \right) (p_M - p_D)}{p_M - d_{MD}^Y} + \frac{(h_{D0} - h_{J0}) \left(\frac{1+p_0}{\pi_0} \right) (p_M)}{p_M - d_{MD}^Y} + \frac{(h_{J0} - h_{MM})(1 + p_0)}{p_M - d_{MD}^Y} = 1$$

$h_{MM} \in (h_{J0}, h_{D0})$:

$$\frac{(h_{M0} - h_{D0}) \left(\frac{1+p_0}{p_0-p_D} \right) (p_M - p_D)}{p_M - d_{MD}^Y} + \frac{(h_{D0} - h_{MM}) \left(\frac{1+p_0}{\pi_0} \right) (p_M)}{p_M - d_{MD}^Y} = 1$$

$h_{MM} \in (h_{D0}, h_{M0})$:

$$\frac{(h_{M0} - h_{MM}) \left(\frac{1+p_0}{p_0-p_D} \right) (p_M - p_D)}{p_M - d_{MD}^Y} = 1$$

Time $t = 1$, State D Asset: We do not know whether $h_{DD} \in (0, h_{J0})$ or $h_{DD} \in (h_{J0}, h_{D0})$. In the first case, we have:

$$\frac{(h_{D0} - h_{J0}) \left(\frac{1+p_0}{\pi_0} \right) p_D}{p_D - d_{DD}^Y} + \frac{(h_{J0} - h_{DD})(1 + p_0)}{p_D - d_{DD}^Y} = 1.$$

In the second case, we have

$$\frac{(h_{D0} - h_{DD}) \left(\frac{1+p_0}{\pi_0} \right) p_D}{p_D - d_{DD}^Y} = 1.$$

Thus, we obtain the following possible cases in equilibrium:

1. h_{D0} holds risky asset at time 1 in state M . (a) h_{J0} holds risky asset in state M and risky asset

in state D . This implies that $h_{MM}, h_{DD} \in (0, h_{J0})$. (b) h_{J0} holds **safe** asset in state M and **safe** asset in state D . This implies that $h_{MM}, h_{DD} \in (h_{J0}, h_{D0})$. (c) h_{J0} holds **risky** asset in state M and **safe** asset in state D . This implies that $h_{MM} \in (0, h_{J0})$ and $h_{DD} \in (h_{J0}, h_{D0})$. (d) h_{J0} holds **safe** asset in state M and **risky** asset in state D . This implies that $h_{MM} \in (h_{J0}, h_{D0})$ and $h_{DD} \in (0, h_{J0})$.

2. h_{D0} holds safe asset at time 1 in state M . This implies that $h_{MM} \in (h_{D0}, h_{M0})$ and h_{J0} holds safe asset in state M . (a) h_{J0} holds **safe** asset in state D . This implies that $h_{DD} \in (h_{J0}, h_{D0})$. (b) h_{J0} holds **risky** asset in state D . This implies that $h_{DD} \in (0, h_{J0})$.

C.5.2 Equilibrium Conditions with Collateralization: Section C.3

The equations defining equilibrium are as follows

Marginal Investors, known

\hat{h}_{M0} : indifferent between holding risky asset, leveraged against state M and risky debt leveraged against state D at time 0

$$\frac{\gamma_U(\hat{h}_{M0})(1 - \hat{p}_M)}{\hat{p}_0 - \hat{\pi}_0} = \frac{\gamma_U(\hat{h}_{M0})(\hat{p}_M - \hat{p}_D)}{\hat{\pi}_0 - \hat{p}_D} + \frac{\gamma_M(\hat{h}_i)(\hat{p}_M - \hat{p}_D)}{\hat{\pi}_0 - \hat{p}_D} \left(\frac{\gamma_U(1 - d_{MD}^Y)}{\hat{p}_M - d_{MD}^Y} \right)$$

\hat{h}_{MM} : Indifferent between holding risky asset and safe asset at time 1, state M .

$$\frac{\gamma_U(\hat{h}_{MM})(1 - d_{MD}^Y)}{\hat{p}_M - d_{MD}^Y} = 1$$

\hat{h}_{DD} : Indifferent between holding risky asset and safe asset at time 1, state D .

$$\frac{\gamma_U(\hat{h}_{DD})(1 - d_{DD}^Y)}{\hat{p}_D - d_{DD}^Y} = 1$$

Marginal Investors, unknown

\hat{h}_{J0} : Indifferent between holding risky debt with leverage and holding the safe asset. There are two possibilities for this agent: at time $t = 1$ in state M , the agent can either hold the safe or risky asset; at time $t = 1$ in state D , \hat{h}_{J0} will be the most optimistic agent still in the market, forcing him to hold the risky asset. Thus, we have the following two possibilities

M safe.

$$\frac{(\gamma_U(\hat{h}_{J0}) + \gamma_M(\hat{h}_{J0}))(\hat{p}_M - \hat{p}_D)}{\hat{\pi}_0 - \hat{p}_D} = \gamma_U(\hat{h}_{J0}) + \gamma_M(\hat{h}_{J0}) + \gamma_D(\hat{h}_{J0}) \left(\frac{\gamma_U(\hat{h}_{J0})(1 - d_{DD}^Y)}{p_D - d_{DD}^Y} \right)$$

M risky.¹⁹

$$\begin{aligned} & \frac{\gamma_U(\hat{h}_{J0})(\hat{p}_M - \hat{p}_D)}{\hat{\pi}_0 - \hat{p}_D} + \left(\frac{\gamma_M(\hat{h}_{J0})(\hat{p}_M - \hat{p}_D)}{\hat{\pi}_0 - \hat{p}_D} \right) \left(\frac{\gamma_U(\hat{h}_{J0})(1 - d_{MD}^Y)}{\hat{p}_M - d_{MD}^Y} \right) \\ &= \gamma_U(\hat{h}_{J0}) + \gamma_M(\hat{h}_{J0}) \left(\frac{\gamma_U(\hat{h}_{J0})(1 - d_{MD}^Y)}{\hat{p}_M - d_{MD}^Y} \right) + \gamma_D(\hat{h}_{J0}) \left(\frac{\gamma_U(\hat{h}_{J0})(1 - d_{DD}^Y)}{p_D - d_{DD}^Y} \right) \end{aligned}$$

Market Clearing, known

Time $t = 0$, Risky Asset:

$$(1 - \hat{h}_{M0}) \left(\frac{1 + \hat{p}_0}{\hat{p}_0 - \hat{\pi}_0} \right) = 1$$

Time $t = 1$, state D , Risky Asset:

$$(\hat{h}_{J0} - \hat{h}_{DD}) \left(\frac{1 + \hat{p}_0}{\hat{p}_D - d_{DD}^Y} \right) = 1$$

Time $t = 0$, Risky Debt:

$$(1 - \hat{h}_{M0}) \left(\frac{1 + \hat{p}_0}{\hat{p}_0 - \hat{\pi}_0} \right) = (\hat{h}_{M0} - \hat{h}_{J0}) \left(\frac{1 + \hat{p}_0}{\hat{\pi}_0 - \hat{p}_D} \right)$$

Market Clearing, unknown

Time $t = 1$, State M Asset: We are unsure whether $\hat{h}_{MM} \in (0, \hat{h}_{J0})$ or $\hat{h}_{MM} \in (\hat{h}_{J0}, \hat{h}_{M0})$. This issue can be resolved by considering both cases, solving for equilibrium, and checking that \hat{h}_{MM} is indeed in the specified interval.

$$\hat{h}_{MM} \in (0, \hat{h}_{J0}):$$

$$\frac{(\hat{h}_{M0} - \hat{h}_{J0}) \left(\frac{1 + \hat{p}_0}{\hat{\pi}_0 - \hat{p}_D} \right) (\hat{p}_M - \hat{p}_D)}{\hat{p}_M - d_{MD}^Y} + \frac{(\hat{h}_{J0} - \hat{h}_{MM})(1 + \hat{p}_0)}{\hat{p}_M - d_{MD}^Y} = 1$$

¹⁹the payout of the asset in state M at time 1 is multiplied by the continuation value of the asset in time 2 since we have specified that \hat{h}_{J0} will hold the risky asset.

$$\hat{h}_{MM} \in (\hat{h}_{J0}, \hat{h}_{M0}):$$

$$\frac{(\hat{h}_{M0} - \hat{h}_{MM}) \left(\frac{1 + \hat{p}_0}{\pi - \hat{p}_D} \right) (\hat{p}_M - \hat{p}_D)}{\hat{p}_M - d_{MD}^Y} = 1$$

D Empirical Implications

Our model offers several empirical implications regarding how explicit and implicit uses of debt as collateral affect returns and leverage: (i) capital structures are affected by funding markets and ease of financing, designed in part to stretch collateral; (ii) debt collateralization decreases economy-wide margins as borrowers shift to using high-leverage, risky contracts; (iii) debt collateralization decreases risk premia, increases defaults, and tends to increase asset prices and volatility; (iv) these effects are strongest when the economy has larger demand for negative-beta assets.

D.1 Static Implications

Capital Structures The main result of our analysis is that debt collateralization (implicit or explicit) leads investors to take maximal leverage, or more broadly fewer investors use low leverage. While the corporate finance literature has emphasized the role of capital structure in mitigating informational frictions, our results imply that capital structures are defined in part to stretch scarce collateral: when “collateral is tight” capital structures should be designed to further stretch collateral.²⁰ An empirical test of this prediction could be to use measures of ease of financing (e.g. loan margins or haircuts in funding markets, or measures that typically correlate with risk and other determinants of credit conditions) to see how capital structures (for LBOs, syndicated loans, mortgages, etc.) respond to changes in funding markets, controlling for incentive conflicts.²¹

Empirically, growth in the origination of CDOs, CDO-squareds, etc., should all else equal translate into changes in capital structures in the ABS deals underlying CDO structures. Implicit or

²⁰Critically, our analysis assumes that default is “costless,” which is obviously problematic in many situations. [Asquith et al. \(1994\)](#) find that debt structure affects restructuring decisions, and [Alderson and Betker \(1995\)](#) find that when liquidation costs are high, firms choose capital structure to minimize firm distress.

²¹For example, [Axelson et al. \(2013\)](#) find that public firms have high leverage when credit spreads are high (when credit is expensive), whereas LBO deals have high leverage when credit spreads are low; they interpret LBO’s “buy expensive when credit is cheap” to reflect agency problems between private equity sponsors and their investors. [Rauh and Sufi \(2010\)](#) show that low-credit-quality firms more likely to have multi-tiered capital structure, with secured bank debt (tight covenants) and subordinated non-bank debt (loose covenants), in order to reduce incentive conflicts.

explicit debt collateralization would imply that subordinated tranches would be larger and riskier, with the largest effect on the most subordinated tranches. However, rather than affecting capital structure, the effect of greater collateralization could manifest itself by changing the underlying composition of ABS collateral. If higher prices would change incentives for mortgage lenders, then debt collateralization could contribute to those incentives. While [Benmelech et al. \(2012\)](#) find no evidence that corporate loans securitized in CLOs were riskier, there is evidence that securitization affected underlying loans in the subprime mortgage market.²² Our analysis takes as given the collateral quality and ignores any informational asymmetries.

Risk Premia Our results imply that risky debt that can be used as collateral should experience decreased risk premia. Empirically, [Nadauld and Weisbach \(2012\)](#) find evidence that securitization of corporate loans was associated with lower spreads by 17 basis points, consistent with a reduction in cost of capital. [Lemmon et al. \(2014\)](#) find that securitization of receivables by nonfinancial firms decreases financing costs by providing access to segmented markets, and innovations in capital structure increase firm value even for large, mid-tier credit firms. Measures of how funding markets treat derivative debt contracts should correspond to increases in debt riskiness together with lower risk premia. In our model the face values of promises are fixed, but one might expect more generally to see debt collateralization leading to larger face values as well, and thus riskier debt.

We show that under sufficient conditions, debt collateralization increases asset prices. Thus, increased ability to use debt as collateral should increase prices of underlying collateral (or push down risk premia). However, as noted earlier, to the extent that collateral quality can change or is subject to informational frictions, prices may stay the same and instead composition changes.

Origination and Securitization Volume Debt collateralization may provide incentives to produce collateral.²³ [Shivdasani and Wang \(2011\)](#) find evidence that the leveraged buyout (LBO) boom of 2004 to 2007 was fueled by growth in CDOs and other forms of securitization, which facilitated

²²[Keys et al. \(2010\)](#) show that securitization does not change the interest rate or the LTV ratio for mortgages, but nonetheless affects the subsequent performance of mortgages through reduced screening by lenders. [Nadauld and Weisbach \(2012\)](#) find that securitization led to less screening for subprime borrowers. Additionally, [Wang and Xia \(2014\)](#) find that banks active in CLO securitization exert less effort on ex-post monitoring.

²³see [Fostel and Geanakoplos 2016](#).

much larger LBOs than historically possible, potentially because it helped relax balance sheet constraints that banks faced in financing large LBOs.

Securitization decreased spectacularly after the crisis (see [Chernenko et al. 2014](#)). While there are many explanations for why this occurred, one possibility is that investors realized many ABS tranche payoffs were better described by “pay full or zero”—in this case debt collateralization is not meaningful. It is well understood that correlations in ABS tranches were the primary determinant of CDO quality, and as the mortgage bubble burst it became clear that CDOs would be essentially worthless or of full-value, with very little likelihood of intermediate values. As the comparative statics demonstrate, when uncertainty is less dispersed (M closer to D), there is less debt collateralization in equilibrium. The incentive for debt collateralization would decrease if uncertainty more closely followed a binomial model (in a binomial model all equilibrium debt is risk free). An implication is that the extent of debt collateralization (whether measured by origination volume or margins on securitized debt) should vary with perceptions of dispersion of payoffs conditional on default. Greater risk dispersion should lead to less debt collateralization.

Negative-Beta Assets/Pessimism Finally, an economy with relatively more pessimists, or more demand for negative-beta assets, exhibits more sensitivity to debt collateralization: the increase in prices is greater; the change in risk premia is greater; the composition of margins is more affected. (An empirical proxy for these measures could be risk-premia variation as a function of an asset’s beta.) Critically, with relatively more pessimists, agents with leverage primarily use low leverage by making safe promises, but with debt collateralization agents switch to using high leverage by making risky promises.

D.2 Dynamic Implications

Our analysis suggests that debt collateralization increases volatility, particularly in response to very bad news, which has the following implications:

(i) Assets with easily financed derivative debt should have prices that fluctuate more in response to news or other changes in fundamentals (given the volatility of fundamentals). However, one might expect that less volatile assets receive better treatment as collateral (endogeneity is a concern). Additionally, debt collateralization should be associated with increased default rates (how funding

markets treat collateral is clearly endogenous, reflecting in part expected default rates).

(ii) Since debt collateralization can also be a metaphor for capital structure, the price of firms whose capital structure implicitly provides debt collateralization (through greater use of subordinated tranches, for example) should be more volatile. Since capital structure is endogenous, testing this prediction would require using some exogenous variation in capital structure (perhaps driven by corporate governance or regulation).

(iii) The model with only leverage suggests that when there is greater dispersion in payoffs conditional on default (M and D are more different) then price crashes after really bad news becomes worse because agents make more risky promises. Measures of uncertainty should correlate with larger price crashes, holding fixed expectations about worst-case scenarios.